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A Short Note on Local Fractional Calculus of Function of One Variable

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Abstract – Local fractional calculus (LFC) handles everywhere continuous but nowhere differentiable functions in fractal space. This note investigates the theory of local fractional derivative and integral of function of one variable. We first introduce the theory of local fractional continuity of function and history of local fractional calculus. We then consider the basic theory of local fractional derivative and integral, containing the local fractional Rolle's theorem, L'Hospital's rule, mean value theorem, anti-differentiation and related theorems, integration by parts and Taylor' theorem. Finally, we study the efficient application of local fractional derivative to local fractional extreme value of non-differentiable functions, and give the theory of local fractional extreme value, containing local fractional extreme value theorem, Fermat's theorem, increasing/decreasing test and the derivative test. It is of great significances for us to process optimization problems of the non-differentiable functions on Cantor set.

Keywords –Local fractional calculus; Local fractional Taylor theorem; Mean value theorem; Local fractional Fermat's theorem; Local extreme value theorem; Cantor set

1. Introduction

Since the by-now classical textbook of Mandelbrot [1], fractals have been revealed a useful tool in several areas ranging from fundamental science to engineering, from microphysics to macrophysics, from chaotic dynamics to non-random phenomena. A fractal phenomenon is characterized by striking irregularities, and as a result, it is described by non-differentiable functions.

In the 1990's, Kolwankar and Gangal [2-6] proposed local fractional derivative operator through renormalization of Riemann-Liouville definition. The classical local fractional derivative of the Hölder continuous functions defined on cantor's set were discussed and Hölder exponent is exactly the order of local fractional derivative of the functions. It is important, as a useful tool, to deal with the non-differentiable functions, which are irregular in the real world, defined on fractal set [2-10, 12-26].

Recently, Jumarie, dealing with the fractional derivative of a constant being zero, had proposed the modified Riemann–Liouville derivative, arrived at the same identification, which was started from a fractional derivative defined as the limit of fractional difference, which allowed us to obtain the generalized fractional Taylor's series, and found that the Hölder exponent is exactly the order of the fractional derivative of the function under consideration [28-29].

More recently, local fractional calculus was modified, and a short definition of the local fractional derivative and integral were proposed [22-26]. Local fractional transforms based on the local fractional calculus were discussed [24]. Here, based on local fractional calculus, the local fractional extreme value of non-differentiable functions continued to be done. Our purpose, in the present short paper, is to provide a short note on the fundamentals of the local fractional calculus.

This paper is organized as follows. In section 2, local fractional calculus is investigated. Section 3 is devoted to the fundamentals of the local fractional calculus. The application of local fractional derivative to a scheme on local fractional extreme value of the non-differentiable functions (the optimization problems of non-differentiable functions on Cantor set) is discussed in section 4. Conclusions are in section 5.

2. Theory of Local Fractional Continuity of Function and History of Local Fractional Derivative and Integral

2.1. The local fractional continuity of the non-differentiable functions

Definition 2.2.1 A function $f:\Re \to \Re, X \to f(X)$, is called a non-differentiable function of exponent α, $0<\alpha\le 1$, which satisfy Hölder function of exponent α, then for $x, y \in X$ such that [2,11]

$$\left|f(x)-f(y)\right| \le C\left|x-y\right|^{\alpha}. \tag{2.1}$$

Definition 2.2.2 A non-differentiable function

$f:\Re \to \Re$, $x \to f(x)$

is called to be local fractional continuous of order α, $0<\alpha\le 1$, or shortly α -local fractional continuous, when we have[24]

$$f(x)-f(x_0)=o\left((x-x_0)^{\alpha}\right). \qquad (2.2)$$

Definition 2.2.3 Suppose that $f(x)$ is defined throughout some interval containing x_0 and all point near x_0, $f(x)$ is local fractional continuous at x_0 if to each positive ε and some positive constant c corresponds some positive δ such that

$$\left|f(x)-f(x_0)\right|<c\varepsilon^{\alpha}$$

whenever $\left|x-x_0\right|<\delta$ and $0<\alpha\leq 1$.

Remark. This form of the condition for continuity is equivalent to definition 2.2.2. That is to say, $f(x)$ is local fractional continuous at x_0 in closed interval if and only if

$$\lim_{x\to x_0} f(x)=f(x_0),$$

$$\lim_{x\to a^-} f(x)=f(a)$$

and

$$\lim_{x\to b^-} f(x)=f(b).$$

If a function $f(x)$ is said in the space $C_{\alpha}[a,b]$ if and only if it can be written as

$$f(x)-f(x_0)=o\left((x-x_0)^{\alpha}\right)$$

with any $x_0\in C_{\alpha}[a,b]$ and $0<\alpha\leq 1$.

Definition 2.2.4 For all number x of a fractal set we have $m<x<M$, the fractal set is called bounded.

Lemma 2.2.1[39, 40]

Every bounded infinite fractal set has at least one limit point.

Remark. Because every bounded infinite fractal set is a special bounded infinite set, this expression of **Lemma 2. 2. 1** is always given.

As a direct result, we have the following results:

Theorem 2.2.2 [39, 40]

Suppose that $f(x)\in C_{\alpha}[a,b]$, then $f(x)$ is bounded.

Proof. Because of $f(x)\in C_{\alpha}[a,b]$, there exist each positive ε and some positive constant c corresponds some positive δ such that

$$\left|f(x)-f(x_0)\right|<c\varepsilon^{\alpha}$$

whenever $\left|x-x_0\right|<\delta$ and $0<\alpha\leq 1$.

When $m=-c\varepsilon^{\alpha}+f(x_0)$ and $M=c\varepsilon^{\alpha}+f(x_0)$, we have

$$m<f(x)<M.$$

Theorem 2.2.3 [39, 40]

Suppose that $f(x), g(x)\in C_{\alpha}[a,b]$, then we have that $f(x)\pm g(x)$, $f(x)g(x)$ and $f(x)/g(x)$, the last only if $g(x)\neq 0$.

Theorem 2.2.4 [39, 40]

Suppose that $f(x)\in C_{\alpha}[a,b]$ and if $f(a)=A$ and $f(b)=B$, then corresponding to any number C between A and B there exists at least one number C in $[a,b]$ such that $f(c)=C$.

Theorem 2.2.5 [39, 40]

Suppose that $f(x)\in C_{\alpha}[a,b]$. If $f(a)$ and $f(b)$ have opposite signs, there is at one number for

$$f(c)=0$$

where $a<c<b$.

Theorem 2.2.6 [39, 40]

Suppose that $f(x)\in C_{\alpha}[a,b]$, then $f(x)$ has a maximum value M for at least one value of x in the interval and a minimum value m for at least one value of x in the interval.

Theorem 2.2.7 [39, 40]

Suppose that $f(x)\in C_{\alpha}[a,b]$. If M and m are respectively the least upper bound and greatest lower bound of $f(x)$, there exists at least one value of x in the interval for which $f(x)=m$ or $f(x)=M$.

Remark. Such expressions are precisely the classical continuous theorem in case of fractal dimension $\alpha=1$.

2.2. History of local fractional derivative and integral

2.2. 1 Local fractional derivative

Definition 2.2.1 Let $f(x)\in C_{\alpha}[a,b]$, K-G local fractional derivative of the function $f(x)$ is given by [2-3]

$$\begin{aligned} &{}_{x_0}D_x^{\alpha} f(x) \\ &= f^{(\alpha)}(x_0) = \frac{d^{\alpha} f(x)}{dx^{\alpha}}\Big|_{x=x_0} \\ &=: \lim_{x \to x_0} \frac{d^{\alpha}\left[f(x) - f(x_0)\right]}{\left[d(x - x_0)\right]^{\alpha}}, 0 < \alpha \le 1. \end{aligned} \tag{2.3}$$

Remark. It is found that, in this expression, α is precisely the Hölder exponent of function defined cantor's set. That is to say, $\left[d(x - x_0)\right]^{\alpha}$, which is a fractal span, is a fractal geometrical meaning. K-G local fractional derivative of a constant is zero.

Definition 2.2.2 Let $f(x) \in C_{\alpha}[a,b]$ or $C_1[a,b]$, Modified fractional derivative of the function $f(x)$ is given by [28-29]

$$\begin{aligned} &{}_{x_0}D_x^{\alpha} f(x) \\ &= f^{(\alpha)}(x_0) = \frac{d^{\alpha} f(x)}{dx^{\alpha}}\Big|_{x=x_0} \\ &=: \lim_{h \to 0} \frac{\Delta^{\alpha}\left[f(x) - f(x_0)\right]}{h^{\alpha}} \end{aligned} \tag{2.4}$$

with

$$\Delta^{\alpha} f(x) = \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} f\left(x + (\alpha - k)h\right)$$

and $0 < \alpha \le 1$.

Remark. Modified fractional derivative of a constant is zero. Such an expression, irrespective of any fractal set, was started from the expression of the fractional derivative as the limit of a fractional difference involving an infinite number of terms, and having been come across the Hölder's exponent, fractional generalized Taylor's series was obtained. Here, h is a constant discretization span.

Definition 2.2.3 Let $f(x) \in C_{\alpha}[a,b]$, P- G local fractional derivative of the function $f(x)$ is given by [9-10, 16, 17]

$$\begin{aligned} &{}_{x_0}D_x^{\alpha} f(x) \\ &= f^{(\alpha)}(x_0) \\ &= \frac{d^{\alpha} f(x)}{dx^{\alpha}}\Big|_{x=x_0} \\ &= F - \lim_{x \to x_0} \frac{f(x) - f(x_0)}{S_F^{\alpha}(x) - S_F^{\alpha}(x_0)} \end{aligned}, \tag{2.5}$$

where $F - \lim\limits_{x \to x_0}$ is the notion of the limit of $f(x)$ through the points of fractal set F.

Remark. This expression had extended to K-G local fractional derivative. In other word, this is equivalent to K-G local fractional derivative. As a simple example, it is said that, the fractional integral of the characteristic function of a Cantor set should be the corresponding Cantor staircase function, while the local fractional derivative of the latter should be the former.

Definition 2.2.4 Let $f(x) \in C_1[a,b]$, F-J local fractional derivative of the function $f(x)$ is given by [7, 13, 14]

$$\begin{aligned} &{}_{x_0}D_x^{\alpha} f(x) \\ &= f^{(\alpha)}(x_0) \\ &= \frac{d^{\alpha} f(x)}{dx^{\alpha}}\Big|_{x=x_0} \\ &=: \lim_{x \to x_0^{\sigma}} D_{y,-\sigma}^{\alpha}\left[\sigma\left(f - f(x_0)\right)(x)\right] \end{aligned} \tag{2.6}$$

with $\sigma = \pm$ and $D_{y,-\sigma}^{\alpha}$ is Riemann-Liouville derivative operator.

Definition 2.2.5 Let $f(x) \in C_{\alpha}[a,b]$, local fractional derivative of the function $f(x)$ is given by [21-26, 31-37]

$$\begin{aligned} &{}_{x_0}D_x^{\alpha} f(x) \\ &= f^{(\alpha)}(x_0) \\ &= \frac{d^{\alpha} f(x)}{dx^{\alpha}}\Big|_{x=x_0} \\ &=: \lim_{x \to x_0} \frac{\Delta^{\alpha}\left[f(x) - f(x_0)\right]}{(x - x_0)^{\alpha}} \end{aligned}, \tag{2.7}$$

where

$$\Delta^{\alpha}\left[f(x) - f(x_0)\right] = \Gamma(1+\alpha)\left[f(x) - f(x_0)\right]$$

and $0 < \alpha \le 1$.

Remark. It is found that, in this expression, α is precisely the Hölder exponent of function defined any fractal set because

$$\begin{aligned} &d^{\alpha}\left[f(x) - f(x_0)\right] \\ &= \Delta^{\alpha}\left[f(x) - f(x_0)\right] \\ &= \Gamma(1+\alpha)\left[f(x) - f(x_0)\right] \end{aligned}$$

and

$$\left[d(x - x_0)\right]^{\alpha} = (x - x_0)^{\alpha}$$

with the notion of fractal discretization span. Such an expression is short of the K-G local fractional derivative if the non-differentiable functions are defined cantor set and is referred to Such an expression is classical derivative in case of $\alpha = 1$ and local fractional derivative of a constant is zero. Meanwhile,

$$\frac{d^{\alpha}\left(\sum_{k=0}^{n}\frac{f^{(k)}(x_0)(x-x_0)^k}{\Gamma(1+k)}\right)}{dx^{\alpha}}=0$$

and therefore

$$\frac{d^{\alpha}f(x)}{dx^{\alpha}}=0$$

for $|f(y)-f(x)|\le C|y-x|$.

As further results, the form of the local fractional derivative was obtained based on the K-G local fractional derivative [20].

2.2.2 Local fractional integral

Definition 2.2.6 Let $f(x)\in C_{\alpha}[a,b]$, K- G local fractional integral of the function $f(x)$ is given by [4, 9, 10]

$$ {}_aI_b^{(\alpha)}f(x)=\lim_{N\to\infty}\sum_{i=0}^{N-1}f(x_i^*)\frac{d^{-\alpha}1_{dx_i(x)}}{d(x_{i+1}-x_i)}, \tag{2.8}$$

with $1_{dx_i(x)}$ is the unit function defined upon $[x_i,x_{i+1}]$ and x_i^* some suitable point of the interval $[x_i,x_{i+1}]$, where $[x_i,x_{i+1}]$, $i=0,.....,N-1$, $x_0=a,x_N=m$, is a partition of the interval $[a,b]$.

Definition 2.2.7 Let $f(x)\in C_{\alpha}[a,b]$, P- G local fractional integral of the function $f(x)$ is given by [16-17]

$$\begin{aligned}&{}_aI_b^{(\alpha)}f(x)\\&=\int_a^b f(x)d_F^{\alpha}x\\&=\sum_{j=0}^{N-1}f(x_j)\left(S_F^{\alpha}(x_{j+1})-S_F^{\alpha}(x_j)\right)\end{aligned},0<\alpha\le1, \tag{2.9}$$

where $S_F^{\alpha}(b)=\frac{1}{\Gamma(1+\alpha)}\lim_{\Delta x\to0}\sum_{i=0}^{N-1}(\Delta x_i)^{\alpha}$ with $\Delta x=\max(\Delta x_i)$ and $\Delta x_i=x_{i+1}-x_i$, where $[x_{i+1},x_i]$, $i=0,.....,N-1$, $x_0=a$, $t_N=b$, is a partition of the interval $[a,b]$.

Remark. In the form, this expression is equivalent to the K-G local fractional integral.

Definition 2.2.8 Let $f(x)\in C_{\alpha}[a,b]$ or $C_1[a,b]$, Jumarie fractional integral of the function $f(x)$ is given by [28-29]

$$\begin{aligned}&{}_0I_x^{(\alpha)}f(x)\\&=\int_0^x f(t)(dt)^{\alpha}\\&=:\alpha\int_0^x x(t-t)^{\alpha-1}f(t)dt\end{aligned}, 0<\alpha\le1, \tag{2.10}$$

Remark. Such an expression is precisely derived from fractional integral.

Definition 2.2.9 Let $f(x)\in C_{\alpha}[a,b]$, local fractional integral of the function $f(x)$ is given by [21-26, 31-37]

$$\begin{aligned}&{}_aI_b^{(\alpha)}f(x)\\&=\frac{1}{\Gamma(1+\alpha)}\int_a^b f(t)(dt)^{\alpha}\\&=\frac{1}{\Gamma(1+\alpha)}\lim_{\Delta t\to0}\sum_{j=0}^{N-1}f(t_j)(\Delta t_j)^{\alpha}\end{aligned} \tag{2.11}$$

with $0<\alpha\le1$, $\Delta t=\max\{\Delta t_1,\Delta t_2,\Delta t_j,...\}$ and $\Delta t_j=t_{j+1}-t_j$, where $[t_j,t_{j+1}]$, $j=0,.....,N-1$, $t_0=a$, $t_N=b$, is a partition of the interval $[a,b]$.

For convenience, we assume that

$${}_aI_a^{(\alpha)}f(x)=0 \text{ if } a=b$$

and

$${}_aI_b^{(\alpha)}f(x)=-{}_bI_a^{(\alpha)}f(x) \text{ if } a<b.$$

Remark. As a matter of fact, such an expression has difference from the definition of the Jumarie's fractional integral of the function in despite of similar forms of them. $(\Delta t_j)^{\alpha}$ is an any element of fractal discretization span in the interval $[t_j,t_{j+1}]$. In a sense, when

$$(\Delta t_j)^{\alpha}=\Gamma(1+\alpha)\left(S_F^{\alpha}(x_{j+1})-S_F^{\alpha}(x_j)\right)$$

this expression could be equivalent to P-G local fractional integral. It is found that there is more significance of expression to deal with non-differentiable functions. Hence, from the perspective of fractal geometry, fractal discretization span is number of operations on fractal set. Such an expression is classical integral in case of $\alpha=1$.

A fractal set of points on real line is said to have *generalized measure zero* if the sum of the lengths of intervals enclosing all the points can be made arbitrary small.

Lemma 2.2.1 [39, 40]

Suppose that $f(x)$ is defined on fractal set of fractal dimension α and is bounded on $[a,b]$, then a necessary and sufficient condition for the existence of

$$\frac{1}{\Gamma(1+\alpha)}\int_a^b f(t)(dt)^{\alpha}$$

is that the fractal set of discontinuities of $f(t)$ have measure zero.

3. The Fundamental Theory of the Local Fractional Derivative and Integral

3.1. The element theory of the local fractional derivative of the non-differentiable functions

Definition 3.1.1 If $f:[a,b]\mapsto R$ is local fractional continuous. For $1\geq\alpha>0$, $\delta>0$ and $x\in(x_0-\delta,x_0)$, the limit [22]

$$\begin{aligned}&{}_{x_0^-}D_x^{\alpha}f(x)\\&=f^{(\alpha)}(x_0^-)\\&=\frac{d^{\alpha}f(x)}{(dx)^{\alpha}}\Big|_{x=x_0^-}\\&=\frac{d^{\alpha}f(x)}{dx^{\alpha}}\Big|_{x=x_0^-}\\&=\lim_{x\to x_0^-}\frac{\Gamma(1+\alpha)\left[f(x)-f(x_0^-)\right]}{(x-x_0^-)^{\alpha}}\end{aligned}\tag{3.1}$$

exists and is finite, then $f(x)$ is said to have the left-hand local fractional derivative of order α at $x=x_0$.

Definition 3.1.2 If $f:[a,b]\mapsto R$ is local fractional continuous. For $1\geq\alpha>0$, $\delta>0$ and $x\in(x_0,x_0+\delta)$, the limit [22]

$$\begin{aligned}&{}_{x_0^+}D_x^{\alpha}f(x)\\&=f^{(\alpha)}(x_0^+)\\&=\frac{d^{\alpha}f(x)}{(dx)^{\alpha}}\Big|_{x=x_0^+}\\&=\frac{d^{\alpha}f(x)}{dx^{\alpha}}\Big|_{x=x_0^+}\\&=\lim_{x\to x_0^+}\frac{\Gamma(1+\alpha)\left[f(x)-f(x_0^+)\right]}{(x-x_0^+)^{\alpha}}\end{aligned}\tag{3.2}$$

exists and is finite, then $f(x)$ is said to have the right-hand local fractional derivative of order α at $x=x_0$.

Remark. The similar forms of the right-left hand local fractional derivative of the function were given [13].

As a direct application, we have the following result:

Property 3.1.1 [22]

If ${}_{x_0}D_x^{\alpha}f(x)\big|_{x=x^-_0}$ and ${}_{x_0}D_x^{\alpha}f(x)\big|_{x=x^+_0}$ exist and

$${}_{x_0}D_x^{\alpha}f(x)\Big|_{x=x^-_0}={}_{x_0}D_x^{\alpha}f(x)\Big|_{x=x^+_0},\tag{3.3}$$

then

$${}_{x_0}D_x^{\alpha}f(x)\Big|_{x=x_0}={}_{x_0}D_x^{\alpha}f(x)\Big|_{x=x^-_0}={}_{x_0}D_x^{\alpha}f(x)\Big|_{x=x^+_0}.\tag{3.3}$$

Property 3.1.2 [22]

Suppose that $f:[a,b]\mapsto R$ is local fractional derivative at $x=x_0$, then $f(x)$ is local fractional continuous at $x=x_0$.

Remark. For $0<\alpha\leq1$, the expression

$$\Delta^{\alpha}f(x)=f^{(\alpha)}(x)(\Delta x)^{\alpha}+\lambda(\Delta x)^{\alpha}$$

is called the increment of $f(x)$, where Δx is increment of x and $\lambda\to0$ as $\Delta x\to0$.

Remark. For $0<\alpha\leq1$, the expression

$$d^{\alpha}f=f^{(\alpha)}(x)(dx)^{\alpha}$$

is called the α-local fractional differential of $f(x)$.

Remark. If there exists any point $x_0\in(a,b)$ such that

$$\frac{d^{\alpha}f(x)}{dx^{\alpha}}|x=x_0=f^{(\alpha)}(x_0),$$

then $D_{\alpha}(a,b)$ is a α- local fractional derivative set.

Remark. If $f(x)\in D_{\alpha}(a,b)$, then $f(x)$ is α-local fractional differentiable on (a,b).

Remark. If

$$\begin{aligned}&\frac{d^{\alpha}}{dx^{\alpha}}\left[\frac{d^{\alpha}}{dx^{\alpha}}f(x)\right]\\&=\left(D_x^{\alpha}\bullet D_x^{\alpha}\right)f(x)\\&=D_x^{2\alpha}\left[f(x)\right]\\&=f^{(2\alpha)}(x)\end{aligned}\tag{3.5}$$

denotes the 2α- local fractional derivative of $f(x)$, for $0<\alpha\leq1$, at $x=x_0$ we have

$$\begin{aligned}&\left(D_x^{\alpha}\bullet D_x^{\alpha}\right)f(x)\Big|_{x=x_0}\\&=D_x^{2\alpha}\left[f(x)\right]\Big|_{x=x_0}\\&=f^{(2\alpha)}(x)\Big|_{x=x_0}\\&=f^{(2\alpha)}(x_0)\end{aligned}.\tag{3.6}$$

As a direct application, we have

$$\begin{aligned}&\overbrace{\left(D_x^{\alpha}\bullet\ldots\bullet D_x^{\alpha}\right)}^{n\ times} f(x)\Big|_{x=x_0}\\&=D_x^{n\alpha}\left[f(x)\right]\Big|_{x=x_0}\\&=f^{(n\alpha)}(x)\Big|_{x=x_0}\\&=f^{(n\alpha)}(x_0)\end{aligned} \qquad (3.7)$$

Lemma 3.1.3 [39]

Suppose that $f(x)\in C_\alpha[a,b]$ and $f(x)\in D_\alpha(a,b)$, then for $1\geq\alpha>0$ we have a α -differential form

$$d^\alpha f(x)=f^\alpha(x)(dx)^\alpha.$$

Rules of local fractional differentiation

Suppose that $f(x)$ and $g(x)$ are differentiable functions, the following differentiation rules are valid.

(I)

$$\frac{d^\alpha\left(f(x)\pm g(x)\right)}{dx^\alpha}=\frac{d^\alpha f(x)}{dx^\alpha}\pm\frac{d^\alpha g(x)}{dx^\alpha}; \qquad (3.8)$$

(II)

$$\frac{d^\alpha\left(f(x)g(x)\right)}{dx^\alpha}=g(x)\frac{d^\alpha f(x)}{dx^\alpha}+f(x)\frac{d^\alpha g(x)}{dx^\alpha}; \qquad (3.9)$$

(III)

$$\frac{d^\alpha\left(\dfrac{f(x)}{g(x)}\right)}{dx^\alpha}=\frac{g(x)\dfrac{d^\alpha f(x)}{dx^\alpha}+f(x)\dfrac{d^\alpha g(x)}{dx^\alpha}}{g(x)^2},$$

$g(x)\neq 0$; (3.10)

(V) $\dfrac{d^\alpha\left(Cf(x)\right)}{dx^\alpha}=C\dfrac{d^\alpha f(x)}{dx^\alpha}$, C is a constant;

(VI) If $y(x)=(f\circ u)(x)$ where $u(x)=g(x)$, then

$$\frac{d^\alpha y(x)}{dx^\alpha}=f^{(\alpha)}\left(g(x)\right)\left(g^{(1)}(x)\right)^\alpha. \qquad (3.11)$$

Remark. In different form of the local fractional derivative, some of above results were discussed [7].

Some useful formulas

(I) $E_\alpha\left(x^\alpha\right)=\sum_{k=0}^{\infty}\dfrac{x^{\alpha k}}{\Gamma(1+k\alpha)}$

where $0<\alpha\leq 1$ [24-26].

(II)

$$\frac{d^\alpha x^{k\alpha}}{dx^\alpha}=\frac{\Gamma(1+k\alpha)}{\Gamma(1+(k-1)\alpha)}x^{(k-1)\alpha}; \qquad (3.12)$$

(III)

$$\frac{d^\alpha E_\alpha\left(x^\alpha\right)}{dx^\alpha}=E_\alpha\left(x^\alpha\right); \qquad (3.13)$$

(V)

$$\frac{d^\alpha E_\alpha\left((cx)^\alpha\right)}{dx^\alpha}=c^\alpha E_\alpha\left((cx)^\alpha\right); \qquad (3.14)$$

(VI)

$$\frac{d^\alpha E_\alpha\left(x^{2\alpha}\right)}{dx^\alpha}=(2x)^\alpha E_\alpha\left(x^{2\alpha}\right). \qquad (3.15)$$

Theorem 3.1.4 [22] (Uniqueness of local fractional derivative)

Suppose that for $1\geq\alpha>0$ ${}_{x_0}D_x^\alpha f(x)$ exists, then it is unique.

Theorem 3.1.5 [22] (Local fractional Rolle's theorem)

Suppose that $f(x)\in C_\alpha[a,b]$ and $f(x)\in D_\alpha(a,b)$. If $f(a)=f(b)$ then there exists a point $c\in(a,b)$ with $f^{(\alpha)}(c)=0$, where $\alpha\in(0,1]$.

Proof. Case 1: $f(x)=0$ in $[a,b]$. Then

$$f^{(\alpha)}(x)=0 \text{ for all } x \text{ in } (a,b).$$

Case 2: $f(x)\neq 0$ in $[a,b]$. Since $f(x)$ is continuous there are points at which $f(x)$ attains its maximum and minimum values, denoted by M and m respectively. Because $f(x)\neq 0$, at least one of the values M , m is not zero. Suppose, for instance, $M\neq 0$ and that $f(c)=M$. For this case, $f(c+\Delta x)\leq f(c)$ is under consideration.

If $\Delta x>0$, then we have

$$\frac{\Gamma(1+\alpha)\left[f(c+\Delta x)-f(c)\right]}{(\Delta x)^{(\alpha)}}\leq 0$$

and

$$\lim_{\Delta x\to 0^+}\frac{\Gamma(1+\alpha)\left[f(c+\Delta x)-f(c)\right]}{(\Delta x)^{(\alpha)}}\leq 0; \qquad (3.16)$$

If $\Delta x<0$, then we have

$$\frac{\Gamma(1+\alpha)\left[f(c+\Delta x)-f(c)\right]}{(\Delta x)^{(\alpha)}}\geq 0$$

and

$$\lim_{\Delta x\to 0^-}\frac{\Gamma(1+\alpha)\left[f(c+\Delta x)-f(c)\right]}{(\Delta x)^{(\alpha)}}\geq 0. \qquad (3.17)$$

Since $f(x) \in D_\alpha(a,b)$, applying **Property 3.1.1**, we find it happen only if the right-hand and left-hand derivatives are both equal to zero, in which case $f^{(\alpha)}(c) = 0$ as required. As similar argument can be used in case $M = 0$ and $m \neq 0$.

Remark. Similar generalized local fractional Rolle's theorem was discussed [7].

Theorem 3.1.6 [22] (Local fractional mean value theorem)

Suppose that $f(x) \in C_\alpha[a,b]$ and $f(x) \in D_\alpha(a,b)$. Then there exists a point $\xi \in (a,b)$ with

$$f(b) - f(a) = \frac{f^{(\alpha)}(\xi)(b-a)^\alpha}{\Gamma(1+\alpha)}, \tag{3.18}$$

where $\alpha \in (0,1]$.

Proof. Define

$$\begin{aligned} &F(x) \\ &= \Gamma(1+\alpha)\left[f(x) - f(a)\right] \\ &- \frac{(x-a)^\alpha \Gamma(1+\alpha)\left[f(b) - f(a)\right]}{(b-a)^\alpha} \end{aligned}, \tag{3.19}$$

then we have

$$F(a) = 0 \text{ and } F(b) = 0.$$

Applying **Theorem 3.1.5** to the function $F(x)$, then we have

$$\begin{aligned} &F^{(\alpha)}(\xi) \\ &= f^{(\alpha)}(\xi) - \frac{\Gamma(1+\alpha)\left[f(b) - f(a)\right]}{(b-a)^\alpha} = 0, \\ &a < \xi < b. \end{aligned} \tag{3.20}$$

That is

$$f(b) - f(a) = \frac{f^{(\alpha)}(\xi)(b-a)^\alpha}{\Gamma(1+\alpha)}, a < \xi < b. \tag{3.21}$$

Theorem 3.1.7 [22] (Local fractional Cauchy's generalized mean value theorem)

Suppose that $f(x), g(x) \in C_\alpha[a,b]$ and $f(x)$, $g(x) \in D_\alpha(a,b)$. If $g(b) \neq g(a)$ then there exists a point $c \in (a,b)$ with

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f^{(\alpha)}(c)}{g^{(\alpha)}(c)}. \tag{3.22}$$

Theorem 3.1.8 [22] (Local fractional L'Hospital's rule)

Suppose that $f(x) \in C_\alpha[a,b]$ and $f(x) \in D_\alpha(a,b)$. If $f(x) \to 0$ and $g(x) \to 0$ as $x \to x_0$ and A denotes either a real number or one of the symbols $+\infty, -\infty$. Suppose that $\lim_{x \to x_0} \frac{f^{(\alpha)}(x)}{g^{(\alpha)}(x)} = A$. Then it is also true that

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = A.$$

3.2. The element theory of the local fractional integral of the non-differentiable functions

Property 3.2.1 [23]

Suppose that $f(x), g(x) \in C_\alpha[a,b]$, then we have

$${}_aI_b^{(\alpha)}\left[f(x) \pm g(x)\right] = {}_aI_b^{(\alpha)} f(x) \pm {}_aI_b^{(\alpha)} g(x). \tag{3.23}$$

Property 3.2.2 [23]

Suppose that $f(x) \in C_\alpha[a,b]$ and C is a constant, then

$${}_aI_b^{(\alpha)}\left[Cf(x)\right] = C\,{}_aI_b^{(\alpha)} f(x). \tag{3.24}$$

Property 3.2.3 [23]

Suppose that $f(x) = 1$, then

$${}_aI_b^{(\alpha)} 1 = \frac{(b-a)^\alpha}{\Gamma(\alpha+1)}. \tag{3.25}$$

Property 3.2.4 [23]

Suppose that $f(x) \in C_\alpha[a,b]$ and $f(x) \geq 0$, then we have

$${}_aI_b^{(\alpha)} f(x) \geq 0 \tag{3.26}$$

with $b - a > 0$.

Property 3.2.5 [23]

Suppose that $f(x), g(x) \in C_\alpha[a,b]$ and $f(x) \geq g(x)$, then we have

$${}_aI_b^{(\alpha)} f(x) \geq {}_aI_b^{(\alpha)} g(x) \text{ with } b - a > 0. \tag{3.27}$$

Property 3.2.6 [23]

Suppose that $f(x)\in C_\alpha[a,b]$. Let M and m are the maximum and minimum values of $f(x)$ over the interval $[a,b]$, respectively. Then we have

$$M\frac{(b-a)^\alpha}{\Gamma(\alpha+1)}\geq {}_aI_b^{(\alpha)}f(x)\geq m\frac{(b-a)^\alpha}{\Gamma(\alpha+1)} \quad (3.28)$$

with $b-a>0$.

Property 3.2.7 [23]

Suppose that $f(x)\in C_\alpha[a,b]$ and $f(x)\geq g(x)$. Then we have

$$\left|{}_aI_b^{(\alpha)}f(x)\right|\leq {}_aI_b^{(\alpha)}\left|f(x)\right| \quad (3.29)$$

with $b-a>0$.

Property 3.2.7 [23]

Suppose that $f(x)\in C_\alpha[a,b]$ and $a<c<b$. Then we have

$${}_aI_b^{(\alpha)}f(x)={}_aI_c^{(\alpha)}f(x)+{}_cI_b^{(\alpha)}f(x). \quad (3.30)$$

Remark. Applying **Property 3.2.7**, we have

$$(x\pm x_0)^\alpha=x^\alpha\pm x_0{}^\alpha.$$

This result is obtained if and only if the characteristic functions defined on fractal set. In this case, the expression of the local fractional derivative is relative to Hausdorff derivative of the function [27], which is local fractional continuous, with respect to fractal measure when $(x-x_0)^\alpha=x^\alpha-x_0{}^\alpha$.

Theorem 3.2.8 [23] (The mean value theorem for local fractional integrals)

Suppose that $f(x)\in C_\alpha[a,b]$, there is a point ξ in (a,b) such that

$${}_aI_b^{(\alpha)}f(x)=f(\xi)\frac{(b-a)^\alpha}{\Gamma(\alpha+1)}. \quad (3.31)$$

Theorem 3.2.9 [23]

Suppose that $f(x)\in C_\alpha[a,b]$, then there is a Function

$$\Pi(x)={}_aI_x^{(\alpha)}f(x),$$

the function has its derivative with respect to $(dx)^\alpha$,

$$\frac{d^\alpha\Pi(x)}{(dx)^\alpha}=f(x),\ a\leq x\leq b. \quad (3.32)$$

Theorem 3.2.10 [23] (Local fractional integration is anti-differentiation)

Suppose that $f(x)=g^{(\alpha)}(x)\in C_\alpha[a,b]$, then we have

$${}_aI_b^{(\alpha)}f(x)=g(b)-g(a). \quad (3.33)$$

Theorem 3.2.11 [23]

Suppose that $g(x)\in C_1[a,b]$ and $(f\circ g)(s)\left[g^{(1)}(s)\right]^\alpha\in C_\alpha[a,b]$. Then we have

$${}_{g(a)}I_{g(b)}^{(\alpha)}f(x)={}_aI_b^{(\alpha)}(f\circ g)(s)\left[g'(s)\right]^\alpha. \quad (3.34)$$

As a direct result, we have the following result:

Theorem 3.2.12 [23] (Local fractional integration by parts)

Suppose that $f(x),g(x)\in D_\alpha(a,b)$ and $f^{(\alpha)}(x)$, $g^{(\alpha)}(x)\in C_\alpha[a,b]$. Then we have

$$\begin{aligned}&{}_aI_b^{(\alpha)}f(t)g^{(\alpha)}(t)\\&=\left[f(t)g(t)\right]_a^b-{}_aI_b^{(\alpha)}f^{(\alpha)}(t)g(t)\end{aligned}. \quad (3.35)$$

As a direct result of **Theorem 3.2.10**, we have this following result:

Theorem 3.2.13 [38, 39, 40]

Suppose that $f^{(k\alpha)}(x)\in C_\alpha(a,b)$, for $0<\alpha\leq 1$, then we have

$$\left({}_{x_0}I_x^{(k\alpha)}f(x)\right)^{(k\alpha)}=f(x), \quad (3.36)$$

where

$${}_{x_0}I_x^{(k\alpha)}f(x)=\overbrace{{}_{x_0}I_x^{(\alpha)}\cdots{}_{x_0}I_x^{(\alpha)}}^{k\ times}f(x)$$

and

$$f^{(k\alpha)}(x)=\overbrace{D_x^{(\alpha)}\ldots D_x^{(\alpha)}}^{k\ times}f(x). \quad (3.37)$$

Theorem 3.2.14 [38, 39, 40]

Suppose that $f^{(k\alpha)}(x),f^{((k+1)\alpha)}(x)\in C_\alpha(a,b)$, for $0<\alpha\leq 1$, then we have

$$ {}_{x_0}I_x^{(k\alpha)}[f^{(k\alpha)}(x)] - {}_{x_0}I_x^{((k+1)\alpha)}[f^{((k+1)\alpha)}(x)] = f^{(k\alpha)}(x_0)\frac{(x-x_0)^{k\alpha}}{\Gamma(k\alpha+1)}, \quad (3.38) $$

where

$$ {}_{x_0}I_x^{((k+1)\alpha)} f(x) = \overbrace{{}_{x_0}I_x^{(\alpha)} \cdots {}_{x_0}I_x^{(\alpha)}}^{k+1 \ times} f(x) $$

and

$$ f^{((k+1)\alpha)}(x) = \overbrace{D_x^{(\alpha)} \ldots D_x^{(\alpha)}}^{k+1 \ times} f(x). \quad (3.39) $$

Proof. Applying **Theorem 3.2.10**, we have

Hence, considering the formula

$$ \begin{aligned} & {}_{x_0}I_x^{(k\alpha)} f^{(k\alpha)}(x_0) \\ & = f^{(k\alpha)}(x_0)\, {}_{x_0}I_x^{(k\alpha)} 1 \\ & = f^{(k\alpha)}(x_0)\, {}_{x_0}I_x^{((k-1)\alpha)}\left[\frac{1}{\Gamma(1+\alpha)}(x-x_0)^{\alpha}\right] \\ & = f^{(k\alpha)}(a)\, {}_{x_0}I_x^{((k-2)\alpha)}\left[\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)} \bullet \frac{1}{\Gamma(1+\alpha)}(x-x_0)^{2\alpha}\right] \\ & = f^{(k\alpha)}(x_0)\frac{(x-x_0)^{k\alpha}}{\Gamma(k\alpha+1)}, \end{aligned} \quad (3.40) $$

we have

$$ {}_{x_0}I_x^{(k\alpha)}[f^{(k\alpha)}(x)] - {}_{x_0}I_x^{((k+1)\alpha)}[f^{((n+1)\alpha)}(x)] = f^{(k\alpha)}(x_0)\frac{(x-x_0)^{k\alpha}}{\Gamma(k\alpha+1)}. \quad (3.41) $$

Remark. When $k = 0$, considering the formula

$$ {}_{x_0}I_x^{0}[f^{(0)}(x)] - f(x) + f(x_0) = f(x_0), $$

we have

$$ {}_aI_x^{0}[f^{(0)}(x)] = f(x). $$

Theorem 3.2.15 [38, 39, 40] (local fractional Taylor' theorem)

Suppose that $f^{((k+1)\alpha)}(x) \in C_\alpha(a,b)$, for $k = 0,1,\ldots,n$ and $0 < \alpha \le 1$, then we have

$$ f(x) = \sum_{k=0}^{n} \frac{f^{(k\alpha)}(x_0)}{\Gamma(1+k\alpha)}(x-x_0)^{k\alpha} + \frac{f^{((n+1)\alpha)}(\xi)}{\Gamma(1+(n+1)\alpha)}(x-x_0)^{(n+1)\alpha} \quad (3.42) $$

with $a < x_0 < \xi < x < b$, $\forall x \in (a,b)$, where

$$ f^{((k+1)\alpha)}(x) = \overbrace{D_x^{(\alpha)} \ldots D_x^{(\alpha)}}^{k+1 \ times} f(x). $$

Proof. Form **Theorem 3.2.14**, we have

$$ {}_aI_x^{(k\alpha)}[f^{(k\alpha)}(x)] - {}_aI_x^{((k+1)\alpha)}[f^{((k+1)\alpha)}(x)] = f^{(k\alpha)}(a)\frac{(x-a)^{k\alpha}}{\Gamma(k\alpha+1)}, \quad (3.43) $$

Applying **Theorem 3.2.8**, we have

$$ \begin{aligned} & {}_aI_x^{((n+1)\alpha)}[f^{((n+1)\alpha)}(x) \\ & = \frac{1}{\Gamma(1+\alpha)}\int_a^x {}_aI_x^{(n\alpha)} f^{((n+1)\alpha)}(x)(dt)^{\alpha} \\ & = \frac{{}_aI_x^{(n\alpha)}\left[f^{((n+1)\alpha)}(\xi)(x-a)^{\alpha}\right]}{\Gamma(1+\alpha)} \\ & = f^{((n+1)\alpha)}(\xi)\frac{{}_aI_x^{(n\alpha)}(x-a)^{\alpha}}{\Gamma(1+\alpha)} \\ & = \frac{f^{((n+1)\alpha)}(\xi)(x-a)^{(n+1)\alpha}}{\Gamma(1+(n+1)\alpha)} \end{aligned} \quad (3.44) $$

with $a < \xi < x$, $\forall x \in [a,b]$.

Remark. Such an expression reduces to the classical Taylor's formula in case of $\alpha = 1$. Comparing with local fractional Taylor's formulas of F-J and K-G local fractional derivative, the difference is led to the definitions of higher order local fractional derivatives [2, 6-8]. Because of critical index $N+1 > p > N$, which is defined K-G, for positive $N \ge 2$, here there is

$$ \alpha = \frac{p}{N+1}. $$

Comparing with the form of the generalized fractional Taylor's formula [3], the difference is led to the definitions of the fractional derivatives.

Theorem 3.2.16 [38, 39, 40]

Suppose that $f^{((k+1)\alpha)}(x) \in C_\alpha(a,b)$, for $k = 0,1,\ldots,n$ and $0 < \alpha \le 1$, then we have

$$ f(x) = \sum_{k=0}^{n} \frac{f^{(k\alpha)}(x_0)}{\Gamma(1+k\alpha)}(x-x_0)^{k\alpha} + R_{n\alpha}(x-x_0) \quad (3.45) $$

with $a<x_0<\xi<x<b, \forall x \in (a,b)$, where

$$ f^{((k+1)\alpha)}(x_0) = \overbrace{D_x^{(\alpha)} \ldots D_x^{(\alpha)}}^{k+1 \ times} f(x) $$

and

$$ R_{n\alpha}(x-x_0) = O\left((x-x_0)^{n\alpha}\right). $$

Proof. Applying **Theorem 3.2.15**, we have

$$\left|\frac{R_{n\alpha}\left(x-x_0\right)}{\left(x-x_0\right)^{n\alpha}}\right|$$
$$=\left|\frac{f^{((n+1)\alpha)}\left(\xi\right)\left(x-x_0\right)^{(n+1)\alpha}}{\Gamma\left(1+(n+1)\alpha\right)\left(x-x_0\right)^{n\alpha}}\right|, \tag{3.46}$$
$$=\left|\frac{f^{((n+1)\alpha)}\left(\xi\right)}{\Gamma\left(1+(n+1)\alpha\right)}\left(x-x_0\right)^{\alpha}\right|$$

that is

$$\lim_{x\to x_0}\left|\frac{R_{n\alpha}\left(x-x_0\right)}{\left(x-x_0\right)^{n\alpha}}\right|$$
$$=\lim_{x\to x_0}\left|\frac{f^{((n+1)\alpha)}\left(\xi\right)}{\Gamma\left(1+(n+1)\alpha\right)}\left(x-x_0\right)^{\alpha}\right| \tag{3.47}$$
$$=0.$$

Remark. This result was discussed when $k=1$ [20, 21] and the different form was presented [7].

Theorem 3.2.17 [38, 39, 40]

Suppose that $f^{((k+1)\alpha)}\left(x\right)\in C_{\alpha}\left(a,b\right)$,
for $k=0,1,\ldots,n$ and $0<\alpha\le 1$, then we have

$$f\left(x\right)=\sum_{k=0}^{n}\frac{f^{(k\alpha)}\left(0\right)}{\Gamma\left(1+k\alpha\right)}x^{k\alpha}+\frac{f^{((n+1)\alpha)}\left(\theta x\right)x^{(n+1)\alpha}}{\Gamma\left(1+(n+1)\alpha\right)} \tag{3.48}$$

with $0<\theta<1$, $\forall x\in(a,b)$, where

$$f^{((k+1)\alpha)}\left(x\right)=\overbrace{D_x^{(\alpha)}\ldots D_x^{(\alpha)}}^{k+1\ times}f\left(x\right). \tag{3.49}$$

Proof. Applying **Theorem 3.2.15**,
for $x_0=0$ and $a<x_0<\xi<x<b$, we have

$$f\left(x\right)=\sum_{k=0}^{n}\frac{f^{(k\alpha)}\left(0\right)}{\Gamma\left(1+k\alpha\right)}x^{k\alpha}+\frac{f^{((n+1)\alpha)}\left(\xi\right)x^{(n+1)\alpha}}{\Gamma\left(1+(n+1)\alpha\right)}. \tag{3.50}$$

If $\xi=\theta x$, then we have

$$\frac{f^{((n+1)\alpha)}\left(\xi\right)x^{(n+1)\alpha}}{\Gamma\left(1+(n+1)\alpha\right)}=\frac{f^{((n+1)\alpha)}\left(\theta x\right)x^{(n+1)\alpha}}{\Gamma\left(1+(n+1)\alpha\right)} \tag{3.51}$$

with $0<\theta<1$.
Hence, we have the result.

4. The Application of Local Fractional Derivative to Local Fractional Extreme Value of Non-differentiable Functions

Some of the most important applications of local fractional differential calculus are optimization problems, in which we are required to find the optimal way of doing something. Some result was discussed by F-J local fractional derivative [7]. Here, optimization problems of non-differentiable functions in fractal space are re-considered.

Definition 4.1 [38, 39, 40] A function has an absolute maximum at c if

$$f\left(c\right)\ge f\left(x\right)$$

for x all in D, where D is domain of $f\left(x\right)$. The number $f\left(c\right)$ is called the maximum value of $f\left(x\right)$ on D .Similarly, $f\left(x\right)$ has an absolute minimum at c if

$$f\left(c\right)\le f\left(x\right)$$

for x all in D and the number $f\left(c\right)$ is called the minimum value of $f\left(x\right)$ on D. The maximum and minimum values of $f\left(x\right)$ are called the extreme values of $f\left(x\right)$.

Definition 4.2 [38, 39, 40] A function has a local maximum at c if $f\left(c\right)\ge f\left(x\right)$ when x is near to c. Similarly, $f\left(x\right)$ has a local minimum at c if

$$f\left(c\right)\le f\left(x\right)$$

when x is near to c.

Theorem 4.1 [38, 39, 40] (The local fractional extreme value theorem)

Suppose that $f\left(x\right)\in C_{\alpha}\left[a,b\right]$, then $f\left(x\right)$ attains an absolute maximum value $f\left(M\right)$ and an absolute minimum value $f\left(m\right)$ at some numbers M,m and in $\left[a,b\right]$.

Theorem 4.2 [38, 39, 40] (Local fractional Fermat's Theorem)

If $f\left(x\right)$ has a local maximum or minimum at c, and if $f^{(\alpha)}\left(c\right)$ exists, then

$$f^{(\alpha)}\left(c\right)=0.$$

Proof. Suppose that $f\left(x\right)$ has a local maximum at c. Then, according to Definition 2, if $f\left(c\right)\ge f\left(x\right)$ is sufficiently close to c. This implies that if h is sufficiently close to 0, with h being positive or negative, then

$$f\left(c\right)\ge f\left(c+h\right)$$

and therefore

$$f(c+h)-f(c)\le 0.$$

We can divide both sides of an inequality by a positive number. Thus, if $h>0$ and is h sufficiently small, we have

$$\frac{\Gamma(1+\alpha)\left(f(c+h)-f(c)\right)}{h^{\alpha}}\le 0. \quad (4.1)$$

Taking the right-hand limit of both sides of this inequality, we get

$$\lim_{h\to 0^{+}}\frac{\Gamma(1+\alpha)\left(f(c+h)-f(c)\right)}{h^{\alpha}}\le 0. \quad (4.2)$$

But since $f^{(\alpha)}(c)$ exists, we have

$$\begin{aligned} &f^{(\alpha)}(c)\\ &=\lim_{h\to 0}\frac{\Gamma(1+\alpha)\left(f(c+h)-f(c)\right)}{h^{\alpha}}\\ &=\lim_{h\to 0^{+}}\frac{\Gamma(1+\alpha)\left(f(c+h)-f(c)\right)}{h^{\alpha}}\le 0 \end{aligned} \quad (4.3)$$

and so we have shown that $f^{(\alpha)}(c)\le 0$.

If $h<0$, considering an inequality

$$f(c)\le f(c+h),$$

we have

$$\frac{\Gamma(1+\alpha)\left(f(c+h)-f(c)\right)}{h^{\alpha}}\ge 0, h<0. \quad (4.4)$$

So, taking the left-hand limit, we have

$$\begin{aligned} &f^{(\alpha)}(c)\\ &=\lim_{h\to 0}\frac{\Gamma(1+\alpha)\left(f(c+h)-f(c)\right)}{h^{\alpha}}.\\ &=\lim_{h\to 0^{-}}\frac{\Gamma(1+\alpha)\left(f(c+h)-f(c)\right)}{h^{\alpha}}\ge 0 \end{aligned} \quad (4.5)$$

Since both of these inequalities

$$f^{(\alpha)}(c)\ge 0 \text{ and } f^{(\alpha)}(c)\le 0$$

must be true, the only possibility is that

$$f^{(\alpha)}(c)=0.$$

Definition 4.3 [38, 39, 40] A critical number of a function $f(x)$ is a number in the domain o $f^{(\alpha)}(c)$ such that either $f^{(\alpha)}(c)=0$ or $f^{(\alpha)}(c)$ does not exist.

As a direct application of local fractional Fermat's Theorem, we have this following:

Theorem 4.3 [38, 39, 40]

If $f(x)$ has a local maximum or minimum at c, then c is a critical number of $f(x)$.

Theorem 4.4 [38, 39, 40] (Increasing/Decreasing test)

(a) If $f^{(\alpha)}(x)>0$ on an interval, then $f(x)$ is increasing on that interval.

(b) If $f^{(\alpha)}(x)<0$ on an interval, then $f(x)$ is decreasing on that interval.

Proof. (a) Let x_1 and x_2 be any two numbers in the interval with $x_1<x_2$. According to the definition of an increasing function we have to show that

$$f(x_1)<f(x_2).$$

Because we are given that $f^{(\alpha)}(x)>0$, we know that is local fractional differentiable on (x_1,x_2). So, by the local fractional mean value theorem there is a number c between x_1 and x_2 such that

$$f(x_2)-f(x_1)=f^{(\alpha)}(c)(x_2-x_1)^{\alpha}. \quad (4.6)$$

By assumption, because $x_2>x_1$, we have

$$f^{(\alpha)}(c)>0 \text{ and } x_2-x_1>0.$$

Thus, we have $f(x_2)-f(x_1)>0$. This shows that $f(x)$ is increasing.

Part (b) is proved similarly.

As direct application, we have this following:

Theorem 4.5 [38, 39, 40] (The α derivative test)

Suppose that c is a critical number of a local fractional function $f(x)$.

(a) If $f^{(\alpha)}(x)$ changes from positive to negative at c, then has a local maximum at c.

(b) If $f^{(\alpha)}(x)$ changes from negative to positive at c, then has a local minimum at c.

(c) If $f^{(\alpha)}(x)$ does not change sign at c (for example, if $f^{(\alpha)}(x)$ is positive on both sides of c or negative on both sides), then $f(x)$ has no local maximum or minimum at c.

Theorem 4.6 [38, 39, 40] (The 2α derivative test)

Suppose that $f^{(2\alpha)}(x)$ is local fractional continuous near c.

(a) If $f^{(\alpha)}(c)=0$ and $f^{(2\alpha)}(c)>0$, then $f(x)$ has a local minimum at c.

(b) If $f^{(\alpha)}(c)=0$ and $f^{(2\alpha)}(c)<0$, then $f(x)$ has a local maximum at c.

Proof. (a) Applying the definition of $f^{(2\alpha)}(c)$, we have

$$\frac{\Gamma(1+\alpha)\left(f^{(\alpha)}(x)-f^{(\alpha)}(c)\right)}{(x\text{-}c)^{(\alpha)}}>0 \text{ for } x>c \quad (4.7)$$

and

$$\frac{\Gamma(1+\alpha)\left(f^{(\alpha)}(x)-f^{(\alpha)}(c)\right)}{(x\text{-}c)^{(\alpha)}}>0 \text{ for } c>x \quad (4.8)$$

because $f^{(2\alpha)}(x)>0$.

Thus, we have $f^{(\alpha)}(x)>0$, $x>c$ and $f^{(\alpha)}(x)<0$, $c>x$. The function $f(x)$ has a local minimum at c.

Part (b) is proved similarly.

5. Conclusions

The work suggests applications of local fractional derivative to optimization problems of non-differentiable functions on Cantor set. It is devoted to the theory of local fractional calculus of function of one variable. The theory of local fractional calculus of function of several variables is investigated in the book [40].

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