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Local fractional Fourier analysis

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Abstract – Local fractional calculus (LFC) deals with everywhere continuous but nowhere differentiable functions in fractal space. In this letter we point out local fractional Fourier analysis in generalized Hilbert space. We first investigate the local fractional calculus and complex number of fractional-order based on the complex Mittag-Leffler function in fractal space. Then we study the local fractional Fourier analysis from the theory of local fractional functional analysis point of view. We finally propose the fractional-order trigonometric and complex Mittag-Leffler functions expressions of local fractional Fourier series.

Keywords –Local fractional calculus; Fractal space; Complex Mittag-Leffler function; Local fractional Fourier analysis; Local fractional functional analysis

1. Introduction

Local fractional calculus (fractal calculus), which was dealing with fractal functions, had been proposed and developed in[1-12]. Local fractional calculus was successfully applied in the fractal elasticity [2, 6], the fractal release equation [15], the fractal wave equation [21], the fractal signal [17], the Yang-Laplace transforms [20-22], the Yang-Fourier transforms [16, 17], the discrete Fourier transform[18], the local fractional short time transform [11,12] and the local fractional wavelet transform[11,12].

In this letter, our aim is to research the local fractional Fourier analysis from the theory of local fractional functional analysis point of view. The organization of this paper is as follows. In section 2, the preliminary results are presented. The complex number of fractional-order is investigated in section 3. Generalization of local fractional Fourier series in generalized Hilbert space is studied in section 4. Applications to local fractional Fourier series are shown in section 5.

2. Preliminaries

2.1 Local fractional continuity

Definition 1 If there exists [11, 12, 16, 21]

$$\left|f(x)-f(x_0)\right|<\varepsilon^{\alpha} \tag{2.1}$$

with $\left|x-x_0\right|<\delta$,for $\varepsilon,\delta>0$ and $\varepsilon,\delta\in\mathbb{R}$, now $f(x)$ is called local fractional continuous at $x=x_0$, denote by $\lim_{x\to x_0} f(x)=f(x_0)$. Then $f(x)$ is called local fractional continuous on the interval (a,b) , denoted by

$$f(x)\in C_{\alpha}(a,b). \tag{2.2}$$

Definition 2 A function $f(x)$ is called a non-differentiable function of exponent α , $0<\alpha\le1$, which satisfies Hölder function of exponent α , then for $x,y\in X$ such that [11,12]

$$\left|f(x)-f(y)\right|\le C\left|x-y\right|^{\alpha}. \tag{2.3}$$

Definition 3 A function $f(x)$ is called to be continuous of order α , $0<\alpha\le1$, or shortly α continuous, when we have that [11,12]

$$f(x)-f(x_0)=o\left(\left(x-x_0\right)^{\alpha}\right). \tag{2.4}$$

Remark 1. Compared with (2.4), (2.1) is standard definition of local fractional continuity. Here (2.3) is unified local fractional continuity.

2.2 Local fractional calculus

Definition 4 Let $f(x)\in C_{\alpha}(a,b)$. Local fractional derivative of $f(x)$ of order α at $x=x_0$ is defined as [11,12,16,21]

$$f^{(\alpha)}(x_0)=\left.\frac{d^{\alpha}f(x)}{dx^{\alpha}}\right|_{x=x_0}=\lim_{x\to x_0}\frac{\Delta^{\alpha}\left(f(x)-f(x_0)\right)}{\left(x-x_0\right)^{\alpha}}, \tag{2.5}$$

where $\Delta^{\alpha}\left(f(x)-f(x_0)\right)\cong\Gamma(1+\alpha)\Delta\left(f(x)-f(x_0)\right)$.

For any $x\in(a,b)$, there exists

$$f^{(\alpha)}(x)=D_x^{(\alpha)}f(x),$$

denoted by

$$f(x)\in D_x^{(\alpha)}(a,b).$$

Definition 5 Let $f(x) \in C_\alpha(a,b)$. Local fractional integral of $f(x)$ of order α in the interval $[a,b]$ is given [11-14, 16, 19, 2 1]

$$
\begin{aligned}
&{}_aI_b^{(\alpha)} f(x) \\
&= \frac{1}{\Gamma(1+\alpha)} \int_a^b f(t)(dt)^\alpha \\
&= \frac{1}{\Gamma(1+\alpha)} \lim_{\Delta t \to 0} \sum_{j=0}^{j=N-1} f(t_j)(\Delta t_j)^\alpha
\end{aligned}, \qquad (2.6)
$$

where $\Delta t_j = t_{j+1} - t_j$, $\Delta t = \max\{\Delta t_1, \Delta t_2, \Delta t_j, ...\}$ and $[t_j, t_{j+1}]$, $j = 0, ..., N-1$, $t_0 = a, t_N = b$, is a partition of the interval $[a,b]$. For convenience, we assume that

$${}_aI_a^{(\alpha)} f(x) = 0 \text{ if } a = b \text{ and}$$

$${}_aI_b^{(\alpha)} f(x) = -{}_bI_a^{(\alpha)} f(x) \text{ if } a < b.$$

For any $x \in (a,b)$, we get

$${}_aI_x^{(\alpha)} f(x), \qquad (2.7)$$

denoted by

$$f(x) \in I_x^{(\alpha)}(a,b).$$

Remark 2. If $f(x) \in D_x^{(\alpha)}(a,b)$, or $I_x^{(\alpha)}(a,b)$, we have that

$$f(x) \in C_\alpha(a,b). \qquad (2.8)$$

Remark 3. The following relations hold

$$\frac{1}{\Gamma(1+\alpha)} \int_a^b E_\alpha(x^\alpha)(dx)^\alpha = E_\alpha(b^\alpha) - E_\alpha(a^\alpha); \qquad (2.9)$$

$$\frac{1}{\Gamma(1+\alpha)} \int_a^b \sin_\alpha x^\alpha (dx)^\alpha = \sin_\alpha a^\alpha - \sin_\alpha b^\alpha; \qquad (2.10)$$

$$\frac{1}{\Gamma(1+\alpha)} \int_a^b \cos_\alpha x^\alpha (dx)^\alpha = \sin_\alpha b^\alpha - \sin_\alpha a^\alpha; \qquad (2.11)$$

$$\frac{1}{\Gamma(1+\alpha)} \int_a^b x^{k\alpha}(dx)^\alpha = \frac{\Gamma(1+k\alpha)}{\Gamma(1+(k+1)\alpha)}\left(b^{(k+1)\alpha} - a^{(k+1)\alpha}\right); \qquad (2.12)$$

3. Complex number of fractional-order

Definition 6 Fractional-order complex number is defined by [11, 12]

$$I^\alpha = x^\alpha + i^\alpha y^\alpha,\ x, y \in \mathbb{R},\ 0 < \alpha \le 1, \qquad (3.1)$$

where its conjugate of complex number shows that

$$\overline{I^\alpha} = x^\alpha - i^\alpha y^\alpha, \qquad (3.2)$$

, and where the fractional modulus is derived as

$$|I^\alpha| = I^\alpha \overline{I^\alpha} = \overline{I^\alpha} I^\alpha = \sqrt{x^{2\alpha} + y^{2\alpha}}. \qquad (3.3)$$

Definition 7 Complex Mittag-Leffler function in fractal space is defined by [11, 12]

$$E_\alpha(z^\alpha) := \sum_{k=0}^{\infty} \frac{z^{\alpha k}}{\Gamma(1+k\alpha)}, \qquad (3.4)$$

for $z \in C$ (complex number set) and $0 < \alpha \le 1$.
The following rules hold:

$$E_\alpha(z_1^\alpha) E_\alpha(z_2^\alpha) = E_\alpha\left((z_1 + z_2)^\alpha\right); \qquad (3.5)$$

$$E_\alpha(z_1^\alpha) E_\alpha(-z_2^\alpha) = E_\alpha\left((z_1 - z_2)^\alpha\right); \qquad (3.6)$$

$$E_\alpha(i^\alpha z_1^\alpha) E_\alpha(i^\alpha z_2^\alpha) = E_\alpha\left(i^\alpha (z_1^\alpha + z_2^\alpha)^\alpha\right). \qquad (3.7)$$

When $z^\alpha = i^\alpha x^\alpha$, the complex Mittag-Leffler function is [11, 12]

$$E_\alpha(i^\alpha x^\alpha) = \cos_\alpha x^\alpha + i^\alpha \sin_\alpha x^\alpha \qquad (3.8)$$

with

$$\cos_\alpha x^\alpha := \sum_{k=0}^{\infty} (-1)^k \frac{x^{2\alpha k}}{\Gamma(1+2\alpha k)}$$

and

$$\sin_\alpha x^\alpha := \sum_{k=0}^{\infty} (-1)^k \frac{x^{\alpha(2k+1)}}{\Gamma[1+\alpha(2k+1)]},$$

for $x \in \mathbb{R}$ and $0 < \alpha \le 1$, we have that

$$E_\alpha(i^\alpha x^\alpha) E_\alpha(i^\alpha y^\alpha) = E_\alpha\left(i^\alpha (x+y)^\alpha\right) \qquad (3.9)$$

and

$$E_\alpha(i^\alpha x^\alpha) E_\alpha(-i^\alpha y^\alpha) = E_\alpha\left(i^\alpha (x-y)^\alpha\right). \qquad (3.10)$$

4. Generalization of local fractional Fourier series in generalized Hilbert space

4.1 Generalized inner product space

Definition 8 Let V be a complex or real vector space. A generalized inner product on a vector space V is a function $\langle x^\alpha, y^\alpha \rangle_\alpha$ on pairs (x^α, y^α) of vectors in $V \times V$ taking values satisfying the following properties [11, 12]:

(1) $\langle x^\alpha, x^\alpha \rangle_\alpha \ge 0$ for all $x^\alpha \in V$ and

$$\langle x^\alpha, x^\alpha \rangle_\alpha = 0 \text{ only if } x = 0.$$

(2) $\langle x^\alpha, y^\alpha \rangle_\alpha = \overline{\langle y^\alpha, x^\alpha \rangle_\alpha}$ for all $x^\alpha, y^\alpha \in V$.

(3) For all $x^\alpha, y^\alpha, z^\alpha \in V$ and scalars $a, b \in \mathbb{R}$,

$$\langle a^\alpha x^\alpha + b^\alpha y^\alpha, z^\alpha \rangle_\alpha = a^\alpha \langle x^\alpha, z^\alpha \rangle_\alpha + b^\alpha \langle y^\alpha, z^\alpha \rangle_\alpha. \qquad (4.1)$$

A generalized inner product space is a generalized vector space with an inner product. Given a generalized inner product space, the following definition provides a norm:

$$\|x^\alpha\|_\alpha = \langle x^\alpha, x^\alpha \rangle_\alpha^{\frac{1}{2}} = \sqrt{\sum_{k=1}^{\infty} |x_k^\alpha|^2}. \qquad (4.2)$$

Now we can define a scalar (or dot) product of two T –periodic functions $f(t)$ and $g(t)$ as

$$\langle f, g\rangle_\alpha = \int_0^T f(t)\overline{g(t)}(dt)^\alpha . \quad (4.3)$$

For more materials, we see [11, 12].

4.2 Generalized Hilbert space

Definition 9 A generalized Hilbert space is a complete generalized inner-product space [11, 12]. Suppose $\{e^\alpha{}_n\}$ is an orthonormal system in an inner product space X .The following are equivalent [11, 12]:

(1) $span\{e_1{}^\alpha,...,e_n{}^\alpha\} = X$, ie. $\{e_n{}^\alpha\}$ is a basis;

(2) (**Pythagorean theorem in fractal space**) The equation

$$\sum_{k=1}^\infty \left|a_k{}^\alpha\right|^2 = \|f\|_\alpha^2 \quad (4.4)$$

for all $f \in X$, where $a_k{}^\alpha = \langle f, e_k{}^\alpha\rangle_\alpha$;

(3) **(Generalized Pythagorean theorem in fractal space)**
Generalized equation

$$\langle f, g\rangle = \sum_{k=1}^n a_k{}^\alpha \overline{b_k{}^\alpha} \quad (4.5)$$

for all $f, g \in X$, where $a_k{}^\alpha = \langle f, e_n{}^\alpha\rangle_\alpha$ and $b_k{}^\alpha = \langle g, e_k{}^\alpha\rangle_\alpha$;

(4) $f = \sum_{k=1}^n a_k{}^\alpha e_k{}^\alpha$ with sum convergent in X for all $f \in X$.

For more details, see [11,12].

Here we can take any sequence of T -periodic fractal functions ϕ_k , $k = 0,1,...$ that are

(1) Orthogonal:

$$\langle \phi_k, \phi_j\rangle_\alpha = \int_0^T \phi_k(t)\overline{\phi_j(t)}(dt)^\alpha = 0 \text{ if } k \neq j; \quad (4.6)$$

(2) Normalized:

$$\langle \phi_k, \phi_k\rangle_\alpha = \int_0^T \phi_k{}^2(t)(dt)^\alpha = 1; \quad (4.7)$$

(3) Complete: If a function $x(t)$ is such that

$$\langle x, \phi_k\rangle_\alpha = \int_0^T x(t)\phi_k(t)(dt)^\alpha = 0 \quad (4.8)$$

for all i , then $x(t) \equiv 0$.

4.3 Generalization of local fractional Fourier series in generalized Hilbert space

4.3.1 Generalization of local fractional Fourier series in generalized Hilbert space

Definition 10 Let $\{\phi_k(t)\}_{k=1}^\infty$ be a complete, orthonormal set of functions. Then any T -periodic fractal signal $f(t)$ can be uniquely represented as an infinite series

$$f(t) = \sum_{k=0}^\infty \varphi_k \phi_k(t) \quad (4.9)$$

This is called the local fractional Fourier series representation of $f(t)$ in the generalized Hilbert space. The scalars φ_i are called the local fractional Fourier coefficients of $f(t)$.

4.3.2 Local fractional Fourier coefficients

To derive the formula for φ_k , write

$$f(t)\phi_k(t) = \sum_{i=0}^\infty \varphi_j \phi_j(t)\phi_k(t), \quad (4.10)$$

and integrate over one period by using the generalized Pythagorean theorem in fractal space

$$\begin{aligned} &\langle f, \phi_k\rangle_\alpha \\ &= \int_0^T f(t)\phi_k(t)(dt)^\alpha \\ &= \int_0^T \sum_{j=0}^\infty \varphi_j \phi_j(t)\phi_k(t)(dt)^\alpha \\ &= \sum_{j=0}^\infty \left(\varphi_j\left(\int_0^T \phi_j(t)\phi_k(t)(dt)^\alpha\right)\right) \\ &= \sum_{j=0}^\infty \varphi_j \langle \phi_j, \phi_k\rangle_\alpha \\ &= \varphi_k \end{aligned} \quad (4.11)$$

Because the functions $\phi_k(t)$ form a complete orthonormal system, the partial sums of the local fractional Fourier series

$$f(t) = \sum_{k=0}^\infty \varphi_k \phi_k(t) \quad (4.12)$$

converge to $f(t)$ in the following sense:

$$\lim_{N\to\infty}\left(\frac{1}{\Gamma(1+\alpha)}\int_0^T\left(f(t)-\sum_{k=1}^\infty \varphi_k\phi_k(t)\right)\overline{\left(f(t)-\sum_{k=1}^\infty \varphi_k\phi_k(t)\right)}(dt)^\alpha\right)=0 \quad (4.13)$$

Therefore, we can use the partial sums

$$f_N(t) = \sum_{k=1}^N \varphi_k \phi_k(t) \quad (4.14)$$

to approximate $f(t)$. Meanwhile, we have that

$$\int_0^T f^2(t)(dt)^\alpha = \sum_{k=1}^\infty \varphi_k{}^2 . \quad (4.15)$$

5. Applications of local fractional Fourier series

The sequence of T-periodic functions in fractal space $\{\phi_k(t)\}_{k=0}^{\infty}$ defined by $\phi_0(t)=\left(\frac{1}{T}\right)^{\frac{\alpha}{2}}$ and

$$\phi_k(t)=\begin{cases}\left(\frac{2}{T}\right)^{\frac{\alpha}{2}}\sin_\alpha\left(k^\alpha\omega_0{}^\alpha t^\alpha\right), if\ k\geq 1\ is\ odd\\ \left(\frac{2}{T}\right)^{\frac{\alpha}{2}}\cos_\alpha\left(k^\alpha\omega_0{}^\alpha t^\alpha\right), if\ k>1\ is\ even\end{cases} \quad (5.1)$$

are complete and orthonormal, where $\omega_0=\frac{2\pi}{T}$.

A more common way of writing down the local fractional trigonometric Fourier series of $f(t)$ is given [12, 13]

$$f(t)=a_0+\sum_{i=1}^{\infty}a_k\sin_\alpha\left(k^\alpha\omega_0{}^\alpha t^\alpha\right)+\sum_{i=1}^{\infty}b_k\cos_\alpha\left(k^\alpha\omega_0{}^\alpha t^\alpha\right) \quad (5.2)$$

Then the local fractional Fourier coefficients can be computed by

$$\begin{cases}a_0=\frac{1}{T^\alpha}\int_0^T f(t)(dt)^\alpha,\\ a_k=\left(\frac{2}{T}\right)^\alpha\int_0^T f(t)\sin_\alpha\left(k^\alpha\omega_0{}^\alpha t^\alpha\right)(dt)^\alpha,\\ b_k=\left(\frac{2}{T}\right)^\alpha\int_0^T f(t)\cos_\alpha\left(k^\alpha\omega_0{}^\alpha t^\alpha\right)(dt)^\alpha.\end{cases} \quad (5.3)$$

This result is equivalent to the formulation [11, 12, 16, 17, 21].

Another useful complete orthonormal set is furnished by the Mittag-Leffler functions:

$$\phi_k(t)=\sqrt{\frac{1}{T^\alpha}}E_\alpha\left(i^\alpha k^\alpha\omega_0{}^\alpha t^\alpha\right), k=0,\pm1,\pm2,... \quad (5.4)$$

where $\omega_0=\frac{2\pi}{T}$.

Hence, we get the Mittag-Leffler functions expression of local fractional Fourier series [22]

$$f(x)=\sum_{k=-\infty}^{\infty}C_k E_\alpha\left(\frac{\pi^\alpha i^\alpha (kx)^\alpha}{l^\alpha}\right), \quad (5.5)$$

where the local fractional Fourier coefficients is

$$C_k=\frac{1}{(2l)^\alpha}\int_{-l}^{l}f(x)E_\alpha\left(\frac{-\pi^\alpha i^\alpha (kx)^\alpha}{l^\alpha}\right)(dx)^\alpha \text{ with } k\in\mathbb{Z}. \quad (5.6)$$

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