Mean value theorems for Local fractional integrals on fractal space

Guang-Sheng Chen

Department of Computer Engineering, Guangxi Modern Vocational Technology College, Hechi, Guangxi, 547000, P.R. China

Email: cgswavelets@126.com

Abstract –The theory of calculus was extended to local fractional calculus involving fractional order. Local fractional calculus (also called Fractal calculus) has played a significant part not only in mathematics but also in physics and engineers. The main purpose of this paper is to further extend some mean value theorems in Fractal space, by Abel's lemma, definition of Local fractional integrals and using some properties of Local fractional integral . In the paper, we present some properties of Local fractional integral. By using it, we establish the generalized first mean value theorem and the generalized second mean value theorem for Local fractional integrals in Fractal space.

Keywords –fractal space, Local fractional integral, local fractional Mean value theorems

1. Introduction

local fractional calculus (also called Fractal calculus) has played an important role not only in mathematics but also in physics and engineers [1-15]. Local fractional integral of $f(x)$ [6-7,9] were written in the form

$$ {}_aI_b^{(\alpha)} f(x) = \frac{1}{\Gamma(1+\alpha)} \int_a^b f(t)(dt)^\alpha $$

$$ = \frac{1}{\Gamma(1+\alpha)} \lim_{\Delta t \to 0} \sum_{j=0}^{N-1} f(t_j)(\Delta t_j)^\alpha $$

with $\Delta t_j = t_{j+1} - t_j$ and $\Delta t = \max\{\Delta t_1, \Delta t_2, \ldots, \Delta t_j, \ldots\}$, where for $j = 1, 2, \ldots, N-1$, $t_0 = a$ and $t_N = b$, $[t_j, t_{j+1}]$ is a partition of the interval $[a,b]$. The purpose of this paper is to establish the generalized Mean value theorems for Local fractional integrals in fractal space. We generalize the results of [1].

2. Preliminaries

Now we present some properties of Local fractional integral, that will be used later in this paper.

Theorem 2. 1 [1]Every constant function $f(x) = c$ is integrable from a to b and

$$ {}_aI_b^{(\alpha)} f(x) = \frac{c(b-a)^\alpha}{\Gamma(1+\alpha)}. $$

Theorem 2.2 Every monotone function on $[a,b]$ is integrable.

Theorem 2.3 [1] Every continuous function on $[a,b]$ is integrable

Theorem 2.4 Let $f(x)$ be a bounded function that is integrable on $[a,b]$. Then $f(x)$ is integrable on every subinterval $[c,d]$ of $[a,b]$.

Theorem 2.5. [1] Let $f(x)$ and $g(x)$ be integrable functions on $[a,b]$ and $c \in \mathbb{R}$. Then

(1) $cf(x)$ is integrable and ${}_aI_b^{(\alpha)} cf(x) = c\, {}_aI_b^{(\alpha)} f(x)$;

(2) $f(x) \pm g(x)$ is integrable and ${}_aI_b^{(\alpha)}[f(x) \pm g(x)] = {}_aI_b^{(\alpha)} f(x) \pm {}_aI_b^{(\alpha)} g(x)$.

Theorem 2.6 If $f(x)$ and $g(x)$ are integrable on $[a,b]$, then so is their product $f(x)g(x)$.

Theorem2.7[1] Let $f(x)$ be a function defined on $[a,b]$ and $a < c < b$ .If $f(x)$ is integrable from a to c and from c to b, then $f(x)$ is integrable from a to b and

$$ {}_aI_b^{(\alpha)} f(x) = {}_aI_c^{(\alpha)} f(x) + {}_cI_b^{(\alpha)} f(x). $$

Theorem 2.8 [1] If $f(x)$ and $g(x)$ are integrable on $[a,b]$ and $f(x) \ge g(x)$ for all $x \in [a,b]$, then

$$ {}_aI_b^{(\alpha)} f(x) \ge {}_aI_b^{(\alpha)} g(x). $$

Theorem 2. 9 [1] If $f(x)$ is integrable on $[a,b]$, then so is $|f(x)|$ and

$$ |{}_aI_b^{(\alpha)} f(x)| \le {}_aI_b^{(\alpha)} |f(x)|. $$

3. Mean value theorems for Local fractional integrals

Theorem 3.1 (First Mean Value Theorem). Let $f(x)$ and $g(x)$ be bounded and integrable functions on $[a,b]$, and let $g(x)$ be nonnegative (or nonpositive) on $[a,b]$. Let us set

$m=\inf\{f(x): x\in[a,b]\}$ and $M=\sup\{f(x): x\in[a,b]\}$. Then there exists a point ξ in (a,b) such that

$$ {}_aI_b^{(\alpha)} f(x)g(x) = f(\xi)\, {}_aI_b^{(\alpha)} g(x) \quad . \tag{3.1}$$

Proof. We have

$$ m \le f(x) \le M \quad , \quad \text{for all} \quad x\in[a,b] \quad . \tag{3.2}$$

Suppose $g(x)\ge 0$. Multiplying (3.2) by $g(x)$ we get

$$ mg(x) \le f(x)g(x) \le Mg(x) \quad \text{for all} \quad x\in[a,b]$$

Besides, each of the functions $mg(x)$, $Mg(x)$, and $f(x)g(x)$ is integrable from a to b by Theorem 2.5 and Theorem 2.6. Therefore, we obtain from these inequalities, by using Theorem 2.8,

$$ m\, {}_aI_b^{(\alpha)} g(x) \le {}_aI_b^{(\alpha)} f(x)g(x) \le M\, {}_aI_b^{(\alpha)} g(x) \quad . \tag{3.3}$$

If ${}_aI_b^{(\alpha)} g(x)=0$, it follows from (3.3) that ${}_aI_b^{(\alpha)} f(x)g(x)=0$, and therefore equality (3.1) becomes obvious; if ${}_aI_b^{(\alpha)} g(x)>0$, then (3.3) implies

$$ m \le \frac{{}_aI_b^{(\alpha)} f(x)g(x)}{{}_aI_b^{(\alpha)} g(x)} \le M \; .$$

there exists a point ξ in (a,b) such that

$$ m \le f(\xi) \le M \; ,$$

which yields the desired result (3.1).

In particular, when $g(x)=1$, we get from Theorem 3.1 the following result

Corollary 3.1. Let $f(x)$ be an integrable function on $[a,b]$ and let m and M be the infimum and supremum, respectively, of $f(x)$ on $[a,b]$. Then there exists a point ξ in (a,b) such that

$$ {}_aI_b^{(\alpha)} f(x) = f(\xi)\frac{(b-a)^\alpha}{\Gamma(1+\alpha)} \, .$$

Remark:conditions of Corollary 3.1. is weaker than those of Theorem 2.23 in [1].

In what follows we will make use of the following fact, known as Abel's lemma.

Lemma 3.2. Let the numbers p_i for $1\le i\le n$ satisfy the inequalities $p_1 \ge p_2 \ge \ldots \ge p_n$ and the numbers $S_k=\sum_{i=1}^{k} q_i$ for $1\le k\le n$ satisfy the inequalities $m\le S_k \le M$ for all values of k, where q_i, m, and M are some numbers. Then $mp_1 \le \sum_{i=1}^{n} p_i q_i \le Mp_1$.

Theorem 3.3 (Second Mean Value Theorem I). Let $f(x)$ be a bounded function that is integrable on $[a,b]$. Let further m_F and M_F be the infimum and supremum, respectively, of the function $F(x)=\frac{1}{\Gamma(1+\alpha)}\int_a^x f(t)(dt)^\alpha$ on $[a,b]$.

Then:

(i) If a function $g(x)$ is nonincreasing with $g(x)\ge 0$ on $[a,b]$, then there is some point ξ in (a,b) such that $m_F \le f(\xi) \le M_F$ and

$$ {}_aI_b^{(\alpha)} f(x)g(x) = g(a)F(\xi) \; . \tag{3.4}$$

(ii) If $g(x)$ is any monotone function on $[a,b]$, then there is some point ξ in (a,b) such that $m_F \le F(\xi) \le M_F$ and

$$ {}_aI_b^{(\alpha)} f(x)g(x) = [g(a)-g(b)]F(\xi) + g(b)\, {}_aI_b^{(\alpha)} f(x) \quad . \tag{3.5}$$

Proof. To prove part (i) of the theorem, assume that $g(x)$ is nonincreasing and that $g(x)\ge 0$ for all $x\in[a,b]$. Consider an arbitrary $\varepsilon>0$. Since $f(x)$ and $f(x)g(x)$ are integrable on $[a,b]$, we can choose, by definition of Local fractional integrals, a partition $a=x_0<x_1<\ldots x_{n-1}<x_n=b$ such that

$$ \frac{1}{\Gamma(1+\alpha)}\sum_{i=1}^{n}(M_i-m_i)(x_i-x_{i-1})^\alpha < \varepsilon^\alpha \quad , \tag{3.6}$$

And

$$\begin{aligned} &\frac{1}{\Gamma(1+\alpha)}\int_a^b f(x)g(x)(dx)^\alpha - \varepsilon^\alpha < \\ &\frac{1}{\Gamma(1+\alpha)}\sum_{i=1}^{n} f(x_{i-1})g(x_{i-1})(x_i-x_{i-1})^\alpha \quad , \\ &< \frac{1}{\Gamma(1+\alpha)}\int_a^b f(x)g(x)(dx)^\alpha + \varepsilon^\alpha \end{aligned} \tag{3.7}$$

where m_i and M_i are the infimum and supremum, respectively, of $f(x)$ on $[x_{i-1},x_i)$. Since $g(x_{i-1})\ge 0$, we get from $m\le f(x_{i-1})\le M$ that

$$\begin{aligned} &\frac{1}{\Gamma(1+\alpha)}\sum_{i=1}^{n} m_i g(x_{i-1})(x_i-x_{i-1})^\alpha \\ &\le \frac{1}{\Gamma(1+\alpha)}\sum_{i=1}^{n} f(x_{i-1})g(x_{i-1})(x_i-x_{i-1})^\alpha \quad , \\ &\le \frac{1}{\Gamma(1+\alpha)}\sum_{i=1}^{n} M_i g(x_{i-1})(x_i-x_{i-1})^\alpha \end{aligned} \tag{3.8}$$

holds. Next, by Corollary 3.1, there exist numbers ξ_i for $1\le i\le n$ such that $m_i \le f(\xi_{i-1}) \le M_i$

and

$$ \frac{1}{\Gamma(1+\alpha)}\int_{x_{i-1}}^{x_i} f(x)(dx)^\alpha = f(\xi_i)\frac{(x_i-x_{i-1})^\alpha}{\Gamma(1+\alpha)} \, .$$

Consider the numbers

$$ S_k=\sum_{i=1}^{k} f(\xi_i)\frac{(x_i-x_{i-1})^\alpha}{\Gamma(1+\alpha)} = \frac{1}{\Gamma(1+\alpha)}\int_a^{x_k} f(x)(dx)^\alpha \; .$$

for $1 \le k \le n$. Obviously, $m_F \le S_k \le M_F$, where m_F and M_F are the infimum and supremum, respectively, of $F(x)$ on $[a,b]$. Put

$$p_i = g(x_{i-1}) \text{ and } q_i = f(\xi_i)\frac{(x_i - x_{i-1})^\alpha}{\Gamma(1+\alpha)}.$$

for $1 \le i \le n$. Since $g(x)$ is nonincreasing and $g(x) \ge 0$, we have

$$p_1 \ge p_2 \ge \ldots \ge p_n.$$

The numbers p_i S_i, and q_i satisfy the conditions of Lemma 3.2. Therefore

$$m_F g(a) \le \sum_{i=1}^{n} g(x_{i-1}) f(\xi_i)\frac{(x_i - x_{i-1})^\alpha}{\Gamma(1+\alpha)} \le M_F g(a) \quad (3.9)$$

On the other hand,

$$\sum_{i=1}^{n} m_i g(x_{i-1})\frac{(x_i - x_{i-1})^\alpha}{\Gamma(1+\alpha)} \le \sum_{i=1}^{n} g(x_{i-1}) f(\xi_i)\frac{(x_i - x_{i-1})^\alpha}{\Gamma(1+\alpha)} \le \sum_{i=1}^{n} M_i g(x_{i-1})\frac{(x_i - x_{i-1})^\alpha}{\Gamma(1+\alpha)} \quad (3.10)$$

From (3.8) and (3.10) we have, taking into account the monotonicity of $g(x)$ and (3.6),

$$\left|\frac{1}{\Gamma(1+\alpha)}\sum_{i=1}^{n} g(x_{i-1})[f(x_{i-1}) - f(\xi_i)](x_i - x_{i-1})^\alpha\right| \le \frac{1}{\Gamma(1+\alpha)}\sum_{i=1}^{n}(M_i - m_i)g(x_{i-1})(x_i - x_{i-1})^\alpha \le \frac{g(a)}{\Gamma(1+\alpha)}\sum_{i=1}^{n}(M_i - m_i)(x_i - x_{i-1})^\alpha \le g(a)\varepsilon$$

From this and (3.7) it follows that

$$\frac{1}{\Gamma(1+\alpha)}\sum_{i=1}^{n} g(x_{i-1}) f(\xi_i)(x_i - x_{i-1})^\alpha - (\varepsilon^\alpha + g(a)\varepsilon^\alpha) < \frac{1}{\Gamma(1+\alpha)}\int_a^b f(x)g(x)(dx)^\alpha < \varepsilon^\alpha + g(a)\varepsilon^\alpha + \frac{1}{\Gamma(1+\alpha)}\sum_{i=1}^{n} g(x_{i-1}) f(\xi_i)(x_i - x_{i-1})^\alpha$$

Hence, using (3.9), we obtain

$$-\varepsilon^\alpha - g(a)\varepsilon^\alpha + m_F g(a) < \frac{1}{\Gamma(1+\alpha)}\int_a^b f(x)g(x)(dx)^\alpha < \varepsilon^\alpha + g(a)\varepsilon^\alpha + M_F g(a)$$

Since $\varepsilon > 0$ is arbitrary, we get

$$m_F g(a) \le \frac{1}{\Gamma(1+\alpha)}\int_a^b f(x)g(x)(dx)^\alpha \le M_F g(a) \quad (3.11)$$

If $g(a) = 0$, it follows from (3.11) that $\int_a^b f(x)g(x)(dx)^\alpha = 0$, and therefore equality (3.4) becomes obvious; if $g(a) > 0$, then (3.11) implies

$$m_F \le \frac{{}_aI_b^{(\alpha)} f(x)g(x)}{g(a)} \le M_F.$$

there exists a point ξ in (a,b) such that

$$m_F \le F(\xi) = \frac{{}_aI_b^{(\alpha)} f(x)g(x)}{g(a)} \le M_F.$$

which yields the desired result (3.4).

Let now $g(x)$ be an arbitrary nonincreasing function on $[a,b]$. Then the function h defined by $h(t) = g(t) - g(b)$ is nonincreasing and $h(t) \ge 0$ on $[a,b]$. therefore, applying formula (3.4) to the function $h(t)$, we can write

$$ {}_aI_b^{(\alpha)} f(x)[g(x) - g(b)] = \frac{1}{\Gamma(1+\alpha)}\int_a^b f(x)[g(x) - g(b)](dx)^\alpha = [g(a) - g(b)]F(\xi)$$

which yields the formula (3.5) of part (ii) for nonincreasing functions $g(x)$. If $g(x)$ is nondecreasing, then the function $g_1(x) = -g(x)$ is nonincreasing, and applying the obtained result to $g_1(x)$, we get the same result for nondecreasing functions $g(x)$ as well. Thus, part (ii) is proved for all monotone functions $g(x)$.

The following theorem can be proved in a similar way as Theorem 3.3.

Theorem 3.4 (Second Mean Value Theorem II). Let $f(x)$ be a bounded function that is integrable on $[a,b]$. Let further m_Φ and M_Φ be the infimum and supremum, respectively, of the function $\Phi(x) = \frac{1}{\Gamma(1+\alpha)}\int_x^b f(t)(dt)^\alpha$ on $[a,b]$. Then

(i) If a function $g(x)$ is nonincreasing with $g(x) \ge 0$ on $[a,b]$, then there is some point ξ in (a,b) such that $m_\Phi \le \Phi(\xi) \le M_\Phi$ and

$${}_aI_b^{(\alpha)} f(x)g(x) = g(b)\Phi(\xi).$$

(ii) If $g(x)$ is any monotone function on $[a,b]$, then there is some point ξ in (a,b) such that $m_\Phi \le \Phi(\xi) \le M_\Phi$ and

$${}_aI_b^{(\alpha)} f(x)g(x) = [g(b) - g(a)]\Phi(\xi) + g(a)\,{}_aI_b^{(\alpha)} f(x).$$

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