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Heat Transfer in Discontinuous Media

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Abstract – From the fractal geometry point of view, the interpretations of local fractional derivative and local fractional integration are pointed out in this paper. It is devoted to heat transfer in discontinuous media derived from local fractional derivative. We investigate the Fourier law and heat conduction equation (also local fractional instantaneous heat conduct equation) in fractal orthogonal system based on cantor set, and extent them. These fractional differential equations are described in local fractional derivative sense. The results are efficiently developed in discontinuous media.

Keywords –Local fractional derivative; Local fractional integrals; Fourier law; Local fractional heat conduction equation; Cantor set

1. Introduction

Since the by-now classical textbook of Mandelbrot [1], fractals has been revealed a useful tool in several areas ranging from fundamental science to engineering. A fractal phenomenon is characterized by striking irregularities, and as a result, it is described by non-differentiable functions.

The theory of local fractional calculus (also called fractal calculus[2-7]), as one of useful tools to handle the fractal and continuously non-differentiable functions, was successfully applied in local fractional Fokker–Planck equation [2, 3], anomalous diffusion and relaxation equation in fractal space [8-9], fractal wave equation[10], fractal-time dynamical systems [11,12], fractal elasticity [13-14], the fractal heat conduction equation [15], local fractional diffusion equation [15], local fractional Laplace equation [16], local fractional ordinary differential equations[16-17], local fractional partial differential equation[15-19], local fractional integral equations[20-23], local fractional variational method and algorithms [22, 23], local fractional complex analysis[16, 17, 24], local fractional Z transform in fractal space [24], local fractional Fourier analysis [25], local fractional short time transforms [16, 17, 26], local fractional wavelet transform [16, 17, 26], local fractional Fourier series [16, 17, 26], Yang-Fourier transform [23, 24, 26-28], Yang-Laplace transform [18, 19, 29-32], local fractional Stieltjes transform in fractal space [30], local fractional Mellin transform in fractal space [31], discrete Yang-Fourier transform [33, 34], fast Yang-Fourier transform [35], RG differential equations [36], the multiple local fractional calculus [37], the theory of local fractional calculus of vector functions[38], local fractional calculus of variations[37, 38] , generalized local Taylor's formula with local fractional derivative[16,17, 39], generalized Newton iteration method [40], and mean value theorems[16, 17, 39, 41].

This letter is to investigate the Fourier law of heat conduction and heat conduction equation in fractal orthogonal system based on cantor set. The paper has been organized as follows. Section 2 gives a brief introduction to fractal orthogonal systems. In section 3, we introduce the concepts of local fractional calculus and its fractal geometrical explanation. Section 4 focus on the Fourier law of heat conduction and heat conduction equation in fractal orthogonal system based on cantor set. Conclusions are presented in section 5.

2. Fractal orthogonal systems

2.1. Hausdorff dimension

Definition 1 (Hausdorff measure) The diameter of a non-empty set E is defined by[16, 17]

$$\left|E\right| =: \sup\left\{\left|x-y\right| : x, y \in E\right\}. \tag{2.1}$$

Given $0<\alpha\le 1$. A countable collection $\{E_i\}$ of subsets of $\mathbb{R}$ is said to be a δ -cover of $F \subseteq \mathbb{R}_n$, $F \subseteq \bigcup_i E_i$ and $0<\left|E_i\right|<\delta \;\; \forall i$.

For each $F \subseteq \mathbb{R}_n$, denote

$$H_\delta^\alpha\left(F\right)=\inf\left\{\sum_i\left|E_i\right| : \{E_i\} \text{ is a } \delta-\text{cover of } F\right\}. \tag{2.2}$$

Note that $H_\delta^\alpha\left(F\right)$ increases as δ decreases, then we define

$$H^\alpha\left(F\right) =: \lim_{\delta\to 0} H_\delta^\alpha\left(F\right). \tag{2.3}$$

Here, we call $H^\alpha : P\left(\mathbb{R}_n\right)\to\left[0,\infty\right)$ α -dimensional Hausdorff measure. For more details, see[16, 17].

Definition 2 (Hausdorff dimension) The Hausdorff dimension is [16, 17]

$$\begin{aligned}\dim_H F &= \inf\left\{s : H^s\left(F\right)=0\right\} \\ &= \sup\left\{s : H^s\left(F\right)=\infty\right\}\end{aligned}. \tag{2.4}$$

2.2. Hausdorff dimension of cantor sets

Lemma 1

Let E and F be subsets of R and let F be a cantor set. Then[16, 17, 37]

$$\dim_H\left(E\times F\right)=\dim_H E+\dim_H F. \tag{2.5}$$

As further result, some results read as follows. For further details see [16, 17, 37]

Theorem 2

Let E and F be subsets of R. Let E and F be fractal sets, then [16, 17, 37]

$$\dim_H\left(E\times F\right)=\dim_H E+\dim_H F. \tag{2.6}$$

Theorem 3

Let E , F and G be subsets of R. Let E , F and G be fractal sets, then[16, 17, 37]

$$\dim_H\left(E\times F\times G\right)=\dim_H E+\dim_H F+\dim_H G. \tag{2.7}$$

For more generalized Hausdorff measure, see [16, 17, 42, 43].

2.3. Fractal orthogonal systems

Definition 3 (Two-dimension fractal orthogonal system) In two-dimension fractal space, a fractal surface, the fractal dimension of which satisfies the condition [16, 17, 37]

$$\dim_H\left(E\times F\right)=\dim_H E+\dim_H F, \tag{2.8}$$

is called the two-dimension fractal system. Let[16, 17, 37]

$$\dim_H E=\dim_H F, \tag{2.9}$$

then this system is so-called the two-dimension fractal orthogonal system.

Definition 4 (Three-dimension fractal orthogonal system) In three-dimension fractal space, a fractal body, the fractal dimension of which satisfies the condition [16, 17, 37]

$$\dim_H\left(E\times F\times G\right)=\dim_H E+\dim_H F+\dim_H G, \tag{2.10}$$

is called the three-dimension fractal system. Let[16, 17, 37]

$$\dim_H E=\dim_H F=\dim_H G, \tag{2.11}$$

then this system is so-called the three-dimension fractal orthogonal system.

Suppose

$$\dim_H E=\dim_H F=\dim_H G=\alpha, \tag{2.12}$$

a number in three-dimension fractal orthogonal system can be written in the form[16, 17]

$$\overline{\eta}^\alpha=i^\alpha x^\alpha+j^\alpha y^\alpha+k^\alpha z^\alpha, x^\alpha, y^\alpha, z^\alpha\in\mathbb{R}^\alpha \tag{2.13}$$

that belongs to a generalized vector space. Likewise, two-dimension fractal orthogonal system can be written in the form [16, 17]

$$\overline{\mu}^\alpha=i^\alpha x^\alpha+j^\alpha y^\alpha, x^\alpha, y^\alpha\in\mathbb{R}^\alpha \tag{2.14}$$

that also belongs to a generalized vector space. Notice that the natural fractional coordinate in case of fractal dimension [42, 43] is a special fractal orthogonal system, ie.

$$\frac{\left(x-x_0\right)^\alpha}{\Gamma\left(1+\alpha\right)}\leftrightarrow x^\alpha,$$

where α is a fractal dimension. It means that $\frac{\left(x-x_0\right)^\alpha}{\Gamma\left(1+\alpha\right)}$ in [42, 43] is corresponding to x^α in [16, 17, 22, 26] if α is fractal dimension, and we can write

$$\frac{\left(x-x_0\right)^\alpha}{\Gamma\left(1+\alpha\right)}=\frac{x^\alpha}{\Gamma\left(1+\alpha\right)}-\frac{x_0{}^\alpha}{\Gamma\left(1+\alpha\right)},$$

by using the fractional set theory [16, 17, 37, 40, 44] . $\frac{\left(x-x_0\right)^\alpha}{\Gamma\left(1+\alpha\right)}$ is a fractal mass function of set $C_\alpha\left(a,b\right)$. Hence, this is the Lebesgue-Cantor Staircase function [11, 12, 26] from the fractal geometry point of view if fractal dimension is $\alpha=\frac{\log 2}{\log 3}$. Suppose that $\alpha=H^\alpha\left(F\right)$, $\frac{\left(x-x_0\right)^\alpha}{\Gamma\left(1+\alpha\right)}$ is any Lebesgue-cantor-like staircase function from the fractal geometry point of view.

3. Local Fractional Calculus and Fractal Geometrical Interpretation

3.1. Local fractional continuity of functions

Definition 5 If there is the relation [16, 17, 22]

$$\left|f\left(x\right)-f\left(x_0\right)\right|<\varepsilon^\alpha \tag{3.1}$$

with $\left|x-x_0\right|<\delta$,for $\varepsilon,\delta>0$ and $\varepsilon,\delta\in\mathbb{R}$. Now $f\left(x\right)$ is called local fractional continuous at $x=x_0$, denote by $\lim_{x\to x_0} f\left(x\right)=f\left(x_0\right)$ .Then $f\left(x\right)$ is called local fractional continuous on the interval $\left(a,b\right)$, denoted by[16, 17, 22]

$$f\left(x\right)\in C_\alpha\left(a,b\right). \tag{3.2}$$

Definition 6 A function $f\left(x\right)$ is called a non-differentiable function of exponent α , $0<\alpha\le 1$, which satisfy Hölder function of exponent α , then for $x,y\in X$ such that[16, 17, 22]

$$\left|f(x)-f(y)\right|\le C\left|x-y\right|^{\alpha} . \quad (3.3)$$

Definition 7 A function $f(x)$ is called to be continuous of order α, $0<\alpha\le 1$, or shortly α continuous, when we have the following relation[16, 17, 22]

$$f(x)-f(x_0)=o\left((x-x_0)^{\alpha}\right). \quad (3.4)$$

Lemma4
If (Ω,d) and (Ω',d') are metric spaces, $E\subset\mathbb{R}$ and $f:E\to\Omega'$ satisfies

$$\rho d(x,y)\le d'\left(f(x),f(y)\right)\le\tau d(x,y) \quad (3.5)$$

where ρ and τ are positives and finite constants, then [20, 22, 45]

$$\rho^s H^s(E)\le H^s\left(f(E)\right)\le\tau^s H^s(E) \quad (3.6)$$

where each $s\ge 0$ and H^s is the s-dimensional Hausdorff measures.
Suppose (Ω,d) and (Ω',d') are metric spaces. A bijection $f:(\Omega,d)\to(\Omega',d')$ is said to be a bi-Lipschitz mapping, if there are constants $\rho,\tau>0$ such that for all $x_1,x_2\in\Omega$,[20, 34]

$$\rho d(x_1,x_2)\le d'\left(f(x_1),f(x_2)\right)\le\tau d(x_1,x_2). \quad (3.7)$$

The following lemma is also a standard result in fractal geometry (see for example [20, 45-49]).

Lemma 5
If $f:(\Omega,d)\to(\Omega',d')$ is a bi-Lipschitz mapping, then [20, 34]

$$\dim_H(A)=\dim_H\left(f(A)\right) \quad (3.8)$$

for all $A\in\Omega$.

Lemma 6
Let F be a subset of the real line and be a fractal. If $f:(F,d)\to(\Omega',d')$ is a bi-Lipschitz mapping, then there is for constants $\rho,\tau>0$ and $F\subset\mathbb{R}$,

$$\rho^s H^s(F)\le H^s\left(f(F)\right)\le\tau^s H^s(F)$$

such that for all $x_1,x_2\in F$, [20, 22]

$$\rho^{\alpha}\left|x_1-x_2\right|^{\alpha}\le\left|f(x_1)-f(x_2)\right|\le\tau^{\alpha}\left|x_1-x_2\right|^{\alpha}. \quad (3.9)$$

This result is directly deduced from fractal geometry.
From ***Lemma 4*** and ***Lemma 5*** it is observed that that $\dim_H(F)=\dim_H\left(f(F)\right)=s.$

Theorem 7
Let F be a subset of the real line and be a fractal. If $f:\left((\zeta,\xi),d\right)\to\left((\eta,\upsilon),d'\right)$ is a bi-Lipschitz mapping, then there is for constants $\rho,\tau>0$, [20, 22]

$$\rho^{\alpha}\left|x_1-x_2\right|^{\alpha}\le\left|f(x_1)-f(x_2)\right|\le\tau^{\alpha}\left|\zeta-\xi\right|^{s}. \quad (3.10)$$

where $E=(\eta,\upsilon)$.

Theorem 8
Let F be a subset of the real line and be a fractal. If $f(\Omega)$ is a bi-Lipschitz mapping, then there are any $x_1,x_2\in\Omega\subset\mathbb{R}$ and positive constant υ such that [20, 22]

$$\left|f(x_1)-f(x_2)\right|\le\upsilon\left|x_1-x_2\right|^{s}. \quad (3.11)$$

Remark 1. If $f(x)\in C_\alpha(a,b)$, then $\dim_H\left(F\cap(a,b)\right)=\dim_H\left(C_\alpha(a,b)\right)=\alpha$ and $C_\alpha(a,b)=\{f: f(x)$ is local fractional continuous, $x\in F\cap(a,b)\}$.

3.2. Fractal geometrical explanation of local fractional derivative and integration

3.2.1. Local fractional derivatives

Definition 8 Setting $f(x)\in C_\alpha(a,b)$, local fractional derivative of $f(x)$ of order α at $x=x_0$ is defined [16-30, 37]

$$f^{(\alpha)}(x_0)=\frac{d^{\alpha}f(x)}{dx^{\alpha}}\Big|_{x=x_0}=\lim_{x\to x_0}\frac{\Delta^{\alpha}\left(f(x)-f(x_0)\right)}{(x-x_0)^{\alpha}}, \quad (3.12)$$

Where $\Delta^{\alpha}\left(f(x)-f(x_0)\right)\cong\Gamma(1+\alpha)\Delta\left(f(x)-f(x_0)\right)$.
For any $x\in(a,b)$, there exists[16-30]

$$f^{(\alpha)}(x)=D_x^{(\alpha)}f(x), \quad (3.13)$$

denoted by

$$f(x)\in D_x^{(\alpha)}(a,b). \quad (3.14)$$

Local fractional derivative of high order is written in the form [16, 17]

$$f^{(k\alpha)}(x)=\overbrace{D_x^{(\alpha)}\ldots D_x^{(\alpha)}}^{k\ times}f(x), \quad (3.15)$$

and local fractional partial derivative of high order [16, 17]

$$\frac{\partial^{k\alpha}f(x)}{\partial x^{k\alpha}}=\overbrace{\frac{\partial^{\alpha}}{\partial x^{\alpha}}\ldots\frac{\partial^{\alpha}}{\partial x^{\alpha}}}^{k\ times}f(x). \quad (3.16)$$

Theorem 9

If u, v are differentiable functions, the following differential rules are valid:

Product Rule [16, 17, 37]:

$$\left[u(x)v(x)\right]_x^{(\alpha)} = u_x^{(\alpha)}(x)v(x) + u(x)v_x^{(\alpha)}(x). \quad (3.17)$$

Chain Rule:

$$\left(v\left[u(x)\right]\right)_x^{(\alpha)} = v_u^{(\alpha)}\left(u^{'}(x)\right)^{\alpha} \text{ or}$$

$$\left(v\left[u(x)\right]\right)_x^{(\alpha)} = v_u^{'} u_x^{(\alpha)}(x). \quad (3.18)$$

Exponent Rule:

$$\Gamma(\alpha+1)dv = d^{\alpha}v. \quad (3.19)$$

Differential rule for multivariable functions:

Let $F\left(u_k(x)\right)$ be local fractional continuous. For $\forall u_k$, $k=1,2,3$, $u_k^{'}(x)$ exists. Then [37]

$$\frac{d^{\alpha}F\left(u_1(x),u_2(x),u_3(x)\right)}{dx^{\alpha}} = \frac{d^{\alpha}F(u_1)}{du_1^{\alpha}}\left(\frac{du_1(x)}{dx}\right)^{\alpha} + \frac{d^{\alpha}F(u_2)}{du_2^{\alpha}}\left(\frac{du_k(x)}{dx}\right)^{\alpha} + \frac{d^{\alpha}F(u_3)}{du_3^{\alpha}}\left(\frac{du_3(x)}{dx}\right)^{\alpha} \quad (3.20)$$

Other definitions see [37].

3.2.2. Local fractional integrals

Definition 9 Setting $f(x) \in C_{\alpha}(a,b)$, local fractional integral of $f(x)$ of order α in the interval $[a,b]$ is defined [16, 17, 37, 39]

$$ {}_aI_b^{(\alpha)}f(x) = \frac{1}{\Gamma(1+\alpha)}\int_a^b f(t)(dt)^{\alpha} = \frac{1}{\Gamma(1+\alpha)}\lim_{\Delta t\to 0}\sum_{j=0}^{j=N-1} f(t_j)(\Delta t_j)^{\alpha}, \quad (3.21)$$

where $\Delta t_j = t_{j+1} - t_j$, $\Delta t = \max\{\Delta t_1, \Delta t_2, \Delta t_j, ...\}$ and $[t_j, t_{j+1}]$, $j=0,...,N-1$, $t_0 = a$, $t_N = b$, is a partition of the interval $[a,b]$. For any $x \in (a,b)$, there exists [16, 17]

$$ {}_aI_x^{(\alpha)}f(x), \quad (3.22)$$

denoted by

$$f(x) \in I_x^{(\alpha)}(a,b). \quad (3.23)$$

Remark 2. If $f(x) \in D_x^{(\alpha)}(a,b)$, or $I_x^{(\alpha)}(a,b)$, we have[16, 17]

$$f(x) \in C_{\alpha}(a,b). \quad (3.24)$$

Here, it follows that [16, 17, 37, 39]

$$ {}_aI_a^{(\alpha)}f(x) = 0 \text{ if } a=b;$$

$$ {}_aI_b^{(\alpha)}f(x) = -{}_bI_a^{(\alpha)}f(x) \text{ if } a<b; \quad (3.25)$$

$$\text{and } {}_aI_a^{(0)}f(x) = f(x). \quad (3.26)$$

We have the following results:

For any $f(x) \in C_{\alpha}(a,b)$, $0<\alpha\le 1$, we have local fractional multiple integrals, which is written as[16, 17, 37, 39]

$$ {}_{x_0}I_x^{(k\alpha)}f(x) = \overbrace{{}_{x_0}I_x^{(\alpha)}\ldots{}_{x_0}I_x^{(\alpha)}}^{k\ times} f(x). \quad (3.27)$$

For $0<\alpha\le 1$, $f^{(k\alpha)}(x) \in C_{\alpha}^{k}(a,b)$, then we have[16, 17, 37, 39]

$$\left({}_{x_0}I_x^{(k\alpha)}f(x)\right)^{(k\alpha)} = f(x), \quad (3.28)$$

where

$$ {}_{x_0}I_x^{(k\alpha)}f(x) = \overbrace{{}_{x_0}I_x^{(\alpha)}\ldots{}_{x_0}I_x^{(\alpha)}}^{k\ times} f(x) \quad (3.29)$$

and

$$f^{(k\alpha)}(x) = \overbrace{D_x^{(\alpha)}\ldots D_x^{(\alpha)}}^{k\ times} f(x). \quad (3.30)$$

These results are different from Jumarie's results for modified Riemann-Liouville derivative and integrals [50-53]. For more results, see [16, 17].

If we consider Chen fractal derivative [8, 9], we get a new integral formula

$$ {}_a^FI_b^{(\alpha)}f(x) = \int_a^b f(t)(dt)^{\alpha} = \lim_{\Delta t\to 0}\sum_{j=0}^{j=N-1} f(t_j)(\Delta t_j)^{\alpha}.$$

We find that

$$ {}_a^FI_b^{(\alpha)}f(x) = \Gamma(1+\alpha)\,{}_aI_b^{(\alpha)}f(x).$$

3.3.3. Its fractal geometrical explanation

Definition 10 Let a be an arbitrary but fixed real number. The integral staircase function $S_F^{\alpha}(x)$ of order α for a set F is given by [11, 12, 20, 22, 26]

$$S_F^{\alpha}(x) = \begin{cases} \gamma^{\alpha}[F,a,x], if\ x \ge a; \\ -\gamma^{\alpha}[F,x,a], if\ x < a. \end{cases} \quad (3.31)$$

Then we have the following results:

(a) The fractal mass function $\gamma^{\alpha}[F,a,b]$ can written as [20, 22, 26]

$$\gamma^{\alpha}[F,a,b] = \lim_{\substack{\max\,(x_{i+1}-x_i)\to 0\\ 0<i<n-1}}\sum_{i=0}^{n-1}\frac{(x_{i+1}-x_i)^{\alpha}}{\Gamma(1+\alpha)} = \frac{1}{\Gamma(1+\alpha)}H^{\alpha}(F\cap(a,b)) = \frac{(b-a)^{\alpha}}{\Gamma(1+\alpha)}. \quad (3.32)$$

(b) we have [20, 22, 26]

$$\begin{aligned} & S_F^\alpha(y)-S_F^\alpha(x) \\ & =\gamma^\alpha[F,x,y] \\ & =\lim_{\substack{\max\left(x_{i+1}-x_i\right)\to 0 \\ 0<i<n-1}} \sum_{i=0}^{n-1} \frac{\left(x_{i+1}-x_i\right)^\alpha}{\Gamma(1+\alpha)}. \\ & =\frac{(y-x)^\alpha}{\Gamma(1+\alpha)} \end{aligned} \tag{3.33}$$

(c) if $a<b<c$, we have [11, 12, 20, 22, 26]

$$\gamma^\alpha[F,a,b]+\gamma^\alpha[F,b,c]=\gamma^\alpha[F,a,c]. \tag{3.34}$$

Remark 3. From formula (a) we obtain that [20, 22, 26]

$$\begin{aligned} \gamma^\alpha[F,a,b] & =\frac{1}{\Gamma(1+\alpha)}\int_a^b (dt)^\alpha \\ & =\frac{1}{\Gamma(1+\alpha)}\lim_{\Delta t\to 0}\sum_{j=0}^{j=N-1}\left(\Delta t_j\right)^\alpha. \\ & =\frac{(b-a)^\alpha}{\Gamma(1+\alpha)} \end{aligned} \tag{3.35}$$

Remark 4. From formula (c) we deduce to $(b-a)^\alpha+(c-b)^\alpha=(c-a)^\alpha$, which is called the theory of fractional set [18, 38].

Hence, we can understand it by fractal geometry [20, 22, 26]:

$$H^\alpha\left(F\cap(a,b)\right)+H^\alpha\left(F\cap(b,c)\right)=H^\alpha\left(F\cap(a,c)\right), \tag{3.36}$$

ie. $1^\alpha+2^\alpha=3^\alpha$. That is, the fractal geometric representation is that cantor set $[0,3]$ is equivalent to the sum of cantor set $[0,1]$ and cantor set $[1,3]$. The dimension of cantor set is α, for $0<\alpha\le 1$ and, 1^α, 2^α and 3^α are real line numbers on a fractional set [16, 17, 37, 40, 44]. Hence,

$$\frac{1}{\Gamma(1+\alpha)}\int_a^b (dt)^\alpha=\frac{(b-a)^\alpha}{\Gamma(1+\alpha)} \tag{3.37}$$

is any Lebesgue-cantor-like staircase function from the fractal geometry point of view if α ($0<\alpha\le 1$) is any fractal dimension. In one-dimension fractal orthogonal system we can write

$$\frac{\left(x-x_0\right)^\alpha}{\Gamma(1+\alpha)}=\frac{x^\alpha}{\Gamma(1+\alpha)}-\frac{{x_0}^\alpha}{\Gamma(1+\alpha)}, \tag{3.38}$$

where $\frac{x^\alpha}{\Gamma(1+\alpha)}$ is a any Lebesgue-cantor-like staircase function. If the set is cantor set, $\frac{x^\alpha}{\Gamma(1+\alpha)}$ is a Lebesgue-cantor staircase function[11, 12, 20, 22].

4. Heat Transfer in Discontinuous Media

4. 1. Fourier law of heat conduction in fractal media

The temperature field reads [54]

$$T(x,y,z,\tau)=f(x,y,z,\tau) \tag{4.1}$$

at $\tau>\tau_0$ and in Ω, where $f(x,y,z,\tau)$ is local fractional continuous at fractal domain Ω.

For a given temperature field T, there is a local fractional temperature gradient [54]

$$\nabla^\alpha T=\frac{\partial^\alpha T}{\partial u_1^\alpha}\vec{e}_1^{\,\alpha}+\frac{\partial^\alpha T}{\partial u_2^\alpha}\vec{e}_2^{\,\alpha}+\frac{\partial^\alpha T}{\partial u_3^\alpha}\vec{e}_3^{\,\alpha}. \tag{4.2}$$

We consider the heat flux per unit fractal area $\vec{q}$ is proportional to the temperature gradient in fractal medium. Fourier law of heat conduction in fractal medium is expressed by [54]

$$\vec{q}(x,y,z,t)=-K^{2\alpha}\nabla^\alpha T(x,y,z,t), \tag{4.3}$$

where $K^{2\alpha}$ denotes the thermal conductivity of the fractal material.

Fourier law of two-dimensional heat conduction in fractal media is

$$q(x,y,t)=-K^{2\alpha}\left[\frac{d^\alpha T(x,y,t)}{dx^\alpha}+\frac{d^\alpha T(x,y,t)}{dy^\alpha}\right], \tag{4.4}$$

at $\tau>\tau_0$ and in $\sum$, where K^α denotes the thermal conductivity of the fractal material.

Fourier law of one-dimensional heat conduction in fractal media is

$$q(x,t)=-K^{2\alpha}\frac{d^\alpha T(x,t)}{dx^\alpha}, \tag{4.5}$$

at $\tau>\tau_0$ and in A, where K^α denotes the thermal conductivity of the fractal material.

4.2. Local fractional heat conduction equations

Local fractional heat-conduction equation with heat generation in fractal media can be written as [54]

$$K^{2\alpha}\nabla^{2\alpha}T+g-\rho_\alpha c_\alpha\frac{\partial^\alpha T}{\partial t^\alpha}=0 \tag{4.6}$$

at $\tau>\tau_0$ and in Ω, or

$$K^{2\alpha}\left(\frac{\partial^{2\alpha}T}{\partial x^{2\alpha}}+\frac{\partial^{2\alpha}T}{\partial y^{2\alpha}}+\frac{\partial^{2\alpha}T}{\partial z^{2\alpha}}\right)+g-\rho_\alpha c_\alpha\frac{\partial^\alpha T}{\partial t^\alpha}=0 \tag{4.7}$$

at $\tau>\tau_0$ and in Ω.

Local fractional two-dimensional heat conduction equation with heat generation in fractal media can be written as

$$K^{2\alpha}\nabla^{2\alpha}T+g-\rho_\alpha c_\alpha\frac{\partial^\alpha T}{\partial t^\alpha}=0 \tag{4.8}$$

at $\tau > \tau_0$ and in $\sum$, or

$$K^{2\alpha}\left(\frac{\partial^{2\alpha}T}{\partial x^{2\alpha}}+\frac{\partial^{2\alpha}T}{\partial y^{2\alpha}}\right)+g-\rho_\alpha c_\alpha\frac{\partial^\alpha T}{\partial t^\alpha}=0 \quad (4.9)$$

at $\tau > \tau_0$ and in $\sum$.

Local fractional one-dimensional heat conduction equation with heat generation in fractal media can be written as

$$K^{2\alpha}\frac{d^{2\alpha}T}{dx^{2\alpha}}+g-\rho_\alpha c_\alpha\frac{d^\alpha T}{dt^\alpha}=0 \quad (4.10)$$

at $\tau > \tau_0$ and in A.

Local fractional heat-conduction equation without heat generation in fractal media is [54]

$$K^{2\alpha}\nabla^{2\alpha}T-\rho_\alpha c_\alpha\frac{\partial^\alpha T}{\partial t^\alpha}=0 \quad (4.11)$$

at $\tau > \tau_0$ and in Ω, or

$$K^{2\alpha}\left(\frac{\partial^{2\alpha}T}{\partial x^{2\alpha}}+\frac{\partial^{2\alpha}T}{\partial y^{2\alpha}}+\frac{\partial^{2\alpha}T}{\partial z^{2\alpha}}\right)-\rho_\alpha c_\alpha\frac{\partial^\alpha T}{\partial t^\alpha}=0 \quad (4.12)$$

at $\tau > \tau_0$ and in Ω, where $\nabla^{2\alpha}$ is local fractional Laplace operator[16, 17, 37, 54].

Local fractional two-dimensional heat conduction equation without heat generation in fractal media can be written as [15]

$$K^{2\alpha}\nabla^{2\alpha}T-\rho_\alpha c_\alpha\frac{\partial^\alpha T}{\partial t^\alpha}=0 \quad (4.13)$$

at $\tau > \tau_0$ and in $\sum$, or

$$K^{2\alpha}\left(\frac{\partial^{2\alpha}T}{\partial x^{2\alpha}}+\frac{\partial^{2\alpha}T}{\partial y^{2\alpha}}\right)-\rho_\alpha c_\alpha\frac{\partial^\alpha T}{\partial t^\alpha}=0 \quad (4.14)$$

at $\tau > \tau_0$ and in $\sum$.

Local fractional one-dimensional heat conduction equation without heat generation in fractal media can be written as

$$K^{2\alpha}\frac{d^{2\alpha}T}{dx^{2\alpha}}-\rho_\alpha c_\alpha\frac{d^\alpha T}{dt^\alpha}=0 \quad (4.15)$$

at $\tau > \tau_0$ and in A.

5. Conclusions

In the present paper, we investigate the interpretations of local fractional derivative and local fractional integration from the fractal geometry point of view. We focus on the Fourier law of heat conduction and heat conduction equation in fractal orthogonal system based on cantor set and extent them. These fractional differential equations (with fractional derivative and fractional partial derivative) are described in local fractional derivative. The results are efficiently developed, and it is of great significance to heat transfer from continuous media to discontinuous media.

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Vitae

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