Advances in Mechanical Engineering and its Applications
Vol. 1, No. 1, March 2012

www.worldsciencepublisher.org

A Single Integral Involving Kampé de Fériet Function

Shoukat Ali

Department of Mathematics, Govt. Engineering College Bikaner, Bikaner - 334 001, Rajasthan State, India

E-mail: dr.alishoukat@rediffmail.com

Abstract: The aim of this research note is to obtain an interesting single integral involving Kampé de Fériet function. The result is derived with the help of the classical Watson's theorem obtained earlier by Lavoie, Grondin and Rathie. For obtaining our main result we express Kampé de Fériet function as a series, change the order of integration and ummation due to uniformly convergence of the series after it evaluate the integral so obtained by well known integral due to Erdelyi et al. and summing up the series and finally using the classical Watson's theorem earlier obtained by Lavoie et al.

Keywords: Kampé de Fériet function; Watson's theorem.

1. Introduction

We recall the definition of generalized Kampé de Fériet's function as follows [5]

$$F^{p:q;k}_{\ell:m;n}\left[\begin{matrix}(a_p):(b_q);(c_k)\\(\alpha_\ell):(\beta_m);(\gamma_n)\end{matrix}\;\middle|\;\begin{matrix}x\\y\end{matrix}\right]=$$

$$\sum_{r,s=0}^{\infty}\frac{\prod_{j=1}^{p}(a_j)_{r+s}\prod_{j=1}^{q}(b_j)_r\prod_{j=1}^{k}(c_j)_s\,x^r y^s}{\prod_{j=1}^{\ell}(\alpha_j)_{r+s}\prod_{j=1}^{m}(\beta_j)_r\prod_{j=1}^{n}(\gamma_j)_s\,r!s!}\qquad ...(1.1)$$

Where for convergence

(i) $p+q<\ell+m+1,\ p+k<\ell+n+1,\ |x|<\infty,\ |y|<\infty$
or
(ii) $p+q<\ell+m+1,\ p+k<\ell+n+1$ and

$$\begin{cases}|x|^{\frac{1}{p-\ell}}+|y|^{\frac{1}{p-\ell}}<1, \text{if } p>\ell\\[2ex] \max\{|x|,|y|\}<1, \text{if } p\le\ell\end{cases}$$

Although the double hypergeometric series defined by (1.1) reduces to the Kampé de Fériet function in the special case:

$q=k$ and $m=n$

yet it is usually referred to in the literature as the Kampé de Fériet series.

The following are the cases in which the Kampé de Fériet function defined in (1.1) can be expressed in terms of generalized hypergeometric series.

$$F^{p:0;0}_{q:0;0}\left[\begin{matrix}\alpha_1,\ldots,\alpha_p\\\beta_1,\ldots,\beta_q\end{matrix}\;\middle|\;\begin{matrix}x\\y\end{matrix}\right]=$$

$${}_pF_q\left[\begin{matrix}\alpha_1,\ldots,\alpha_p\\\beta_1,\ldots,\beta_q\end{matrix}\;\middle|\;x+y\right]\qquad ...(1.2)$$

$$F^{0:p;r}_{0:q;s}\left[\begin{matrix}-;\alpha_1,\ldots,\alpha_p;\gamma_1,\ldots,\gamma_r\\-;\beta_1,\ldots,\beta_q;\delta_1,\ldots,\delta_s\end{matrix}\;\middle|\;\begin{matrix}x\\y\end{matrix}\right]$$

$$={}_pF_q\left[\begin{matrix}\alpha_1,\ldots,\alpha_p\\\beta_1,\ldots,\beta_q\end{matrix}\;\middle|\;x\right]$$

$${}_rF_s\left[\begin{matrix}\gamma_1,\ldots,\gamma_r\\\delta_1,\ldots,\delta_s\end{matrix}\;\middle|\;y\right]\qquad ...(1.3)$$

$$F^{p:1;1}_{q:0;0}\left[\begin{matrix}\alpha_1,\ldots,\alpha_p;\nu;\sigma\\\beta_1,\ldots,\beta_q;-;-\end{matrix}\;\middle|\;\begin{matrix}x\\x\end{matrix}\right]=$$

$${}_{p+1}F_q\left[\begin{matrix}\alpha_1,\ldots,\alpha_p;\nu+\sigma\\\beta_1,\ldots,\beta_q\end{matrix}\;\middle|\;x\right]\qquad ...(1.4)$$

and

$$F^{p:0;0}_{q:1;1}\left[\begin{matrix}\alpha_1,\ldots,\alpha_p;-;-\\\beta_1,\ldots,\beta_q;\nu;\sigma\end{matrix}\;\middle|\;\begin{matrix}x\\x\end{matrix}\right]=$$

$${}_{p+2}F_{q+3}\left[\begin{matrix}\alpha_1,\ldots,\alpha_p,\Delta(2;\nu+\sigma-1)\\\beta_1,\ldots,\beta_q,\nu,\sigma,\nu+\sigma-1\end{matrix}\;\middle|\;4x\right]$$

...(1.5)

where, and in what follows, $\Delta(\ell;\lambda)$ abbreviates the array of ℓ parameters

$$\frac{\lambda}{\ell},\frac{(\lambda+1)}{\ell},\ldots,\frac{(\lambda+\ell-1)}{\ell},\quad \ell=1,2,3,\ldots$$

For more detail see [5, pp. 28-32].

We have

$$\int_0^1 x^{\alpha-1}\,(1-x)^{\beta-1}\,dx = \frac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\alpha+\beta)} \qquad \ldots(1.6)$$

Let us consider the integral

$$I = \int_0^1 x^{\alpha-1}\,(1-x)^{\beta-1}\,{}_2F_1(a,b,c;x)\,dx$$

On expressing ${}_2F_1$ as a series, we have

$$I = \int_0^1 x^{\alpha-1}\,(1-x)^{\beta-1}\sum_{r=0}^{\infty}\frac{(a)_r\,(b)_r\,x^r}{(c)_r\,r!}dx$$

On changing the order of integration and summation due to uniformly convergence of the series, we have

$$I = \sum_{r=0}^{\infty}\frac{(a)_r\,(b)_r}{(c)_r\,r!}\int_0^1 x^{\alpha+r-1}\,(1-x)^{\beta-1}\,dx$$

Evaluating the integral by (1.6), we have

$$I = \sum_{r=0}^{\infty}\frac{(a)_r\,(b)_r}{(c)_r\,r!}\,\frac{\Gamma(\alpha+r)\,\Gamma(\beta)}{\Gamma(\alpha+r+\beta)}$$

$$= \sum_{r=0}^{\infty}\frac{(a)_r\,(b)_r}{(c)_r\,r!}\,\frac{\Gamma(\alpha+r)\,\Gamma(\beta)}{\Gamma(\alpha+r+\beta)}\,\frac{\Gamma(\alpha)\,\Gamma(\alpha+\beta)}{\Gamma(\alpha)\,\Gamma(\alpha+\beta)}$$

$$= \frac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\alpha+\beta)}\sum_{r=0}^{\infty}\frac{(a)_r\,(b)_r\,(\alpha)_r}{(c)_r\,(\alpha+\beta)_r\,r!}$$

Hence on summing up the series, we have

$$I = \int_0^1 x^{\alpha-1}\,(1-x)^{\beta-1}\,{}_2F_1(a,b,c;x)\,dx$$

$$= \frac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\alpha+\beta)}\,{}_3F_2\left[\begin{matrix}a,b,\alpha\\ c,\alpha+\beta\end{matrix}\;\middle|\;1\right]$$

In this, if we set a = A, b = B, α = C, β = C & c = $\frac{1}{2}$ (A + B + 1), we get

$$\int_0^1 x^{C-1}\,(1-x)^{C-1}\,{}_2F_1\left[\begin{matrix}A,B\\ \frac{1}{2}(A+B+1)\end{matrix}\;\middle|\;x\right]dx$$

$$= \frac{\Gamma(C)\,\Gamma(C)}{\Gamma(2C)}\,{}_3F_2\left[\begin{matrix}A,B,C\\ \frac{1}{2}(A+B+1),2C\end{matrix}\;\middle|\;1\right] \qquad \ldots(1.7)$$

To find the single integral involving Kampé de Fériet Function we express Kampé de Fériet Function [1] as a series, change the order of integration and summation due to uniformly convergence of the series, evaluate the integral so obtained by well known integral due to Erdelyi, A. et al. [2, 3] and summing up the series and finally use of generalized Watson's theorem earlier obtained by Lavoie et al. [4].

2. Results Required

The following results will be required in our present investigations.

(i)
$$ {}_3F_2\left[\begin{matrix}a,b,c\\ \frac{1}{2}(a+b+1),2c\end{matrix}\;\middle|\;1\right] = \frac{\Gamma(\frac{1}{2})\,\Gamma(c+\frac{1}{2})\,\Gamma(\frac{1}{2}+\frac{1}{2}a+\frac{1}{2}b)\,\Gamma(c-\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2})}{\Gamma(\frac{1}{2}a+\frac{1}{2})\,\Gamma(\frac{1}{2}a+\frac{1}{2}b)\,\Gamma(c-\frac{1}{2}a+\frac{1}{2})\,\Gamma(c-\frac{1}{2}b+\frac{1}{2})}$$

provided $\Re e\,(2c-a-b) > -1$. …(2.1)

(ii)
$$\Gamma(2m) = \frac{2^{2m-1}\,\Gamma(m)\,\Gamma(m+\frac{1}{2})}{\Gamma(\frac{1}{2})} \qquad \ldots(2.2)$$

(iii)
$$(\alpha)_n = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)} \qquad \ldots(2.3)$$

3. Main Result

The following result will be established in this section.

$$\int_0^1 x^{c-1}\,(1-x)^{c-1}\,{}_2F_1\left[\begin{matrix}a,b\\ \frac{1}{2}(a+b+1)\end{matrix}\;\middle|\;x\right]$$

$$F^{\lambda,\mu}_{\upsilon,\rho}\left[\begin{matrix}(\alpha_\lambda):(\beta_\mu);(\beta'_\mu)\\(\gamma_\upsilon):(\delta_\rho);(\delta'_\rho)\end{matrix}\;\middle|\;\begin{matrix}z_1x(1-x)\\z_2x(1-x)\end{matrix}\right]dx \;=$$

$$\frac{2^{a+b-2c-1}\;\Gamma(\frac{1}{2}(a+b+1))\;\Gamma(\frac{1}{2}a)\;\Gamma(\frac{1}{2}b)\;\Gamma(c)\;\Gamma(c-\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2})}{\Gamma(a)\;\Gamma(b)\;\Gamma(c-\frac{1}{2}a+\frac{1}{2})\;\Gamma(c-\frac{1}{2}b+\frac{1}{2})}$$

$$F^{\lambda+2,\mu}_{\upsilon+2,\rho}\left[\begin{matrix}(\alpha_\lambda),c,c+\frac{1}{2}(1-a-b):(\beta_\mu);(\beta'_\mu)\\(\gamma_\upsilon),c+\frac{1}{2}(1-a),c+\frac{1}{2}(1-b):(\delta_\rho);(\delta'_\rho)\end{matrix}\;\middle|\;\begin{matrix}\frac{z_1}{4}\\\frac{z_2}{4}\end{matrix}\right] \qquad \ldots(3.1)$$

4. Derivation

To prove (3.1), we proceed as follows:

Let

$$\int_0^1 x^{c-1}\;(1-x)^{c-1}\;{}_2F_1\left[\begin{matrix}a,b\\\frac{1}{2}(a+b+1)\end{matrix}\;\middle|\;x\right]$$

$$F^{\lambda,\mu}_{\upsilon,\rho}\left[\begin{matrix}(\alpha_\lambda):(\beta_\mu);(\beta'_\mu)\\(\gamma_\upsilon):(\delta_\rho);(\delta'_\rho)\end{matrix}\;\middle|\;\begin{matrix}z_1x(1-x)\\z_2x(1-x)\end{matrix}\right]dx$$

Express Kampé de Fériet function as a double series, we have

$$I=\int_0^1 x^{c-1}\;(1-x)^{c-1}\;{}_2F_1\left[\begin{matrix}a,b\\\frac{1}{2}(a+b+1)\end{matrix}\;\middle|\;x\right]$$

$$\sum_{p,q=0}^{\infty}A_{p,q}z_1{}^pz_2{}^qx^p(1-x)^px^q(1-x)^qdx$$

Changing the order of integration and summation due to uniformly convergence of the series, we have

$$I=\sum_{p,q=0}^{\infty}A_{p,q}z_1{}^pz_2{}^q$$

$$\int_0^1 x^{c+p+q-1}\;(1-x)^{c+p+q-1}\;{}_2F_1\left[\begin{matrix}a,b\\\frac{1}{2}(a+b+1)\end{matrix}\;\middle|\;x\right]dx$$

On using (1.7), we get

$$I=\sum_{p,q=0}^{\infty}A_{p,q}z_1{}^pz_2{}^q$$

$$\frac{\Gamma(c+p+q)\;\;\Gamma(c+p+q)}{\Gamma(2c+2p+2q)}$$

$${}_3F_2\left[\begin{matrix}a,b,(c+p+q)\\\frac{1}{2}(a+b+1),(2c+2p+2q)\end{matrix}\;\middle|\;1\right]$$

On using (2.1), we get

$$I=\sum_{p,q=0}^{\infty}A_{p,q}z_1{}^pz_2{}^q$$

$$\frac{\Gamma(c+p+q)\;\;\Gamma(c+p+q)}{\Gamma(2c+2p+2q)}$$

$$\frac{2^{a+b-2}\;\Gamma(\frac{1}{2}(a+b+1))\;\Gamma(\frac{1}{2}a)\;\Gamma(\frac{1}{2}b)\;\Gamma(c+p+q-\frac{1}{2}(a+b-1))\;\Gamma(c+p+q+\frac{1}{2})}{\Gamma(\frac{1}{2})\;\Gamma(a)\;\Gamma(b)\;\Gamma(c+p+q-\frac{1}{2}b+\frac{1}{2})\;\Gamma(c+p+q-\frac{1}{2}a+\frac{1}{2})}$$

On using (2.2) and (2.3), we finally have

$$I=\sum_{p,q=0}^{\infty}A_{p,q}(\frac{1}{4}z_1)^p(\frac{1}{4}z_2)^q$$

$$\frac{2^{a+b-2c-1}\;\Gamma(\frac{1}{2}(a+b+1))\;\Gamma(\frac{1}{2}a)\;\Gamma(\frac{1}{2}b)\;\Gamma(c)\;\;\Gamma(c-\frac{1}{2}(a+b-1))\;(c)_{p+q}\;(c-\frac{1}{2}(a+b-1))_{p+q}}{\Gamma(a)\;\Gamma(b)\;\Gamma(c-\frac{1}{2}a+\frac{1}{2})\;\Gamma(c-\frac{1}{2}b+\frac{1}{2})\;\;(c-\frac{1}{2}b+\frac{1}{2})_{p+q}\;\;(c-\frac{1}{2}a+\frac{1}{2})_{p+q}}$$

Interpreting the result with the definition of Kampé de Fériet function [1], we get the desired result (3.1).

References

1. Appell, P. and Kampé de Fériet, J., *Fonctions Hypergéométriques et Hypersphériques*; *Polynomes d'Hermites.* Paris: Gauthier-Villars (1926).
2. Erdélyi, A. et al., Tables of Integral Transforms, Vol. I (McGraw-Hill, New York, 1954).
3. Erdélyi, A. et al., Tables of Integral Transforms, Vol. II (McGraw-Hill, New York, 1954).
4. Lavoie, J. L., Grondin, F. and Rathie, A. K., Generalizations of Watson's theorem on the sum of a ${}_3F_2$. Indian J. Math., 32, No. 1, 23-32 (1992).
5. Srivastava, H. M. and Karlsson, P. K., *Multiple Gaussian Hypergeometric Series*. Ellis Horwood Limited, New York (1985).
6. Hedi Jedli, Noureddine Hidouri, A Power Drive Scheme for an Isolated Pitched Wind Turbine Water Pumping System based on DC machine, Advances in Mechanical Engineering and its Applications, vol.1, no.1, pp. 1-4, 2012
7. Yudong Zhang, Lenan Wu, Artificial Bee Colony for Two Dimensional Protein Folding, Advances in Electrical Engineering Systems, vol.1, no.1, pp. 19-23, 2012
8. Armin Ghabousian, Mousa Shamsi, Segmentation of Apple Color Images Utilizing Fuzzy Clustering Algorithms, Advances in Digital Multimedia, vol.1, no.1, pp.59-63, 2012