

Theory and Applications of Local Fractional Fourier Analysis

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Abstract –Local fractional Fourier analysis is a generalized Fourier analysis in fractal space. The local fractional calculus is one of useful tools to process the local fractional continuously non-differentiable functions (fractal functions). Based on the local fractional derivative and integration, the present work is devoted to the theory and applications of local fractional Fourier analysis in generalized Hilbert space. We investigate the local fractional Fourier series, the Yang-Fourier transform, the generalized Yang-Fourier transform, the discrete Yang-Fourier transform and fast Yang-Fourier transform.

Keywords –Fractal space; Local fractional Fourier analysis; Local fractional calculus; Non-differentiable functions

1. Introduction

Fourier analysis [1-6] is a mathematical method applied to transform a periodic function with many applications in physics and engineering. It had been used to a wider variety of field in the sciences and in engineering, image and signal processing, containing electrical engineering, quantum mechanics, neurology, optics, acoustics, and oceanography, and so on, and after improved and expanded upon it, its general field was come to be known as the field of harmonic analysis [7, 8].

In mathematics, in the area of harmonic analysis, the fractional Fourier transform (FRFT) [9] is a linear transformation generalizing the Fourier transform. The FRFT [10-18] can be used to define fractional convolution, correlation, and other operations, and can also be further generalized into the linear canonical transformation (LCT).

However, the above referred results can't process the non-differentiable time-frequency functions on a fractal set (also local fractional continuous functions). The theory of local fractional calculus (also called fractal calculus [19-33]) is one of useful tools to handle the fractal and continuously non-differentiable functions, and was successfully applied in describing physical phenomena [34-43]. Local fractional Fourier analysis [44, 45] derived from the local fractional calculus, which is a generalization of the Fourier analysis in fractal space, has played an important role in handling non-differentiable functions.

The aim of this paper is investigated the theory and applications of the local fractional Fourier analysis. The organization of this paper is as follows. In section 2, the preliminary results for the local fractional calculus are investigated. The theory of local fractional Fourier series is presented in section 3. Section 4 is devoted to theory of the Yang-Fourier transform. Theory of the generalized Yang-Fourier transform is considered in section 5. The discrete Yang-Fourier transform is studied in section 6. The Fast Yang-Fourier transform is considered in section 7. The conclusion is in section 8.

2. Preliminary results

2.1. Local fractional continuity of functions

Definition 1 [30-35]

If there is

$$|f(x) - f(x_0)| < \varepsilon^\alpha \quad (2.1)$$

with $|x - x_0| < \delta$, for $\varepsilon, \delta > 0$ and $\varepsilon, \delta \in \mathbb{R}$. Now

$f(x)$ is called local fractional continuous at $x = x_0$,

denote by $\lim_{x \rightarrow x_0} f(x) = f(x_0)$. Then $f(x)$ is called

local fractional continuous on the interval (a, b) ,

denoted by [30-35]

$$f(x) \in C_\alpha(a, b). \quad (2.2)$$

Lemma 1 [33-36]

Let F be a subset of the real line and be a fractal. If $f: (F, d) \rightarrow (\Omega, d')$ is a bi-Lipschitz mapping, then there is for constants $\rho, \tau > 0$ and $F \subset \mathbb{R}$,

$$\rho^s H^s(F) \leq H^s(f(F)) \leq \tau^s H^s(F) \quad (2.3)$$

such that for all $x_1, x_2 \in F$,

$$\rho^\alpha |x_1 - x_2|^\alpha \leq |f(x_1) - f(x_2)| \leq \tau^\alpha |x_1 - x_2|^\alpha. \quad (2.4)$$

As a direct result in the condition of Lemma 1, we have

$$|f(x_1) - f(x_2)| \leq \tau^\alpha |x_1 - x_2|^\alpha \quad (2.5)$$

such that

$$|f(x_1) - f(x_2)| < \varepsilon^\alpha \quad (2.6)$$

Notice that α is fractal dimension. This result is directly deduced from fractal geometry.

2.2. Local fractional derivative and integration

Definition 2 [30-35]

Setting $f(x) \in C_\alpha(a, b)$, local fractional derivative of $f(x)$ of order α at $x = x_0$ is defined by

$$f^{(\alpha)}(x_0) = \frac{d^\alpha f(x)}{dx^\alpha} \Big|_{x=x_0} = \lim_{x \rightarrow x_0} \frac{\Delta^\alpha (f(x) - f(x_0))}{(x - x_0)^\alpha}, \quad (2.7)$$

where $\Delta^\alpha (f(x) - f(x_0)) \cong \Gamma(1 + \alpha) \Delta (f(x) - f(x_0))$.

For any $x \in (a, b)$, there exists [20-24, 48-50]

$$f^{(\alpha)}(x) = D_x^{(\alpha)} f(x), \quad (2.8)$$

denoted by

$$f(x) \in D_x^{(\alpha)}(a, b). \quad (2.9)$$

Definition 3 [30-35]

Setting $f(x) \in C_\alpha(a, b)$, local fractional integral of $f(x)$ of order α in the interval $[a, b]$ is defined

$$\begin{aligned} {}_a I_b^{(\alpha)} f(x) &= \frac{1}{\Gamma(1 + \alpha)} \int_a^b f(t) (dt)^\alpha, \quad (2.10) \\ &= \frac{1}{\Gamma(1 + \alpha)} \lim_{\Delta t \rightarrow 0} \sum_{j=0}^{N-1} f(t_j) (\Delta t_j)^\alpha \end{aligned}$$

where $\Delta t_j = t_{j+1} - t_j$, $\Delta t = \max\{\Delta t_1, \Delta t_2, \Delta t_j, \dots\}$ and $[t_j, t_{j+1}]$, $j = 0, \dots, N - 1$, $t_0 = a$, $t_N = b$, is a partition of the interval $[a, b]$. For any $x \in (a, b)$, there exists [30-35]

$${}_a I_x^{(\alpha)} f(x), \quad (2.11)$$

denoted by

$$f(x) \in I_x^{(\alpha)}(a, b). \quad (2.12)$$

Here, it follows that [30-35]

$${}_a I_a^{(\alpha)} f(x) = 0 \text{ if } a = b; \quad (2.13)$$

$${}_a I_b^{(\alpha)} f(x) = -{}_b I_a^{(\alpha)} f(x) \text{ if } a < b; \quad (2.14)$$

and

$${}_a I_a^{(0)} f(x) = f(x). \quad (2.15)$$

We notice that we have [30-35]

$$f(x) \in C_\alpha(a, b). \quad (2.16)$$

if

$$f(x) \in D_x^{(\alpha)}(a, b), \text{ or } I_x^{(\alpha)}(a, b).$$

For their fractal geometrical explanation of local fractional derivative and integration, we see [30-35].

2.3. Complex number of fractional-order

Definition 4

Fractional-order complex number is defined by [30, 31, 38]

$$I^\alpha = x^\alpha + i^\alpha y^\alpha, \quad x, y \in \mathbb{R}, \quad 0 < \alpha \leq 1, \quad (2.17)$$

where its conjugate of complex number shows that

$$\overline{I^\alpha} = x^\alpha - i^\alpha y^\alpha, \quad (2.18)$$

, and where the fractional modulus is derived as

$$|I^\alpha| = I^\alpha \overline{I^\alpha} = \overline{I^\alpha} I^\alpha = \sqrt{x^{2\alpha} + y^{2\alpha}}. \quad (2.19)$$

Definition 5

Complex Mittag-Leffler function in fractal space is defined by [30, 31, 38]

$$E_\alpha(z^\alpha) := \sum_{k=0}^{\infty} \frac{z^{\alpha k}}{\Gamma(1 + k\alpha)}, \quad (2.20)$$

for $z \in \mathbb{C}$ (complex number set) and $0 < \alpha \leq 1$.

The following rules hold [30, 31, 38]:

$$E_\alpha(z_1^\alpha) E_\alpha(z_2^\alpha) = E_\alpha((z_1 + z_2)^\alpha); \quad (2.21)$$

$$E_\alpha(z_1^\alpha) E_\alpha(-z_2^\alpha) = E_\alpha((z_1 - z_2)^\alpha); \quad (2.22)$$

$$E_\alpha(i^\alpha z_1^\alpha) E_\alpha(i^\alpha z_2^\alpha) = E_\alpha(i^\alpha (z_1^\alpha + z_2^\alpha)^\alpha). \quad (2.23)$$

When $z^\alpha = i^\alpha x^\alpha$, the complex Mittag-Leffler function is [30, 31, 38]

$$E_\alpha(i^\alpha x^\alpha) = \cos_\alpha x^\alpha + i^\alpha \sin_\alpha x^\alpha \quad (2.24)$$

with

$$\cos_\alpha x^\alpha = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2\alpha k}}{\Gamma(1 + 2\alpha k)}$$

and

$$\sin_\alpha x^\alpha = \sum_{k=0}^{\infty} (-1)^k \frac{x^{\alpha(2k+1)}}{\Gamma[1 + \alpha(2k+1)]},$$

for $x \in \mathbb{R}$ and $0 < \alpha \leq 1$, we have that [30, 31, 38]

$$E_\alpha(i^\alpha x^\alpha) E_\alpha(i^\alpha y^\alpha) = E_\alpha(i^\alpha (x + y)^\alpha) \quad (2.25)$$

and

$$E_\alpha(i^\alpha x^\alpha) E_\alpha(-i^\alpha y^\alpha) = E_\alpha(i^\alpha (x - y)^\alpha). \quad (2.26)$$

2.4. Generalized Hilbert space

Definition 6[30, 31, 38, 43, 44]

A generalized Hilbert space is a complete generalized inner-product space.

Definition 7[30, 31, 38, 43, 44]

A scalar (or dot) product of two T -periodic functions

$f(t)$ and $g(t)$ is defined by

$$\langle f, g \rangle_\alpha = \int_0^T f(t) \overline{g(t)} (dt)^\alpha. \quad (2.27)$$

Suppose $\{e_n^\alpha\}$ is an orthonormal system in an inner

product space X . The following results are equivalent [30, 31, 38, 43, 44]:

(1) $span\{e_1^\alpha, \dots, e_n^\alpha\} = X$, ie. $\{e_n^\alpha\}$ is a basis;

(2) **(Pythagorean theorem in fractal space)**

The equation

$$\sum_{k=1}^{\infty} |a_k^\alpha|^2 = \|f\|_\alpha^2 \quad (2.28)$$

for all $f \in X$, where $a_k^\alpha = \langle f, e_k^\alpha \rangle_\alpha$;

(3) (Generalized Pythagorean theorem in fractal space)

Generalized equation

$$\langle f, g \rangle = \sum_{k=1}^n a_k^\alpha \overline{b_k^\alpha} \quad (2.29)$$

for all $f, g \in X$, where

$$a_k^\alpha = \langle f, e_k^\alpha \rangle_\alpha \text{ and } b_k^\alpha = \langle g, e_k^\alpha \rangle_\alpha;$$

(4) $f = \sum_{k=1}^n a_k^\alpha e_k^\alpha$ with sum convergent in X for all $f \in X$.

For more details, we see [30, 31, 38, 43, 44].

Here we can take any sequence of T -periodic local fractional continuous functions ϕ_k , $k = 0, 1, \dots$ that are [43, 44]

(1) Orthogonal:

$$\langle \phi_k, \phi_j \rangle_\alpha = \int_0^T \phi_k(t) \overline{\phi_j(t)} (dt)^\alpha = 0 \text{ if } k \neq j; \quad (2.30)$$

(2) Normalized:

$$\langle \phi_k, \phi_k \rangle_\alpha = \int_0^T \phi_k^2(t) (dt)^\alpha = 1; \quad (2.31)$$

(3) Complete: If a function $x(t)$ is such that

$$\langle x, \phi_k \rangle_\alpha = \int_0^T x(t) \phi_k(t) (dt)^\alpha = 0 \quad (2.32)$$

for all i , then $x(t) \equiv 0$.

2.5. Local fractional Fourier series in generalized Hilbert space

2.5.1. Local fractional Fourier series in generalized Hilbert space

Definition 8 [43, 44]

Let $\{\phi_k(t)\}_{k=1}^\infty$ be a complete, orthonormal set of functions. Then any T -periodic fractal signal $f(t)$ can be uniquely represented as an infinite series

$$f(t) = \sum_{k=0}^{\infty} \varphi_k \phi_k(t) \quad (2.33)$$

This is called the local fractional Fourier series representation of $f(t)$ in the generalized Hilbert space.

The scalars φ_i are called the local fractional Fourier coefficients of $f(t)$.

2.5.2. Local fractional Fourier coefficients

To derive the formula for φ_k , write [43, 44]

$$f(t) \phi_k(t) = \sum_{i=0}^{\infty} \varphi_i \phi_j(t) \phi_k(t), \quad (2.34)$$

and integrate over one period by using the generalized Pythagorean theorem in fractal space [43, 44]

$$\begin{aligned} & \langle f, \phi_k \rangle_\alpha \\ &= \int_0^T f(t) \phi_k(t) (dt)^\alpha \\ &= \int_0^T \sum_{j=0}^{\infty} \varphi_j \phi_j(t) \phi_k(t) (dt)^\alpha \\ &= \sum_{j=0}^{\infty} \left(\varphi_j \left(\int_0^T \phi_j(t) \phi_k(t) (dt)^\alpha \right) \right) \\ &= \sum_{j=0}^{\infty} \varphi_j \langle \phi_j, \phi_k \rangle_\alpha \\ &= \varphi_k \end{aligned} \quad (2.35)$$

Because the functions $\phi_k(t)$ form a complete orthonormal system, the partial sums of the local fractional Fourier series

$$f(t) = \sum_{k=0}^{\infty} \varphi_k \phi_k(t) \quad (2.36)$$

converge to $f(t)$ in the following sense:

$$\begin{aligned} & \lim_{N \rightarrow \infty} \left(\frac{1}{\Gamma(1+\alpha)} \int_0^T \left(f(t) - \sum_{k=1}^N \varphi_k \phi_k(t) \right) \overline{\left(f(t) - \sum_{k=1}^N \varphi_k \phi_k(t) \right)} (dt)^\alpha \right) \\ &= 0 \end{aligned} \quad (2.37)$$

Therefore, we can use the partial sums

$$f_N(t) = \sum_{k=1}^N \varphi_k \phi_k(t) \quad (2.38)$$

to approximate $f(t)$.

Hence, we have that

$$\int_0^T f^2(t) (dt)^\alpha = \sum_{k=1}^{\infty} \varphi_k^2. \quad (2.39)$$

The sequence of T -periodic functions in fractal space $\{\phi_k(t)\}_{k=0}^\infty$ defined by

$$\phi_0(t) = \left(\frac{1}{T} \right)^{\frac{\alpha}{2}}$$

and

$$\phi_k(t) = \begin{cases} \left(\frac{2}{T} \right)^{\frac{\alpha}{2}} \sin_\alpha(k^\alpha \omega_0^\alpha t^\alpha), & \text{if } k \geq 1 \text{ is odd} \\ \left(\frac{2}{T} \right)^{\frac{\alpha}{2}} \cos_\alpha(k^\alpha \omega_0^\alpha t^\alpha), & \text{if } k > 1 \text{ is even} \end{cases} \quad (2.40)$$

are complete and orthonormal, where $\omega_0 = \frac{2\pi}{T}$.

Another useful complete orthonormal set is furnished by the Mittag-Leffler functions:

$$\phi_k(t) = \left(\frac{1}{T}\right)^{\frac{\alpha}{2}} E_{\alpha} \left(i^{\alpha} k^{\alpha} \omega_0^{\alpha} t^{\alpha} \right), k = 0, \pm 1, \pm 2, \dots \quad (2.41)$$

where $\omega_0 = \frac{2\pi}{T}$.

3. Local Fractional Fourier Series

3.1. Notations

Definition 9[30, 31, 38, 43, 44]

Local fractional trigonometric Fourier series of $f(t)$ is given by

$$f(t) = a_0 + \sum_{k=1}^{\infty} a_k \sin_{\alpha} \left(k^{\alpha} \omega_0^{\alpha} t^{\alpha} \right) + \sum_{k=1}^{\infty} b_k \cos_{\alpha} \left(k^{\alpha} \omega_0^{\alpha} t^{\alpha} \right) \quad (3.1)$$

Then the local fractional Fourier coefficients can be computed by

$$\begin{cases} a_0 = \frac{1}{T^{\alpha}} \int_0^T f(t) (dt)^{\alpha}, \\ a_k = \left(\frac{2}{T}\right)^{\alpha} \int_0^T f(t) \sin_{\alpha} \left(k^{\alpha} \omega_0^{\alpha} t^{\alpha} \right) (dt)^{\alpha}, \\ b_k = \left(\frac{2}{T}\right)^{\alpha} \int_0^T f(t) \cos_{\alpha} \left(k^{\alpha} \omega_0^{\alpha} t^{\alpha} \right) (dt)^{\alpha}. \end{cases} \quad (3.2)$$

When $\omega_0 = 1$, we get the short form

$$f(t) = a_0 + \sum_{k=1}^{\infty} a_k \sin_{\alpha} \left(k^{\alpha} t^{\alpha} \right) + \sum_{k=1}^{\infty} b_k \cos_{\alpha} \left(k^{\alpha} t^{\alpha} \right)$$

Then the local fractional Fourier coefficients can be computed by

$$\begin{cases} a_0 = \frac{1}{T^{\alpha}} \int_0^T f(t) (dt)^{\alpha}, \\ a_k = \left(\frac{2}{T}\right)^{\alpha} \int_0^T f(t) \sin_{\alpha} \left(k^{\alpha} t^{\alpha} \right) (dt)^{\alpha}, \\ b_k = \left(\frac{2}{T}\right)^{\alpha} \int_0^T f(t) \cos_{\alpha} \left(k^{\alpha} t^{\alpha} \right) (dt)^{\alpha}. \end{cases}$$

The Mittag-Leffler functions expression of local fractional Fourier series is given by [30, 31, 38, 43, 44]

$$f(x) = \sum_{k=-\infty}^{\infty} C_k E_{\alpha} \left(\frac{\pi^{\alpha} i^{\alpha} (kx)^{\alpha}}{l^{\alpha}} \right), \quad (3.3)$$

where the local fractional Fourier coefficients is

$$C_k = \frac{1}{(2l)^{\alpha}} \int_{-l}^l f(x) E_{\alpha} \left(\frac{-\pi^{\alpha} i^{\alpha} (kx)^{\alpha}}{l^{\alpha}} \right) (dx)^{\alpha} \text{ with } k \in \mathbb{Z}. \quad (3.4)$$

For local fractional Fourier series (3.4), the weights of the Mittag-Leffler functions are written in the form [43, 44]

$$C_k = \frac{\frac{1}{(2l)^{\alpha}} \int_{-l}^l f(x) E_{\alpha} \left(\frac{-\pi^{\alpha} i^{\alpha} (kx)^{\alpha}}{l^{\alpha}} \right) (dx)^{\alpha}}{\frac{1}{(2l)^{\alpha}} \int_{-l}^l E_{\alpha} \left(\frac{-\pi^{\alpha} i^{\alpha} (kx)^{\alpha}}{l^{\alpha}} \right) E_{\alpha} \left(\frac{-\pi^{\alpha} i^{\alpha} (kx)^{\alpha}}{l^{\alpha}} \right) (dx)^{\alpha}}. \quad (3.5)$$

Above is generalized to calculate local fractional Fourier series.

3.2. Properties of local fractional Fourier series

We have the following results [30, 31]:

Property 2 (Linearity)

Suppose that local fractional Fourier coefficients of $f(x)$ and $g(x)$ are f_n and g_n respectively, then we has for two constants a and b

$$af(x) + bg(x) \leftrightarrow af_n + bgf_n. \quad (3.6)$$

Property 3(Conjugation)

Suppose that C_n is Fourier coefficients of $f(x)$. Then we have

$$\overline{f(x)} \leftrightarrow \overline{C_{-n}}. \quad (3.7)$$

Property 4 (Shift in time)

Suppose that C_n is Fourier coefficients of $f(x)$. Then we have

$$f(x - x_0) \leftrightarrow E_{\alpha} \left(-i^{\alpha} (nx)^{\alpha} \right) C_n. \quad (3.8)$$

Property 5 (Time reversal)

Suppose that C_n is Fourier coefficients of $f(x)$. Then we have

$$f(-x) \leftrightarrow C_{-n}. \quad (3.9)$$

3.3. The basic theorems of local fractional Fourier series

We have the following results [30, 31]:

Theorem 6 (Local fractional Bessel inequality)

Suppose that $f(t)$ is 2π -periodic, bounded and local fractional integral on $[-\pi, \pi]$. If both a_n and b_n are Fourier coefficients of $f(t)$, then there exists the inequality

$$\frac{a_0^2}{2} + \sum_{k=1}^n (a_k^2 + b_k^2) \leq \frac{1}{\pi^{\alpha}} \int_{-\pi}^{\pi} f^2(t) (dt)^{\alpha}. \quad (3.10)$$

Theorem 7 (Local fractional Riemann -Lebesgue theorem)

Suppose that $f(x)$ is 2π -periodic, bounded and local fractional integral on $[-\pi, \pi]$. Then we have

$$\lim_{n \rightarrow +\infty} \frac{1}{(2\pi)^\alpha} \int_{-\pi}^{\pi} f(t) \sin_\alpha(nt)^\alpha (dt)^\alpha = 0 \quad (3.11)$$

and

$$\lim_{n \rightarrow +\infty} \frac{1}{(2\pi)^\alpha} \int_{-\pi}^{\pi} f(t) \cos_\alpha(nt)^\alpha (dt)^\alpha = 0. \quad (3.12)$$

Theorem 8

Suppose that $f(t)$ is 2π -periodic, bounded and local fractional integral on $[-\pi, \pi]$. Then we have

$$\lim_{n \rightarrow +\infty} \frac{1}{(2\pi)^\alpha} \int_0^\pi f(t) \sin_\alpha\left(\frac{2n+1}{2}t\right)^\alpha (dt)^\alpha = 0, \quad (3.13)$$

$$\lim_{n \rightarrow +\infty} \frac{1}{(2\pi)^\alpha} \int_0^\pi f(t) \cos_\alpha\left(\frac{2n+1}{2}t\right)^\alpha (dt)^\alpha = 0, \quad (3.14)$$

$$\lim_{n \rightarrow +\infty} \frac{1}{(2\pi)^\alpha} \int_{-\pi}^0 f(t) \sin_\alpha\left(\frac{2n+1}{2}t\right)^\alpha (dt)^\alpha = 0 \quad (3.15)$$

and

$$\lim_{n \rightarrow +\infty} \frac{1}{(2\pi)^\alpha} \int_{-\pi}^0 f(t) \cos_\alpha\left(\frac{2n+1}{2}t\right)^\alpha (dt)^\alpha = 0. \quad (3.16)$$

Theorem 9

Suppose that

$$T_{n,\alpha}(x) \sim \frac{a_0}{2^\alpha} + \sum_{n=1}^{\infty} \left(a_n \cos_\alpha(nx)^\alpha + b_n \sin_\alpha(nx)^\alpha \right),$$

then we have that

$$T_{n,\alpha}(x) = \frac{1}{\pi^\alpha} \int_{-\pi}^{\pi} T_{n,\alpha}(x+t) D_{n,\alpha}(t) (dt)^\alpha, \quad (3.17)$$

where

$$\begin{aligned} D_{n,\alpha}(t) &= \frac{1}{2} + \sum_{k=1}^n \cos_\alpha(kx)^\alpha \\ &= \frac{\sin_\alpha\left(\frac{(2n+1)x}{2}\right)^\alpha}{\sin_\alpha\left(\frac{x}{2}\right)^\alpha} \end{aligned} \quad (3.18)$$

Theorem 10

Suppose that $f(t)$ is 2π -periodic, bounded and local fractional integral on $[-\pi, \pi]$. If

$$f(t) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} \left(a_k \cos_\alpha(kt)^\alpha + b_k \sin_\alpha(kt)^\alpha \right),$$

we have

$$\frac{1}{\pi^\alpha} \int_{-\pi}^{\pi} f^2(t) (dt)^\alpha = \frac{a_0^2}{2} + \sum_{k=0}^{\infty} (a_k^2 + b_k^2). \quad (3.19)$$

Theorem 11 (Convergence theorem for local fractional Fourier series)

Suppose that $f(t)$ is 2π -periodic, bounded and local fractional integral on $[-\pi, \pi]$. The local fractional series of $f(t)$ converges to $f(t)$ at $t \in [-\pi, \pi]$, and

$$\begin{aligned} &\frac{a_0}{2} + \sum_{k=1}^{\infty} \left(a_k \cos_\alpha(kt)^\alpha + b_k \sin_\alpha(kt)^\alpha \right) \\ &= \frac{f(t+0) + f(t-0)}{2} \end{aligned} \quad (3.20)$$

where

$$a_0 = \frac{1}{\pi^\alpha} \int_{-\pi}^{\pi} f(x) (dx)^\alpha, \quad (3.21)$$

$$a_n = \frac{1}{\pi^\alpha} \int_{-\pi}^{\pi} f(x) \cos_\alpha(nx)^\alpha (dx)^\alpha \quad (3.22)$$

and

$$b_n = \frac{1}{\pi^\alpha} \int_{-\pi}^{\pi} f(x) \sin_\alpha(nx)^\alpha (dx)^\alpha. \quad (3.23)$$

3.4. Applications of local fractional Fourier series

3.4.1. Applications of local fractional Fourier series to fractal signal

Expand fractal signal $X(t) = t^\alpha (-\pi < t \leq \pi)$ in local fractional Fourier series.

Now we find the local fractional Fourier coefficients

$$\begin{aligned} a_0 &= \frac{1}{\pi^\alpha} \int_{-\pi}^{\pi} X(t) (dt)^\alpha \\ &= \frac{1}{\pi^\alpha} \int_{-\pi}^{\pi} t^\alpha (dt)^\alpha \\ &= \frac{\Gamma^2(1+\alpha) t^{2\alpha} \Big|_{-\pi}^{\pi}}{\pi^\alpha \Gamma(1+2\alpha)} \\ &= 0 \end{aligned} \quad (3.24)$$

$$\begin{aligned}
 a_n &= \frac{1}{\pi^\alpha} \int_{-\pi}^{\pi} X(t) \cos_\alpha(nt)^\alpha (dt)^\alpha \\
 &= \frac{1}{\pi^\alpha} \int_{-\pi}^{\pi} t^\alpha \cos_\alpha(nt)^\alpha (dt)^\alpha \\
 &= \frac{\Gamma(1+\alpha)t^\alpha \sin_\alpha(nt)^\alpha}{n^\alpha \pi^\alpha} \Big|_{-\pi}^{\pi} - \frac{1}{\pi^\alpha n^\alpha} \int_{-\pi}^{\pi} \sin_\alpha(nt)^\alpha (dt)^\alpha \\
 &= \frac{\Gamma(1+\alpha)t^\alpha \sin_\alpha(nt)^\alpha}{n^\alpha \pi^\alpha} \Big|_{-\pi}^{\pi} + \frac{\Gamma(1+\alpha)\cos_\alpha(nt)^\alpha}{n^{2\alpha} \pi^\alpha} \Big|_{-\pi}^{\pi} \\
 &= 0
 \end{aligned} \tag{3.25}$$

and b_n

$$\begin{aligned}
 b_n &= \frac{1}{\pi^\alpha} \int_{-\pi}^{\pi} X(t) \sin_\alpha(nt)^\alpha (dt)^\alpha \\
 &= \left[-\frac{\Gamma(1+\alpha)t^\alpha \cos_\alpha(nt)^\alpha}{n^\alpha \pi^\alpha} \right] \Big|_{-\pi}^{\pi} - \frac{1}{\pi^\alpha n^\alpha} \int_{-\pi}^{\pi} \cos_\alpha(nt)^\alpha (dt)^\alpha \\
 &= \left[-\frac{\Gamma(1+\alpha)t^\alpha \cos_\alpha(nt)^\alpha}{n^\alpha \pi^\alpha} \right] \Big|_{-\pi}^{\pi} + \frac{\Gamma(1+\alpha)\sin_\alpha(nt)^\alpha}{n^{2\alpha} \pi^\alpha} \Big|_{-\pi}^{\pi} \\
 &= \frac{2\Gamma(1+\alpha)(-1)^{n+1}}{n^\alpha \pi^\alpha}.
 \end{aligned} \tag{3.26}$$

Therefore, for $-\pi < t \leq \pi$ we have local fractional Fourier series representation of fractal signal

$$X(t) = \sum_{n=1}^{\infty} \left(\frac{2\Gamma(1+\alpha)(-1)^{n+1}}{n^\alpha \pi^\alpha} \sin_\alpha(nt)^\alpha \right). \tag{3.27}$$

3.4.2. Applications of local fractional Fourier series to local fractional partial differential equation

Local fractional partial differential equation is written in the form

$$\frac{\partial^\alpha u}{\partial t^\alpha} = k^{2\alpha} \frac{\partial^{2\alpha} u}{\partial x^{2\alpha}} \tag{3.28}$$

with boundary conditions

$$\frac{\partial^\alpha u(0,t)}{\partial x^\alpha} = 0, \quad \frac{\partial^\alpha u(L,t)}{\partial x^\alpha} = 0,$$

and

$$u(x,0) = f(x). \tag{3.29}$$

Letting $u = XT$ in (2.1) and separating the variables, we find that

$$TX^{(2\alpha)} = k^{2\alpha} XT^{(\alpha)}. \tag{3.30}$$

Setting each side equal to the constant $-\lambda^{2\alpha}$, we find

$$X^{(2\alpha)} + \lambda^{2\alpha} X = 0 \tag{3.31}$$

and

$$T^{(\alpha)} + \lambda^{2\alpha} T = 0. \tag{3.32}$$

So that

$$X = a \cos_\alpha(\lambda^\alpha x^\alpha) + b \sin_\alpha(\lambda^\alpha x^\alpha) \tag{3.33}$$

and

$$T = cE_\alpha(-\lambda^{2\alpha} k^{2\alpha} t^\alpha). \tag{3.34}$$

A solution is thus given by

$$\begin{aligned}
 u(x,t) &= E_\alpha(-\lambda^{2\alpha} k^{2\alpha} t^\alpha) (A \cos_\alpha(\lambda^\alpha x^\alpha) + B \sin_\alpha(\lambda^\alpha x^\alpha))
 \end{aligned} \tag{3.35}$$

where $A = ac, B = bc$.

From $\frac{\partial^\alpha u(0,t)}{\partial x^\alpha} = 0$, we have $B = 0$ so that

$$u(x,t) = AE_\alpha(-\lambda^{2\alpha} k^{2\alpha} t^\alpha) \cos_\alpha(\lambda^\alpha x^\alpha). \tag{3.36}$$

Then from $\frac{\partial^\alpha u(L,t)}{\partial x^\alpha} = 0$, we get

$$\sin_\alpha(\lambda^\alpha x^\alpha) = 0. \tag{3.37}$$

Thus

$$\begin{aligned}
 u(x,t) &= AE_\alpha(-\lambda^{2\alpha} k^{2\alpha} t^\alpha (m\pi/L)^{2\alpha}) \cos_\alpha((m\pi x/L)^\alpha), \\
 m &= 0, 1, 2, 3, \dots
 \end{aligned} \tag{3.38}$$

To satisfy the condition, $u(x,0) = f(x)$, we obtain

$$\begin{aligned}
 u(x,t) &= \frac{A_0}{2} + \sum_{m=1}^{\infty} A_m E_\alpha(-\lambda^{2\alpha} k^{2\alpha} t^\alpha (m\pi/L)^{2\alpha}) \cos_\alpha((m\pi x/L)^\alpha)
 \end{aligned} \tag{3.39}$$

Then from $u(x,0) = f(x)$ we see that

$$\begin{aligned}
 u(x,t) &= \frac{A_0}{2} + \sum_{m=1}^{\infty} A_m E_\alpha(-\lambda^{2\alpha} k^{2\alpha} t^\alpha (m\pi/L)^{2\alpha}) \cos_\alpha((m\pi x/L)^\alpha)
 \end{aligned} \tag{3.40}$$

Thus, from local fractional Fourier series we find

$$A_m = \left(\frac{2}{L}\right)^\alpha \int_0^L f(x) \cos_\alpha((m\pi x/L)^\alpha) (dx)^\alpha \tag{3.41}$$

and

$$\begin{aligned}
 u(x,t) &= \frac{1}{L^\alpha} \int_0^L f(x) (dx)^\alpha \\
 &+ \sum_{m=1}^{\infty} \left(\frac{2}{L}\right)^\alpha \int_0^L f(x) \cos_\alpha((m\pi x/L)^\alpha) (dx)^\alpha
 \end{aligned} \tag{3.42}$$

$$E_\alpha(-\lambda^{2\alpha} k^{2\alpha} t^\alpha (m\pi/L)^{2\alpha}) \cos_\alpha((m\pi x/L)^\alpha)$$

4. The Yang-Fourier Transform in Fractal Space

(4.6)

4.1. Notations

Let us consider the formulas (3.3) and (3.4), and set

$$C_n = \frac{\Gamma(1+\alpha)}{(2l)^\alpha} C_n^t. \text{ We have}$$

$$f(x) = \frac{1}{(2l)^\alpha} \sum_{k=-\infty}^{\infty} C_n^t E_\alpha \left(\frac{\pi^\alpha i^\alpha (nx)^\alpha}{l^\alpha} \right),$$

where its local fractional Fourier coefficients is

$$C_n^t = \frac{1}{\Gamma(1+\alpha)} \int_{-l}^l f(x) E_\alpha \left(\frac{-\pi^\alpha i^\alpha (nx)^\alpha}{l^\alpha} \right) (dx)^\alpha.$$

If we define

$$k_n^\alpha = \left(\frac{\pi n}{l} \right)^\alpha,$$

then we have

$$(\Delta k_n)^\alpha = (k_{n+1} - k_n)^\alpha = \left(\frac{\pi}{l} \right)^\alpha.$$

It is convenient to rewrite

$$\begin{aligned} f(x) &= \frac{1}{(2\pi)^\alpha} \sum_{k=-\infty}^{\infty} C_k E_\alpha (i^\alpha x^\alpha k_n^\alpha) (\Delta k_n)^\alpha \quad (4.1) \\ &= \frac{1}{(2\pi)^\alpha} \int_{-\infty}^{\infty} C_k E_\alpha (i^\alpha x^\alpha k_n^\alpha) (dk_n)^\alpha \end{aligned}$$

as $l \rightarrow \infty$ and

$$C_k = \frac{1}{\Gamma(1+\alpha)} \int_{-\infty}^{\infty} f(x) E_\alpha (i^\alpha x^\alpha k_n^\alpha) (dx)^\alpha. \quad (4.2)$$

When $k_n^\alpha = \omega^\alpha$, from (4.1) and (4.2) this leads to the following results

$$f(x) = \frac{1}{(2\pi)^\alpha} \int_{-\infty}^{\infty} C_k E_\alpha (i^\alpha x^\alpha \omega^\alpha) (d\omega)^\alpha \quad (4.3)$$

and

$$C_k = \frac{1}{\Gamma(1+\alpha)} \int_{-\infty}^{\infty} f(x) E_\alpha (i^\alpha x^\alpha \omega^\alpha) (dx)^\alpha. \quad (4.4)$$

When $\omega^\alpha = \frac{(2\pi)^\alpha}{\Gamma(1+\alpha)} \omega^{*\alpha}$, it follows from (4.1) and

(4.2) that

$$f(x) = \frac{1}{\Gamma(1+\alpha)} \int_{-\infty}^{\infty} C_k E_\alpha \left(i^\alpha x^\alpha \frac{(2\pi)^\alpha}{\Gamma(1+\alpha)} \omega^{*\alpha} \right) (d\omega^*)^\alpha \quad (4.5)$$

and

$$C_k = \frac{1}{\Gamma(1+\alpha)} \int_{-\infty}^{\infty} f(x) E_\alpha \left(-i^\alpha x^\alpha \frac{(2\pi)^\alpha}{\Gamma(1+\alpha)} \omega^{*\alpha} \right) (dx)^\alpha.$$

Definition 10 (Yang-Fourier transform in fractal space)

Suppose that $f(x) \in C_\alpha(-\infty, \infty)$, from (4.4) the

Yang-Fourier transform, denoted by $F_\alpha \{f(x)\} \equiv f_\omega^{F,\alpha}(\omega)$, is written in the form [38-43]:

$$\begin{aligned} F_\alpha \{f(x)\} &= f_\omega^{F,\alpha}(\omega) \\ &= \frac{1}{\Gamma(1+\alpha)} \int_{-\infty}^{\infty} E_\alpha(-i^\alpha \omega^\alpha x^\alpha) f(x) (dx)^\alpha \end{aligned} \quad (4.7)$$

where the latter converges.

Definition 11

If $F_\alpha \{f(x)\} \equiv f_\omega^{F,\alpha}(\omega)$, from (4.3) its inversion formula is written in the form [38-43]

$$\begin{aligned} f(x) &= F_\alpha^{-1} (f_\omega^{F,\alpha}(\omega)) \\ &= \frac{1}{(2\pi)^\alpha} \int_{-\infty}^{\infty} E_\alpha(i^\alpha \omega^\alpha x^\alpha) f_\omega^{F,\alpha}(\omega) (d\omega)^\alpha, x > 0 \end{aligned} \quad (4.8)$$

4.2. The basic theorems of Yang-Fourier transform

The following results are valid [38-43].

Theorem 12

Let $F_\alpha \{f(x)\} = f_\omega^{F,\alpha}(\omega)$, then we have

$$f(x) = F_\alpha^{-1} (f_\omega^{F,\alpha}(\omega)). \quad (4.9)$$

Theorem 13

Let $F_\alpha \{f(x)\} = f_\omega^{F,\alpha}(\omega)$ and $F_\alpha \{g(x)\} = g_\omega^{F,\alpha}(\omega)$, and let a, b be two constants. Then we have

$$F_\alpha \{af(x) + bg(x)\} = aF_\alpha \{f(x)\} + bF_\alpha \{g(x)\}. \quad (4.10)$$

Theorem 14

Let $F_\alpha \{f(x)\} = f_\omega^{F,\alpha}(\omega)$. If $\lim_{|x| \rightarrow \infty} f(x) = 0$, then we have

$$F_\alpha \{f^{(\alpha)}(x)\} = i^\alpha \omega^\alpha F_\alpha \{f(x)\}. \quad (4.11)$$

As a direct result, repeating this process, when

$$f(0) = f^{(\alpha)}(0) = \dots = f^{((k-1)\alpha)}(0) = 0$$

we have

$$F_\alpha \{f^{(k\alpha)}(x)\} = i^{k\alpha} \omega^{k\alpha} F_\alpha \{f(x)\}. \quad (4.12)$$

Theorem 15

Let $F_\alpha \{f(x)\} = f_\omega^{F,\alpha}(\omega)$ and $\lim_{x \rightarrow \infty} I_x^{(\alpha)} f(x) \rightarrow 0$, then we have

$$F_\alpha \left\{ -\infty I_x^{(\alpha)} f(x) \right\} = \frac{1}{i^\alpha \omega^\alpha} F_\alpha \{ f(x) \}. \quad (4.13)$$

Theorem 16

If $F_\alpha \{ f(x) \} = f_\omega^{F,\alpha}(\omega)$, and $a > 0$, then we have

$$F_\alpha \{ f(ax) \} = \frac{1}{a^\alpha} f_\omega^{F,\alpha} \left(\frac{\omega}{a} \right). \quad (4.14)$$

Theorem 17

If $F_\alpha \{ f(x) \} = f_\omega^{F,\alpha}(\omega)$, and c is a constant, then we have

$$F_\alpha \{ f(x-c) \} = E_\alpha \left(-i^\alpha c^\alpha \omega^\alpha \right) F_\alpha \{ f(x) \}. \quad (4.15)$$

Theorem 18

If $F_\alpha \{ f(x) \} = f_\omega^{F,\alpha}(\omega)$, and c is a constant, then we have

$$F_\alpha \left\{ f(x) E_\alpha \left(-i^\alpha x^\alpha \omega_0^\alpha \right) \right\} = f_\omega^{F,\alpha}(\omega - \omega_0). \quad (4.16)$$

Theorem 19

Let $F_\alpha \{ f_1(x) \} = f_{\omega,1}^{F,\alpha}(\omega)$ and $F_\alpha \{ f_2(x) \} = f_{\omega,2}^{F,\alpha}(\omega)$, then we have

$$F_\alpha \{ f_1(x) * f_2(x) \} = f_{\omega,1}^{F,\alpha}(\omega) f_{\omega,2}^{F,\alpha}(\omega). \quad (4.17)$$

Theorem 20

Let $F_\alpha \{ f_1(x) \} = f_{\omega,1}^{F,\alpha}(\omega)$, $F_\alpha \{ f_2(x) \} = f_{\omega,2}^{F,\alpha}(\omega)$ and let a, b be two constants, then we have

$$\begin{aligned} & F_\alpha^{-1} \left\{ a f_{\omega,1}^{F,\alpha}(\omega) + b f_{\omega,2}^{F,\alpha}(\omega) \right\} \\ &= a F_\alpha^{-1} \left\{ f_{\omega,1}^{F,\alpha}(\omega) \right\} + b F_\alpha^{-1} \left\{ f_{\omega,2}^{F,\alpha}(\omega) \right\}. \end{aligned} \quad (4.18)$$

Theorem 21

Let $F_\alpha \{ f_1(x) \} = f_{\omega,1}^{F,\alpha}(\omega)$ and $F_\alpha \{ f_2(x) \} = f_{\omega,2}^{F,\alpha}(\omega)$.

If $\lim_{|\omega| \rightarrow \infty} f_\omega^{F,\alpha}(\omega) = 0$, then we have

$$F_\alpha^{-1} \left\{ \left[f_\omega^{F,\alpha}(\omega) \right]^{(\alpha)} \right\} = i^\alpha x^\alpha f(x). \quad (4.19)$$

Theorem 22

If $F_\alpha \{ f(x) \} = f_\omega^{F,\alpha}(\omega)$, and c is a constant, then for $a > 0$ we have

$$F_\alpha^{-1} \left\{ f_\omega^{F,\alpha}(a\omega) \right\} = \frac{1}{a^\alpha} f \left(\frac{x}{a} \right). \quad (4.20)$$

Theorem 23

If $F_\alpha \{ f(x) \} = f_\omega^{F,\alpha}(\omega)$, and c is a constant, then we have

$$F_\alpha^{-1} \left\{ f_\omega^{F,\alpha}(\omega - c) \right\} = E_\alpha \left(i^\alpha c^\alpha x^\alpha \right) f(x). \quad (4.21)$$

Theorem 24

If $F_\alpha \{ f(x) \} = f_\omega^{F,\alpha}(\omega)$, and c is a constant, then we have

$$F_\alpha^{-1} \left\{ f_\omega^{F,\alpha}(\omega) E_\alpha \left(-i^\alpha \omega^\alpha c^\alpha \right) \right\} = f(x-c). \quad (4.22)$$

Theorem 25

If $F_\alpha \{ f_1(x) \} = f_{\omega,1}^{F,\alpha}(\omega)$ and $F_\alpha \{ f_2(x) \} = f_{\omega,2}^{F,\alpha}(\omega)$, then we have

$$F_\alpha^{-1} \left[f_{\omega,1}^{F,\alpha}(\theta) * f_{\omega,2}^{F,\alpha}(\omega) \right] = f_1(x) f_2(x). \quad (4.23)$$

Theorem 26

If $\lim_{|\omega| \rightarrow \infty} f_\omega^{F,\alpha}(\omega) = 0$, then we have

$$F_\alpha^{-1} \left\{ f_\omega^{F,\alpha(\alpha)}(\omega) \right\} = -i^\alpha t^\alpha f(x). \quad (4.24)$$

Theorem 27

If $F_\alpha \{ f(x) \} = f_\omega^{F,\alpha}(\omega)$, then

$$\frac{1}{\Gamma(1+\alpha)} \int_{-\infty}^{\infty} |f(x)|^2 (dx)^\alpha = \frac{1}{(2\pi)^\alpha} \int_{-\infty}^{\infty} |f_\omega^{F,\alpha}(\omega)|^2 (d\omega)^\alpha \quad (4.25)$$

Theorem 28

If $F_\alpha \{ f(x) \} = f_\omega^{F,\alpha}(\omega)$ and $F_\alpha \{ g(x) \} = g_\omega^{F,\alpha}(\omega)$, then

$$\frac{1}{\Gamma(1+\alpha)} \int_{-\infty}^{\infty} f(x) \overline{g(x)} (dx)^\alpha = \frac{1}{(2\pi)^\alpha} \int_{-\infty}^{\infty} f_\omega^{F,\alpha}(\omega) \overline{g_\omega^{F,\alpha}(\omega)} (d\omega)^\alpha. \quad (4.26)$$

4.3. Applications of local fractional Fourier transform

4.3.1. Application of local fractional Fourier transform to local fractional ODE

For the local fractional ODE problem [30, 31]

$$y^{(\alpha)}(t) + 2y(t) = E_\alpha(-t^\alpha), \quad 0 < \alpha \leq 1, \quad (4.27)$$

Initial data

$$y(t)|_{t=0} = 0.$$

Taking local fractional Fourier transform we have

$$i^\alpha \omega^\alpha y_\omega^{F,\alpha}(\omega) + 2y_\omega^{F,\alpha}(\omega) = \frac{1}{1 + i^\alpha \omega^\alpha}.$$

Therefore, we have the following identity

$$y_\omega^{F,\alpha}(\omega) = \frac{1}{1 + i^\alpha \omega^\alpha} - \frac{1}{2 + i^\alpha \omega^\alpha}.$$

The inverse local fractional Fourier transform gives

$$f(x) = E_\alpha(-t^\alpha) - E_\alpha(-2t^\alpha).$$

Therefore, we obtain the relation

$$f(x) = E_\alpha(-t^\alpha) - E_\alpha(-2t^\alpha).$$

4.3.2. Application of local fractional Fourier transform to fractal signal

Let a non-periodic signal $X(t)$ be defined by the relation

$$X(t) = \begin{cases} A, & -t_0 \leq t < t_0; \\ 0, & \text{else.} \end{cases} \quad (4.28)$$

Taking the Yang-Fourier transforms, we have

$$\begin{aligned} X_{\omega}^{F,\alpha}(\omega) &= \frac{1}{\Gamma(1+\alpha)} \int_{-\infty}^{\infty} X(t) E_{\alpha}(-i^{\alpha} \omega^{\alpha} x^{\alpha})(dx)^{\alpha} \\ &= \frac{1}{\Gamma(1+\alpha)} \int_{-t}^t A E_{\alpha}(-i^{\alpha} \omega^{\alpha} x^{\alpha})(dx)^{\alpha} \\ &= \frac{A E_{\alpha}(-i^{\alpha} \omega^{\alpha} x^{\alpha}) \Big|_{-t_0}^{t_0}}{-i^{\alpha} \omega^{\alpha}}. \end{aligned} \quad (4.29)$$

Taking into account

$$E_{\alpha}(-i^{\alpha} x^{\alpha}) = \cos_{\alpha} x^{\alpha} - i^{\alpha} \sin_{\alpha} x^{\alpha},$$

we get

$$X_{\omega}^{F,\alpha}(\omega) = \frac{2A \sin_{\alpha} \omega^{\alpha} t_0^{\alpha}}{\omega^{\alpha}} = 2A t^{\alpha} \sin_{\alpha C} \omega^{\alpha} t_0^{\alpha}. \quad (4.30)$$

where

$$\sin_{\alpha C} \omega^{\alpha} t_0^{\alpha} = \frac{\sin_{\alpha} \omega^{\alpha} t_0^{\alpha}}{\omega^{\alpha} t_0^{\alpha}}.$$

5. The Generalized Yang-Fourier Transform in Fractal Space

5.1. Notations

Definition12 (Generalized Yang-Fourier transform)

From (4.6) the generalized Yang-Fourier transform is written in the form [30, 31, 41]

$$\begin{aligned} F_{\alpha}\{f(x)\} &= f_{\omega}^{F,\alpha}(\omega) \\ &= \frac{1}{\Gamma(1+\alpha)} \int_{-\infty}^{\infty} f(x) E_{\alpha}(-i^{\alpha} h_0 x^{\alpha} \omega^{\alpha})(dx)^{\alpha} \end{aligned} \quad (5.1)$$

where $h_0 = \frac{(2\pi)^{\alpha}}{\Gamma(1+\alpha)}$ with $0 < \alpha \leq 1$.

Definition13

From (4.5) the inverse formula of the generalized Yang-Fourier transform is written in the form [30, 31, 41]

$$\begin{aligned} F_{\alpha}^{-1}\left(f_{\omega}^{F,\alpha}(\omega)\right) &= f(x) \end{aligned} \quad (5.2)$$

$$= \frac{1}{\Gamma(1+\alpha)} \int_{-\infty}^{\infty} f_{\omega}^{F,\alpha}(\omega) E_{\alpha}(i^{\alpha} h_0 x^{\alpha} \omega^{\alpha})(d\omega)^{\alpha}$$

where $h_0 = \frac{(2\pi)^{\alpha}}{\Gamma(1+\alpha)}$ with $0 < \alpha \leq 1$.

5.2. The basic theorems of Yang-Fourier transform

The following result is valid [30, 31, 41].

Theorem 29

Let $F_{\alpha}\{f(x)\} = f_{\omega}^{F,\alpha}(\omega)$, then we have

$$f(x) = F_{\alpha}^{-1}\left(f_{\omega}^{F,\alpha}(\omega)\right). \quad (5.3)$$

Theorem 30

Let $F_{\alpha}\{f(x)\} = f_{\omega}^{F,\alpha}(\omega)$ and $F_{\alpha}\{g(x)\} = g_{\omega}^{F,\alpha}(\omega)$, and let a, b be two constants. Then we have

$$F_{\alpha}\{af(x) + bg(x)\} = aF_{\alpha}\{f(x)\} + bF_{\alpha}\{g(x)\}. \quad (5.4)$$

Theorem 31

Let $F_{\alpha}\{f(x)\} = f_{\omega}^{F,\alpha}(\omega)$. If $\lim_{|x| \rightarrow \infty} f(x) = 0$, then we have

$$F_{\alpha}\{f^{(\alpha)}(x)\} = i^{\alpha} h_0 \omega^{\alpha} F_{\alpha}\{f(x)\}. \quad (5.5)$$

As a direct result, repeating this process, when

$$f(0) = f^{(\alpha)}(0) = \dots = f^{((k-1)\alpha)}(0) = 0$$

we have

$$F_{\alpha}\{f^{(k\alpha)}(x)\} = i^{k\alpha} h_0^k \omega^{k\alpha} F_{\alpha}\{f(x)\}. \quad (5.6)$$

Theorem 32

Let $F_{\alpha}\{f(x)\} = f_{\omega}^{F,\alpha}(\omega)$ and $\lim_{x \rightarrow \infty} {}_{-\infty}I_x^{(\alpha)} f(x) \rightarrow 0$, then we have

$$F_{\alpha}\left\{{}_{-\infty}I_x^{(\alpha)} f(x)\right\} = \frac{1}{i^{\alpha} h_0 \omega^{\alpha}} F_{\alpha}\{f(x)\}. \quad (5.7)$$

Theorem 33

If $F_{\alpha}\{f(x)\} = f_{\omega}^{F,\alpha}(\omega)$, and $a > 0$, then we have

$$F_{\alpha}\{f(ax)\} = \frac{1}{a^{\alpha}} f_{\omega}^{F,\alpha}\left(\frac{\omega}{a}\right). \quad (5.8)$$

Theorem 34

If $F_{\alpha}\{f(x)\} = f_{\omega}^{F,\alpha}(\omega)$, and c is a constant, then we have

$$F_\alpha \{ f(x-c) \} = E_\alpha \left(-i^\alpha h_0 c^\alpha \omega^\alpha \right) F_\alpha \{ f(x) \}. \quad (5.9)$$

Theorem 35

If $F_\alpha \{ f(x) \} = f_\omega^{F,\alpha}(\omega)$, and c is a constant, then we have

$$F_\alpha \{ f(x) E_\alpha \left(-i^\alpha h_0 x^\alpha \omega_0^\alpha \right) \} = f_\omega^{F,\alpha}(\omega - \omega_0). \quad (5.10)$$

Theorem 36

Let $F_\alpha \{ f_1(x) \} = f_{\omega,1}^{F,\alpha}(\omega)$ and $F_\alpha \{ f_2(x) \} = f_{\omega,2}^{F,\alpha}(\omega)$, then we have

$$F_\alpha \{ f_1(x) * f_2(x) \} = f_{\omega,1}^{F,\alpha}(\omega) f_{\omega,2}^{F,\alpha}(\omega). \quad (5.11)$$

Theorem 37

Let $F_\alpha \{ f_1(x) \} = f_{\omega,1}^{F,\alpha}(\omega)$, $F_\alpha \{ f_2(x) \} = f_{\omega,2}^{F,\alpha}(\omega)$ and let a, b be two constants, then we have

$$\begin{aligned} & F_\alpha^{-1} \{ a f_{\omega,1}^{F,\alpha}(\omega) + b f_{\omega,2}^{F,\alpha}(\omega) \} \\ &= a F_\alpha^{-1} \{ f_{\omega,1}^{F,\alpha}(\omega) \} + b F_\alpha^{-1} \{ f_{\omega,2}^{F,\alpha}(\omega) \}. \end{aligned} \quad (5.12)$$

Theorem 38

Let $F_\alpha \{ f_1(x) \} = f_{\omega,1}^{F,\alpha}(\omega)$ and $F_\alpha \{ f_2(x) \} = f_{\omega,2}^{F,\alpha}(\omega)$.

If $\lim_{|\omega| \rightarrow \infty} f_\omega^{F,\alpha}(\omega) = 0$, then we have

$$F_\alpha^{-1} \left[\left[f_\omega^{F,\alpha}(\omega) \right]^{(\alpha)} \right] = i^\alpha h_0 x^\alpha f(x). \quad (5.13)$$

Theorem 39

If $F_\alpha \{ f(x) \} = f_\omega^{F,\alpha}(\omega)$, and c is a constant, then for $a > 0$ we have

$$F_\alpha^{-1} \{ f_\omega^{F,\alpha}(a\omega) \} = \frac{1}{a^\alpha} f\left(\frac{x}{a}\right). \quad (5.14)$$

Theorem 40

If $F_\alpha \{ f(x) \} = f_\omega^{F,\alpha}(\omega)$, and c is a constant, then we have

$$F_\alpha^{-1} \{ f_\omega^{F,\alpha}(\omega - c) \} = E_\alpha \left(i^\alpha h_0 c^\alpha x^\alpha \right) f(x). \quad (5.15)$$

Theorem 41

If $F_\alpha \{ f(x) \} = f_\omega^{F,\alpha}(\omega)$, and c is a constant, then we have

$$F_\alpha^{-1} \{ f_\omega^{F,\alpha}(\omega) E_\alpha \left(-i^\alpha h_0 \omega^\alpha c^\alpha \right) \} = f(x-c). \quad (5.16)$$

Theorem 42

If $F_\alpha \{ f_1(x) \} = f_{\omega,1}^{F,\alpha}(\omega)$ and $F_\alpha \{ f_2(x) \} = f_{\omega,2}^{F,\alpha}(\omega)$, then we have

$$F_\alpha^{-1} \left[f_{\omega,1}^{F,\alpha}(\theta) * f_{\omega,2}^{F,\alpha}(\omega) \right] = f_1(x) f_2(x). \quad (5.17)$$

Theorem 43

If $\lim_{|\omega| \rightarrow \infty} f_\omega^{F,\alpha}(\omega) = 0$, then we have

$$F_\alpha^{-1} \{ f_\omega^{F,\alpha(\alpha)}(\omega) \} = -i^\alpha h_0 t^\alpha f(x). \quad (5.18)$$

Theorem 44

If $F_\alpha \{ f(x) \} = f_\omega^{F,\alpha}(\omega)$, then we have

$$\int_{-\infty}^{\infty} |f(x)|^2 (dx)^\alpha = \int_{-\infty}^{\infty} |f_\omega^{F,\alpha}(\omega)|^2 (d\omega)^\alpha. \quad (5.19)$$

Theorem 45

If $F_\alpha \{ f(x) \} = f_\omega^{F,\alpha}(\omega)$ and $F_\alpha \{ g(x) \} = g_\omega^{F,\alpha}(\omega)$, then we have

$$\int_{-\infty}^{\infty} f(x) \overline{g(x)} (dx)^\alpha = \int_{-\infty}^{\infty} f_\omega^{F,\alpha}(\omega) \overline{g_\omega^{F,\alpha}(\omega)} (d\omega)^\alpha. \quad (5.20)$$

6. Discrete Yang-Fourier Transform in Fractal Space

6.1. Notations

Now we determine from our data,

$$\begin{aligned} & \frac{1}{\Gamma(1+\alpha)} \int_{-\frac{1}{2}\Delta t}^{\frac{2N-1}{2}\Delta t} \tilde{f}(t) \phi(t) (dt)^\alpha \\ & \approx \frac{1}{\Gamma(1+\alpha)} \int_{-\frac{1}{2}\Delta t}^{\frac{2N-1}{2}\Delta t} f(t) \phi(t) (dt)^\alpha \end{aligned} \quad (6.1)$$

for any local fractional continuous function on the natural widow. This sampling can be used to complete a corresponding sum approximation for the integration,

$$\begin{aligned} & \frac{1}{\Gamma(1+\alpha)} \int_{-\frac{1}{2}\Delta t}^{\frac{2N-1}{2}\Delta t} f(t) \phi(t) (dt)^\alpha \\ & \approx \frac{1}{\Gamma(1+\alpha)} \sum_{k=0}^{N-1} f(k\Delta t) \phi(k\Delta t) (\Delta t)^\alpha \\ & = \frac{1}{\Gamma(1+\alpha)} \sum_{k=0}^{N-1} f_k \phi(k\Delta t) (\Delta t)^\alpha. \end{aligned} \quad (6.2)$$

Notice, however, that

$$\begin{aligned} & \frac{1}{\Gamma(1+\alpha)} \sum_{k=0}^{N-1} f_k \phi(k\Delta t) (\Delta t)^\alpha \\ & = \frac{1}{\Gamma(1+\alpha)} \sum_{k=0}^{N-1} f_k \left[\frac{1}{\Gamma(1+\alpha)} \int_{-\frac{1}{2}\Delta t}^{\frac{2N-1}{2}\Delta t} \phi(t) \delta_{k\Delta t}(t) (dt)^\alpha \right] (\Delta t)^\alpha \\ & = \frac{1}{\Gamma(1+\alpha)} \int_{-\frac{1}{2}\Delta t}^{\frac{2N-1}{2}\Delta t} \left[\frac{1}{\Gamma(1+\alpha)} \sum_{k=0}^{N-1} f_k \delta_{k\Delta t}(t) (\Delta t)^\alpha \right] \phi(t) (dt)^\alpha \end{aligned} \quad (6.3)$$

where

$$\frac{1}{\Gamma(1+\alpha)} \int_{-\frac{1}{2}\Delta t}^{\frac{2N-1}{2}\Delta t} \phi(t) \delta_{k\Delta t}(t) (dt)^\alpha = \phi(k\Delta t), \tag{6.4}$$

for $k = 0, 1, \dots, N-1$.

So,

$$\begin{aligned} & \frac{1}{\Gamma(1+\alpha)} \int_{-\frac{1}{2}\Delta t}^{\frac{2N-1}{2}\Delta t} f(t) \phi(t) (dt)^\alpha \\ &= \frac{1}{\Gamma(1+\alpha)} \int_{-\frac{1}{2}\Delta t}^{\frac{2N-1}{2}\Delta t} \left[\frac{1}{\Gamma(1+\alpha)} \sum_{k=0}^{N-1} f_k \delta_{k\Delta t}(t) (\Delta t)^\alpha \right] \phi(t) (dt)^\alpha \end{aligned} \tag{6.5}$$

Suggest that, with the natural window, we use

$$\tilde{f}(t) = \frac{1}{\Gamma(1+\alpha)} \sum_{k=0}^{N-1} \tilde{f}_k \delta_{k\Delta t}(t), \tag{6.6}$$

where $\tilde{f}_k = f_k (\Delta t)^\alpha$ for $k = 0, 1, \dots, N-1$.

Now there are two natural choices: Either \tilde{f} define to be 0 outside the nature window, or define \tilde{f} to be periodic with period T equalling the length of the natural window,

$$T = N\Delta t. \tag{6.7}$$

Combing with our definition of \tilde{f} on the natural window, the first choice would be give

$$\tilde{f}(t) = \frac{1}{\Gamma(1+\alpha)} \sum_{k=0}^{N-1} \tilde{f}_k \delta_{k\Delta t}(t), \tag{6.8}$$

while the second choice would be give

$$\tilde{f}(t) = \frac{1}{\Gamma(1+\alpha)} \sum_{k=-\infty}^{\infty} \tilde{f}_k \delta_{k\Delta t}(t) \tag{6.9}$$

with $\tilde{f}_{k+N} = \tilde{f}_k$.

Clearly, the latter is the more clear choice. That is to say, suppose that $\{f_0, f_1, \dots, f_{N-1}\}$ is the N_{th} order regular sampling with spacing Δt of some function f . The corresponding discrete approximation of f is the periodic, regular array

$$\tilde{f}(t) = \frac{1}{\Gamma(1+\alpha)} \sum_{k=-\infty}^{\infty} \tilde{f}_k \delta_{k\Delta t}(t) \tag{6.10}$$

with spacing Δt index period N , and its coefficients

$$\tilde{f}_k = \begin{cases} f_k (\Delta x)^\alpha, & \text{if } k = 0, 1, \dots, N-1. \\ f_{k+N}, & \text{in general.} \end{cases} \tag{6.11}$$

From the Yang-Fourier transform theory, we then know

$$F_\alpha \{f(x)\} = f_\omega^{F,\alpha}(\omega)$$

is a local fractional continuous, given by

$$\begin{aligned} & f_\omega^{F,\alpha}(\omega) \\ &= \frac{1}{\Gamma(1+\alpha)} \int_{-\infty}^{\infty} f(t) E_\alpha(-i^\alpha \omega^\alpha t^\alpha) (dt)^\alpha \\ &= \frac{1}{\Gamma(1+\alpha)} \int_{-\frac{1}{2}\Delta t}^{\frac{2N-1}{2}\Delta t} f(t) E_\alpha(-i^\alpha \omega^\alpha t^\alpha) (dt)^\alpha \\ &\approx \frac{1}{\Gamma(1+\alpha)} \int_{-\frac{1}{2}\Delta t}^{\frac{2N-1}{2}\Delta t} \tilde{f}(t) E_\alpha(-i^\alpha \omega^\alpha t^\alpha) (dt)^\alpha \\ &= \frac{1}{\Gamma(1+\alpha)} \int_{-\frac{1}{2}\Delta t}^{\frac{2N-1}{2}\Delta t} \left(\frac{1}{\Gamma(1+\alpha)} \sum_{k=0}^{N-1} f_k \delta_{k\Delta t}(t) (\Delta t)^\alpha \right) \\ & E_\alpha(-i^\alpha \omega^\alpha t^\alpha) (dt)^\alpha \\ &= \frac{1}{\Gamma(1+\alpha)} \sum_{k=0}^{N-1} f_k (\Delta t)^\alpha \\ & \left(\frac{1}{\Gamma(1+\alpha)} \int_{-\frac{1}{2}\Delta t}^{\frac{2N-1}{2}\Delta t} \delta_{k\Delta t}(t) E_\alpha(-i^\alpha \omega^\alpha t^\alpha) (dt)^\alpha \right) \\ &= \frac{1}{\Gamma(1+\alpha)} \sum_{k=0}^{N-1} f_k (\Delta t)^\alpha E_\alpha(-i^\alpha \omega^\alpha k^\alpha (\Delta t)^\alpha) \end{aligned} \tag{6.12}$$

So, approximation of the formula

$$\frac{1}{\Gamma(1+\alpha)} \int_{-\infty}^{\infty} f(t) E_\alpha(-i^\alpha \omega^\alpha t^\alpha) (dt)^\alpha$$

reduces to

$$f_\omega^{F,\alpha}(\omega) \approx \frac{1}{\Gamma(1+\alpha)} \sum_{k=0}^{N-1} f_k (\Delta t)^\alpha E_\alpha(-i^\alpha \omega^\alpha k^\alpha (\Delta t)^\alpha) \tag{6.13}$$

with $T = N\Delta t$.

Taking $\omega = n\Delta\omega$ and $\frac{2\pi}{T} = \Delta\omega$ in (6.13) implies that

$$\begin{aligned} & \phi(n) \\ &= f_\omega^{F,\alpha}(\omega) \\ &\approx \frac{1}{\Gamma(1+\alpha)} \sum_{k=0}^{N-1} f_k (\Delta t)^\alpha E_\alpha(-i^\alpha \omega^\alpha k^\alpha (\Delta t)^\alpha) \\ &= \frac{1}{\Gamma(1+\alpha)} \frac{T^\alpha}{N^\alpha} \sum_{k=0}^{N-1} f_k E_\alpha(-i^\alpha (2\pi)^\alpha n^\alpha k^\alpha / N^\alpha) \\ &= \frac{1}{\Gamma(1+\alpha)} \frac{T^\alpha}{N^\alpha} \sum_{k=0}^{N-1} \phi(k) E_\alpha(-i^\alpha (2\pi)^\alpha n^\alpha k^\alpha / N^\alpha). \end{aligned} \tag{6.14}$$

In the same manner, if

$$f(t) = \frac{1}{(2\pi)^\alpha} \int_{-\infty}^{\infty} E_\alpha(i^\alpha \omega^\alpha t^\alpha) f_\omega^{F,\alpha}(\omega) (d\omega)^\alpha,$$

then we can write

$$\begin{aligned}
 & f_k(k\Delta t) \\
 & \approx \frac{1}{(2\pi)^\alpha} \sum_{n=0}^{N-1} f_\omega^{F,\alpha}(n\Delta\omega)(\Delta\omega)^\alpha E_\alpha\left(i^\alpha t^\alpha n^\alpha (\Delta\omega)^\alpha\right)
 \end{aligned} \tag{6.15}$$

with $\omega = N\Delta\omega$.

Taking $t = k\Delta t$ and $\frac{2\pi}{T} = \Delta\omega$ in (6.15) implies that

$$\begin{aligned}
 & \varphi(k) \\
 & = f_k(k\Delta t) \\
 & \approx \frac{1}{(2\pi)^\alpha} \sum_{n=0}^{N-1} f_\omega^{F,\alpha}(n\Delta\omega)(\Delta\omega)^\alpha E_\alpha\left(i^\alpha n^\alpha (\Delta t)^\alpha k^\alpha (\Delta\omega)^\alpha\right) \\
 & \approx \frac{1}{(2\pi)^\alpha} \sum_{n=0}^{N-1} f_\omega^{F,\alpha}(n\Delta\omega)(\Delta\omega)^\alpha E_\alpha\left(i^\alpha n^\alpha (\Delta t)^\alpha k^\alpha (\Delta\omega)^\alpha\right) \\
 & = \frac{1}{T^\alpha} \sum_{n=0}^{N-1} \phi(n) E_\alpha\left(i^\alpha n^\alpha k^\alpha (2\pi)^\alpha / N^\alpha\right).
 \end{aligned} \tag{6.16}$$

Combing the formulas (6.14) and (6.16), we have the following results:

$$\phi(n) = \frac{1}{\Gamma(1+\alpha)} \frac{T^\alpha}{N^\alpha} \sum_{k=0}^{N-1} \varphi(k) E_\alpha\left(-i^\alpha (2\pi)^\alpha n^\alpha k^\alpha / N^\alpha\right) \tag{6.17}$$

and

$$\varphi(k) = \frac{1}{T^\alpha} \sum_{n=0}^{N-1} \phi(n) E_\alpha\left(i^\alpha n^\alpha k^\alpha (2\pi)^\alpha / N^\alpha\right). \tag{6.18}$$

Setting $F(n) = \frac{1}{T^\alpha} \phi(n)$ and interchanging k and n ,

$$\varphi(n) = \sum_{k=0}^{N-1} F(k) E_\alpha\left(i^\alpha n^\alpha k^\alpha (2\pi)^\alpha / N^\alpha\right) \tag{6.19}$$

and

$$F(k) = \frac{1}{\Gamma(1+\alpha)} \frac{1}{N^\alpha} \sum_{n=0}^{N-1} \varphi(n) E_\alpha\left(-i^\alpha (2\pi)^\alpha n^\alpha k^\alpha / N^\alpha\right). \tag{6.20}$$

Definition 14 (Discrete Yang-Fourier transform)

Suppose that $f(n)$ be a periodic discrete-time fractal signal with period N . The N -point discrete Yang-Fourier transform (DYFT) of $F(n)$ is written in the form [47]

$$\begin{aligned}
 & F(k) \\
 & = \sum_{n=0}^{N-1} f(n) E_\alpha\left(-i^\alpha (2\pi)^\alpha n^\alpha k^\alpha / N^\alpha\right), \\
 & = \sum_{n=0}^{N-1} f(n) W_{N,\alpha}^{-nk}
 \end{aligned} \tag{6.21}$$

with $W_{N,\alpha}^{-nk} = E_\alpha\left(-\frac{i^\alpha n^\alpha k^\alpha (2\pi)^\alpha}{N^\alpha}\right)$. This is called

N -point discrete Yang-Fourier transform of $F(n)$, denoted by

$$f(n) \leftrightarrow F(k). \tag{6.22}$$

Definition 15 (Inverse discrete Yang-Fourier transform)

The inverse discrete Yang-Fourier transform (IDYFT) is given by is rewritten as [47]

$$\begin{aligned}
 & f(n) \\
 & = \frac{1}{\Gamma(1+\alpha)} \frac{1}{N^\alpha} \sum_{k=0}^{N-1} F(k) E_\alpha\left(i^\alpha n^\alpha k^\alpha (2\pi)^\alpha / N^\alpha\right). \\
 & = \frac{1}{\Gamma(1+\alpha)} \frac{1}{N^\alpha} \sum_{k=0}^{N-1} F(k) W_{N,\alpha}^{kn}
 \end{aligned} \tag{6.23}$$

with $W_{N,\alpha}^{kn} = E_\alpha\left(\frac{i^\alpha n^\alpha k^\alpha (2\pi)^\alpha}{N^\alpha}\right)$.

Taking into account the relation [47]

$$E_\alpha\left(i^\alpha (2\pi)^\alpha (n+1)^\alpha\right) = E_\alpha\left(i^\alpha (2\pi)^\alpha n^\alpha\right),$$

we deduce that

$$E_\alpha\left(i^\alpha (2\pi)^\alpha n^\alpha \left(\frac{k+N}{N}\right)^\alpha\right) = E_\alpha\left(i^\alpha (2\pi)^\alpha \frac{n^\alpha k^\alpha}{N^\alpha}\right) \tag{6.24}$$

for all $n \in \mathbb{Z}$. That is to say,

$$W_{1,\alpha}^{(n+N)} = W_{1,\alpha}^n$$

and

$$W_{N,\alpha}^{(k+N)n} = W_{N,\alpha}^{kn}.$$

6.2. The basic theorems of discrete Yang-Fourier transform

The following results are valid [46, 47]:

Theorem 46

Suppose that $F(k) = \sum_{n=0}^{N-1} f(n) W_{N,\alpha}^{-nk}$, then we have

$$f(n) = \frac{1}{\Gamma(1+\alpha)} \frac{1}{N^\alpha} \sum_{k=0}^{N-1} F(k) W_{N,\alpha}^{nk}. \tag{6.25}$$

Theorem 47

Suppose that $f(n)$ be periodic discrete time signals with period N , then we have [6]

$$\sum_{n=0}^{N-1} f(n) = \sum_{n=j}^{j+N-1} f(n). \tag{6.26}$$

Theorem 48

Suppose that $f_1(n) \leftrightarrow F_1(k)$ and $f_2(n) \leftrightarrow F_2(k)$, then we have [6]

$$af_1(n) + bf_2(n) \leftrightarrow aF_1(k) + bF_2(k). \quad (6.27)$$

Corollary 49

$$F(n) \leftrightarrow N^\alpha \Gamma(1+\alpha) f(-k). \quad (6.28)$$

Corollary 50 (Time reversal rule for DYFT)

$$f(-n) \leftrightarrow F(-k). \quad (6.29)$$

Corollary 51 (Conjugation rule for DYFT)

$$\overline{f(n)} \leftrightarrow \overline{F(-k)}. \quad (6.30)$$

Corollary 52 (Shift in the n -domain rule for DYFT)

$$f(n-l) \leftrightarrow E_\alpha \left(-i^\alpha (2\pi)^\alpha k^\alpha / N^\alpha \right) F(k). \quad (6.31)$$

Corollary 53 (Shift in the k -domain rule for DYFT)

$$E_\alpha \left(i^\alpha (2\pi)^\alpha k^\alpha / N^\alpha \right) f(n) \leftrightarrow F(k-l). \quad (6.32)$$

Definition 16 (Cyclical convolution)

The cyclical convolution product of two periodic discrete time signals $f(n)$ and $g(n)$ with periodic N is the fractal discrete time signal $(f * g)(n)$ defined by

$$(f * g)(n) = \sum_{l=0}^{N-1} f(l) g(n-l). \quad (6.33)$$

Theorem 54 (Convolution in the n -domain rule for DYFT)

Let $f(n)$ and $g(n)$ be periodic discrete time signals with period N . Suppose that $f(n) \leftrightarrow F(k)$ and $g(n) \leftrightarrow G(k)$, then

$$(f * g)(n) \leftrightarrow F(k)G(k). \quad (6.34)$$

Theorem 55 (Convolution in the k -domain rule for DYFT)

Let $f(n)$ and $g(n)$ be periodic discrete time signals with period N . Suppose that $f(n) \leftrightarrow F(k)$ and $g(n) \leftrightarrow G(k)$, then

$$\frac{1}{\Gamma(1+\alpha)} \frac{1}{N^\alpha} f(n) g(n) \leftrightarrow (F * G)(k). \quad (6.35)$$

Theorem 56 (Parseval theorem for DYFT)

Let $f(n)$ and $g(n)$ be periodic discrete time signals with period N . Suppose that $f(n) \leftrightarrow F(k)$ and $g(n) \leftrightarrow G(k)$, then

$$\sum_{n=0}^{N-1} f(n) \overline{g(n)} = \frac{1}{\Gamma(1+\alpha)} \frac{1}{N^\alpha} \sum_{k=0}^{N-1} F(k) \overline{G(k)}. \quad (6.36)$$

Corollary 57

Let $f(n)$ and $g(n)$ be periodic discrete time signals with period N . Suppose that $f(n) \leftrightarrow F(k)$, then

$$\sum_{n=0}^{N-1} |g(n)|^2 = \frac{1}{\Gamma(1+\alpha)} \frac{1}{N^\alpha} \sum_{k=0}^{N-1} |G(k)|^2. \quad (6.37)$$

Theorem 58

Let $f(n)$ and $g(n)$ be periodic discrete time signals with period N . Suppose that $f(n) \leftrightarrow F(k)$ and $g(n) \leftrightarrow G(k)$, then

$$\sum_{n=0}^{N-1} f(n) G(n) = \sum_{k=0}^{N-1} F(k) g(k). \quad (6.38)$$

7. Fast Yang-Fourier Transform

7.1. Fast Yang-Fourier transform of discrete Yang-Fourier transform

The relations

$$\begin{aligned} & [F_N]_{-n,k+1}^\alpha \\ &= \frac{1}{N^\alpha} W_{N,\alpha}^{-(k+1)n} \\ &= \frac{1}{N^\alpha} W_{N,\alpha}^{-kn} W_{N,\alpha}^{-n} \\ &= [F_N]_{-n,k}^\alpha W_{N,\alpha}^{-n} \end{aligned} \quad (7.1)$$

and

$$\begin{aligned} & [F_N]_{n,k+1}^\alpha \\ &= \frac{1}{N^\alpha} W_{N,\alpha}^{(k+1)n} \\ &= \frac{1}{N^\alpha} W_{N,\alpha}^{kn} W_{N,\alpha}^n \\ &= [F_N]_{n,k}^\alpha W_{N,\alpha}^n \end{aligned} \quad (7.2)$$

are the component formulas for the Yang-Fourier transform.

Suppose that $\{V_0, V_1, V_2, \dots, V_{N-1}\}$ is the N_{th} order discrete Yang-Fourier transforms of $\{v_0, v_1, v_2, \dots, v_{N-1}\}$. Starting with the component formulas for the discrete Yang-Fourier transform, we obtain that, for $n = 0, 1, 2, \dots, N-1$,

$$\begin{aligned}
 V_n &= \sum_{k=0}^{N-1} W_{N,\alpha}^{-(k+1)n} v_k \\
 &= \sum_{\substack{k=0 \\ k\text{-even}}}^{N-1} W_{N,\alpha}^{-(k+1)n} v_k + \sum_{\substack{k=0 \\ k\text{-odd}}}^{N-1} W_{N,\alpha}^{-(k+1)n} v_k \\
 &= \frac{1}{2^\alpha} \left(\sum_{j=0}^{M-1} W_{2M,\alpha}^{-n(2j)} v_{2j} + \sum_{j=0}^{M-1} W_{2M,\alpha}^{-n(2j+1)} v_{2j+1} \right) \\
 &= \frac{1}{2^\alpha} \left(\sum_{j=0}^{M-1} W_{2M,\alpha}^{-n(2j)} v_{2j} + W_{M,\alpha}^{\frac{n}{2}} \sum_{j=0}^{M-1} W_{2M,\alpha}^{-n(2j)} v_{2j+1} \right).
 \end{aligned} \tag{7.3}$$

and we have the following relation

$$[F_{NV}]_n^\alpha = \frac{1}{2^\alpha} \left([F_{NV_E}]_n^\alpha + W_{M,\alpha}^{\frac{n}{2}} [F_{NV_O}]_n^\alpha \right), \tag{7.4}$$

where V is the sequence vector corresponding to $\{V_0, V_1, V_2, \dots, V_{N-1}\}$, V_E is the M -th order sequence of even-index v_k 's $\{V_0, V_2, \dots, V_{N-2}\}$ and V_O is the M -th order sequence of odd-index v_k 's $\{V_1, V_3, \dots, V_{N-1}\}$.

Here we can deduce that

$$\begin{aligned}
 W_{M,\alpha}^{-(M+l)} &= E_\alpha \left(-i^\alpha \left(\frac{2\pi}{M} \right)^\alpha (M+l)^\alpha \right) \\
 &= E_\alpha \left(-i^\alpha \left(\frac{2\pi}{M} \right)^\alpha l^\alpha \right) \\
 &= W_{M,\alpha}^{-l}
 \end{aligned} \tag{7.5}$$

and

$$\begin{aligned}
 W_{M,\alpha}^{\frac{M+l}{2}} &= E_\alpha \left(-i^\alpha \left(\frac{\pi}{M} \right)^\alpha (M+l)^\alpha \right) \\
 &= -E_\alpha \left(-i^\alpha \left(\frac{\pi}{M} \right)^\alpha l^\alpha \right) \\
 &= W_{M,\alpha}^{-\frac{l}{2}}
 \end{aligned} \tag{7.6}$$

Hence for $l = 0, 1, 2, \dots, m-1$, we have [48]

$$\begin{aligned}
 V_l &= \frac{1}{2^\alpha} \left(\sum_{j=0}^{M-1} W_{M,\alpha}^{-lj} v_{2j} + W_{M,\alpha}^{-\left(\frac{l}{2}\right)j} \sum_{j=0}^{M-1} W_{M,\alpha}^{-lj} v_{2j+1} \right) \\
 &= \frac{1}{2^\alpha} \left([F_{MV_E^{-1}}]_l^\alpha + W_{M,\alpha}^{-\left(\frac{l}{2}\right)j} [F_{MV_O^{-1}}]_l^\alpha \right)
 \end{aligned} \tag{7.7}$$

and

$$\begin{aligned}
 V_{M+l} &= \frac{1}{2^\alpha} \left(\sum_{j=0}^{M-1} W_{M,\alpha}^{-lj} v_{2j} - W_{M,\alpha}^{-\left(\frac{l}{2}\right)j} \sum_{j=0}^{M-1} W_{M,\alpha}^{-lj} v_{2j+1} \right) \\
 &= \frac{1}{2^\alpha} \left([F_{MV_E^{-1}}]_l^\alpha - W_{M,\alpha}^{-\left(\frac{l}{2}\right)j} [F_{MV_O^{-1}}]_l^\alpha \right)
 \end{aligned} \tag{7.8}$$

Here, formulas (7.7) and (7.8) contain common elements that can be computed once for each l and then used to compute both V_l and V_{M+l} . Hence we can obtain the total number of computations to find all the V_n 's. That is to say, this process of increasing levels to our algorithm can be continued to the K th level provided to $N = 2^K N_0$ for some integer N_0 . Moreover, that integer, $N_0 = 2^{-K} N$ will also be the order of the discrete Yang-Fourier transforms and inverse discrete Yang-Fourier transforms. If $N = 2^K$, it is this final K th level algorithm, fully implemented and refined, that is called a fast Yang-Fourier transform of the discrete Yang-Fourier transforms.

7.2. Fast Yang-Fourier transform of inverse discrete Yang-Fourier transform

Suppose that $\{V_0^{-1}, V_1^{-1}, \dots, V_{N-1}^{-1}\}$ is the N th order discrete Yang-Fourier transforms of $\{v_0^{-1}, v_1^{-1}, \dots, v_{N-1}^{-1}\}$, starting with the component formulas for the inverse discrete Yang-Fourier transform, we obtain that, for $n = 0, 1, 2, \dots, N-1$,

$$\begin{aligned}
 V_n^{-1} &= \frac{1}{\Gamma(1+\alpha)} \frac{1}{N^\alpha} \sum_{k=0}^{N-1} W_{N,\alpha}^{(k+1)n} v_k^{-1} \\
 &= \frac{1}{\Gamma(1+\alpha)} \frac{1}{N^\alpha} \left(\sum_{\substack{k=0 \\ k\text{-even}}}^{N-1} W_{N,\alpha}^{(k+1)n} v_k^{-1} + \sum_{\substack{k=0 \\ k\text{-odd}}}^{N-1} W_{N,\alpha}^{(k+1)n} v_k^{-1} \right) \\
 &= \frac{1}{\Gamma(1+\alpha)} \frac{1}{(2M)^\alpha} \left(\sum_{j=0}^{M-1} W_{2M,\alpha}^{n(2j)} v_{2j}^{-1} + \sum_{j=0}^{M-1} W_{2M,\alpha}^{n(2j+1)} v_{2j+1}^{-1} \right) \\
 &= \frac{1}{\Gamma(1+\alpha)} \frac{1}{(2M)^\alpha} \left(\sum_{j=0}^{M-1} W_{2M,\alpha}^{n(2j)} v_{2j}^{-1} + W_{M,\alpha}^{\frac{n}{2}} \sum_{j=0}^{M-1} W_{2M,\alpha}^{n(2j)} v_{2j+1}^{-1} \right).
 \end{aligned} \tag{7.9}$$

and we have the following relation

$$[F_{NV}^{-1}]_n^\alpha = \frac{1}{\Gamma(1+\alpha)} \frac{1}{(2M)^\alpha} \left([F_{NV_E^{-1}}]_n^\alpha + W_{M,\alpha}^{\frac{n}{2}} [F_{NV_O^{-1}}]_n^\alpha \right), \tag{7.10}$$

where V^{-1} is the sequence vector corresponding to $\{V_0^{-1}, V_1^{-1}, V_2^{-1}, \dots, V_{N-1}^{-1}\}$, V_E^{-1} is the M -th order

sequence of even-index $V_k^{-1}, S \{V_0^{-1}, V_2^{-1}, \dots, V_{N-2}^{-1}\}$ and V_o^{-1} is the $M - th$ order sequence of odd-index $V_k^{-1}, S \{V_1^{-1}, V_3^{-1}, \dots, V_{N-1}^{-1}\}$.

Here we can deduce that

$$\begin{aligned} W_{M,\alpha}^{M+l} &= E_\alpha \left(i^\alpha \left(\frac{2\pi}{M} \right)^\alpha (M+l)^\alpha \right) \\ &= E_\alpha \left(i^\alpha \left(\frac{2\pi}{M} \right)^\alpha l^\alpha \right) \\ &= W_{M,\alpha}^l \end{aligned} \tag{7.11}$$

and

$$\begin{aligned} W_{M,\alpha}^{\frac{M+l}{2}} &= E_\alpha \left(i^\alpha \left(\frac{\pi}{M} \right)^\alpha (M+l)^\alpha \right) \\ &= -E_\alpha \left(i^\alpha \left(\frac{\pi}{M} \right)^\alpha l^\alpha \right) \\ &= W_{M,\alpha}^{\frac{l}{2}} \end{aligned} \tag{7.12}$$

Hence for $l = 0, 1, 2, \dots, m-1$, we have [48]

$$\begin{aligned} V_l^{-1} &= \frac{1}{\Gamma(1+\alpha)} \frac{1}{(2M)^\alpha} \left(\sum_{j=0}^{M-1} W_{M,\alpha}^{lj} V_{2j} + W_{M,\alpha}^{\left(\frac{l}{2}\right)^j} \sum_{j=0}^{M-1} W_{M,\alpha}^{lj} V_{2j+1} \right) \\ &= \frac{1}{\Gamma(1+\alpha)} \frac{1}{2^\alpha} \left([F_{M^l}^{\alpha}] + W_{M,\alpha}^{\left(\frac{l}{2}\right)^j} [F_{M^0}^{\alpha}] \right) \end{aligned} \tag{7.13}$$

and

$$\begin{aligned} V_{M+l}^{-1} &= \frac{1}{\Gamma(1+\alpha)} \frac{1}{(2M)^\alpha} \left(\sum_{j=0}^{M-1} W_{M,\alpha}^{lj} V_{2j} - W_{M,\alpha}^{\left(\frac{l}{2}\right)^j} \sum_{j=0}^{M-1} W_{M,\alpha}^{lj} V_{2j+1} \right) \\ &= \frac{1}{\Gamma(1+\alpha)} \frac{1}{2^\alpha} \left([F_{M^l}^{\alpha}] - W_{M,\alpha}^{\left(\frac{l}{2}\right)^j} [F_{M^0}^{\alpha}] \right) \end{aligned} \tag{7.14}$$

It is shown that, formulas (7.13) and (7.14) contain common elements that can also be computed once for each l and then used to compute both V_l^{-1} and V_{M+l}^{-1} . These can also yield the total number of computations to find all the V_n^{-1}, S . That is to say, this process of increasing levels to our algorithm of inverse discrete Yang-Fourier transforms is similar to that of the discrete Yang-Fourier transforms. Taking into account the relation $N = 2^K$, it is also this final K^{th} level algorithm, fully

implemented and refined, that is called a fast Yang-Fourier transform of the inverse discrete Yang-Fourier transforms.

8. Conclusions

In the paper we investigate the theory of local fractional Fourier analysis, the local fractional Fourier series, the Yang-Fourier transform, the generalized Yang-Fourier transform, the discrete Yang-Fourier transform and fast Yang-Fourier transform, and some applications of local fractional Fourier analysis. Our attention is devoted to the analytical technique of the local fractional Fourier analysis for treating with fractal problems in a way accessible to applied scientists and engineers.

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