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On New Unified Integrals involving Appell Series

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Abstract – In the present paper we derive four new finite integrals involving the products of the $\overline{H}$ -function, Appell series in reduction form and Srivastava Polynomials with essentially arbitrary coefficients. The values of integral are obtained in terms of $\psi(z)$ (The logarithmic derivative of $\Gamma(z)$). By assigning suitably special values to these coefficients, the main results can be reduced to the corresponding integral formulas involving the classical orthogonal polynomials including, for example, Hermite, Jacobi, Legendre and Laguerre polynomials. Furthermore, the $\overline{H}$ -function occurring in each of our main results can be reduced, under various special cases, to such simpler functions as the generalized Wright hypergeometric function and generalized Wright- Bessal function. A specimen of some of these interesting applications of our main integral formulas is presented briefly. For the sake of illustration we record here some special cases of our main results which are also new. The integrals establish here are basic in nature and are likely to find useful applications in the field of science and engineering.

Keywords – $\overline{H}$ -function, Appell series, Srivastava Polynomials.

AMS subject classification: 33C45, 33C60.

1. Introduction

Inayat-Hussain (1987a) introduced generalization form of the Fox H-function, which is popularly known as $\overline{H}$ -function. Now $\overline{H}$ -function stands on fairly firm footing through the research contributions of various authors [1-3, 5, 7-12].

$\overline{H}$ -function is defined and represented in the following manner [9]:

$$\overline{H}_{p,q}^{m,n}[z]=\overline{H}_{p,q}^{m,n}\left[z\left|\begin{matrix}(a_j,\alpha_j;A_j)_{1,n},(a_j,\alpha_j)_{n+1,p}\\(b_j,\beta_j)_{1,m},(b_j,\beta_j;B_j)_{m+1,q}\end{matrix}\right.\right]$$

$$=\frac{1}{2\pi i}\int_L z^{\xi}\overline{\phi}(\xi)\,d\xi,\qquad (z\neq 0) \tag{1.1}$$

where

$$\overline{\phi}(\xi)=\frac{\prod_{j=1}^{m}\Gamma(b_j-\beta_j\xi)\prod_{j=1}^{n}\{\Gamma(1-a_j+\alpha_j\xi)\}^{A_j}}{\prod_{j=m+1}^{q}\{\Gamma(1-b_j+\beta_j\xi)\}^{B_j}\prod_{j=n+1}^{p}\Gamma(a_j-\alpha_j\xi)} \tag{1.2}$$

It may be noted that the $\overline{\phi}(\xi)$ contains fractional powers of some of the gamma function and m,n,p,q are integers such that $1\le m\le q, 1\le n\le p$, $(\alpha_j)_{1,p},(\beta_j)_{1,q}$ are positive real numbers and $(A_j)_{1,n},(B_j)_{m+1,q}$ may take non-integer values, which we assume to be positive for standardization purpose. $(a_j)_{1,p}$ and $(b_j)_{1,q}$ are complex numbers.

The nature of contour L , sufficient conditions of convergence of defining integral (1.1) and other details about the $\overline{H}$ -function can be seen in the papers [9, 10].

The behavior of the $\overline{H}$ -function for small values of $|z|$ follows easily from a result given by Rathie [11]:

$$\overline{H}_{p,q}^{m,n}[z]=o\left(|z|^{\alpha}\right) \tag{1.3}$$

where

$$\alpha=\min_{1\le j\le m}\operatorname{Re}\left(\frac{b_j}{\alpha_j}\right),|z|\to 0$$

The following series representation for the $\overline{H}$ -function given by Saxena et al. [12] will be required later on:

$$\bar{H}_{p,q}^{m,n}\left[z\left|\begin{matrix}(a_j,\alpha_j;A_j)_{1,n},(a_j,\alpha_j)_{n+1,p}\\(b_j,\beta_j)_{1,m},(b_j,\beta_j;B_j)_{m+1,q}\end{matrix}\right.\right]=\sum_{t=0}^{\infty}\sum_{h=1}^{m}\bar{f}(\varsigma)z^{\varsigma} \tag{1.4}$$

where

$$\bar{f}(\varsigma)=\frac{\prod\limits_{\substack{j=1\\j\neq h}}^{m}\Gamma(b_j-\beta_j\varsigma)\prod\limits_{j=1}^{n}\{\Gamma(1-a_j+\alpha_j\varsigma\}^{A_j}}{\prod\limits_{j=m+1}^{q}\{\Gamma(1-b_j+\beta_j\varsigma)\}^{B_j}\prod\limits_{j=n+1}^{p}\Gamma(a_j-\alpha_j\varsigma)}\frac{(-1)^t}{t!\beta_h}, \tag{1.5}$$

$$\varsigma=\frac{b_h+t}{\beta_h} \tag{1.6}$$

$$|\arg(z)|<\frac{1}{2}\pi\Omega \tag{1.7}$$

$$\Omega=\sum_{j=1}^{m}|B_j|+\sum_{j=m+1}^{q}|b_jB_j|-\sum_{j=1}^{n}|a_jA_j|-\sum_{j=n+1}^{p}|A_j|>0,$$
$$0<|z|<\infty. \tag{1.8}$$

The following function which follows as special cases of the $\bar{H}$ -function will be required in the sequel [7]:

$$_p\bar{\psi}_q\left[\begin{matrix}(a_j,\alpha_j;A_j)_{1,p}\\(b_j,\beta_j;B_j)_{1,q}\end{matrix};z\right]=\bar{H}_{p,q+1}^{1,p}\left[-z\left|\begin{matrix}(1-a_j,\alpha_j,A_j)_{1,p}\\(0,1),(1-b_j,\beta_j,B_j)_{1,q}\end{matrix}\right.\right] \tag{1.9}$$

The Srivastava polynomials $S_n^m[x]$ will be defined and represented as follows [13, p.1, Eq. (1)]:

$$S_n^m[x]=\sum_{k=0}^{[n/m]}\frac{(-n)_{mk}}{k!}A_{n,k}x^k \tag{1.10}$$

where $n=0,1,2,...,m$ is an arbitrary positive integer, the coefficients $A_{n,k}\,(n,k\geq 0)$ are arbitrary constants, real or complex. $S_n^m[x]$ yields number of known polynomials as its special cases. These include, among other, the Jacobi Polynomials, the Bessel Polynomials, the Lagurre Polynomials, the Brafman Polynomials and several others [15].

Reduction formulae for Appell series defined as follow [4, p.42, Eq. (4.2)]:

$$F_4\left[\alpha,\beta;\beta,\alpha;\frac{-x}{(1-x)(1-y)},\frac{-y}{(1-x)(1-y)}\right]$$
$$=(1-xy)^{-1}(1-x)^{\alpha}(1-y)^{\beta} \tag{1.11}$$

The following interesting formula is required in our investigation [6, p.145]:

$$\int_0^1\int_0^1\frac{(1-x)^{m-1}y^m(1-y)^{n-1}}{(1-xy)^{m+n-1}}dxdy=B(m,n) \tag{1.12}$$

2. Main Results

Let $\psi(z)$ denote the logarithmic derivative of gamma function $\Gamma(z)$ i.e. $\psi(z)=\dfrac{\Gamma'(z)}{\Gamma(z)}$

First Integral:

$$\int_0^1\int_0^1\left[\frac{(1-x)y}{1-xy}\right]^a\left[\frac{1-y}{1-xy}\right]^b\left[\frac{(1-xy)^2}{(1-x)(1-y)}\right]$$
$$\frac{y^{\alpha}}{(1-xy)^{\alpha+\beta}}F_4\left[\alpha,\beta;\beta,\alpha;\frac{-x}{(1-x)(1-y)},\frac{-y}{(1-x)(1-y)}\right]$$
$$\prod_{i=1}^{r}S_{n_i}^{m_i}\left[C_i\left(\frac{(1-x)y}{1-xy}\right)^{s_i}\left(\frac{1-y}{1-xy}\right)^{t_i}\right]$$
$$\bar{H}_{P,Q}^{M,N}\left[z\left(\frac{(1-x)y}{1-xy}\right)^{\lambda}\left(\frac{1-y}{1-xy}\right)^{\mu}\right]\log\left[\frac{(1-x)y}{1-xy}\right]dxdy$$
$$=\prod_{i=1}^{r}\sum_{k_i=0}^{[n_i/m_i]}\frac{(-n_i)_{m_ik_i}}{k_i!}A_{n_i,k_i}C_i^{k_i}\sum_{t=0}^{\infty}\sum_{h=1}^{m}\bar{f}(\varsigma)z^{\varsigma}$$
$$B\left(a+\alpha+\sum_{i=1}^{r}s_ik_i+\lambda\varsigma,b+\beta+\sum_{i=1}^{r}t_ik_i+\mu\varsigma\right)\times$$
$$\left[\psi\left(a+\alpha+\sum_{i=1}^{r}s_ik_i+\lambda\varsigma\right)-\right.$$
$$\left.\psi\left(a+b+\alpha+\beta+\sum_{i=1}^{r}(s_i+t_i)k_i+(\lambda+\mu)\varsigma\right)\right] \tag{2.1}$$

Second Integral :

$$\int_0^1\int_0^1\left[\frac{(1-x)y}{1-xy}\right]^a\left[\frac{1-y}{1-xy}\right]^b\left[\frac{(1-xy)^2}{(1-x)(1-y)}\right]$$
$$\frac{y^{\alpha}}{(1-xy)^{\alpha+\beta}}F_4\left[\alpha,\beta;\beta,\alpha;\frac{-x}{(1-x)(1-y)},\frac{-y}{(1-x)(1-y)}\right]$$
$$\prod_{i=1}^{r}S_{n_i}^{m_i}\left[C_i\left(\frac{(1-x)y}{1-xy}\right)^{s_i}\left(\frac{1-y}{1-xy}\right)^{t_i}\right]$$
$$\bar{H}_{P,Q}^{M,N}\left[z\left(\frac{(1-x)y}{1-xy}\right)^{\lambda}\left(\frac{1-y}{1-xy}\right)^{\mu}\right]\log\left[\frac{1-y}{1-xy}\right]dxdy$$
$$=\prod_{i=1}^{r}\sum_{k_i=0}^{[n_i/m_i]}\frac{(-n_i)_{m_ik_i}}{k_i!}A_{n_i,k_i}C_i^{k_i}\sum_{t=0}^{\infty}\sum_{h=1}^{m}\bar{f}(\varsigma)z^{\varsigma}$$
$$B\left(a+\alpha+\sum_{i=1}^{r}s_ik_i+\lambda\varsigma,b+\beta+\sum_{i=1}^{r}t_ik_i+\mu\varsigma\right)$$
$$\left[\psi\left(b+\beta+\sum_{i=1}^{r}t_ik_i+\mu\varsigma\right)-\psi\left(a+b+\alpha+\beta+\sum_{i=1}^{r}(s_i+t_i)k_i+(\lambda+\mu)\varsigma\right)\right] \tag{2.2}$$

Third Integral:

$$\int_0^1\int_0^1\left[\frac{(1-x)y}{1-xy}\right]^a\left[\frac{1-y}{1-xy}\right]^b\left[\frac{(1-xy)^2}{(1-x)(1-y)}\right]$$
$$\frac{y^\alpha}{(1-xy)^{\alpha+\beta}}F_4\left[\alpha,\beta;\beta,\alpha;\frac{-x}{(1-x)(1-y)},\frac{-y}{(1-x)(1-y)}\right]$$
$$\prod_{i=1}^r S_{n_i}^{m_i}\left[C_i\left(\frac{(1-x)y}{1-xy}\right)^{s_i}\left(\frac{1-y}{1-xy}\right)^{t_i}\right]$$
$$\bar{H}_{P,Q}^{M,N}\left[z\left(\frac{(1-x)y}{1-xy}\right)^\lambda\left(\frac{1-y}{1-xy}\right)^\mu\right]\log\left[\frac{(1-x)y(1-y)}{(1-xy)^2}\right]dxdy$$
$$=\prod_{i=1}^r\sum_{k_i=0}^{[n_i/m_i]}\frac{(-n_i)_{m_ik_i}}{k_i!}A_{n_i,k_i}C_i^{k_i}\sum_{t=0}^\infty\sum_{h=1}^m\bar{f}(\varsigma)z^\varsigma$$
$$B\left(a+\alpha+\textstyle\sum_{i=1}^r s_ik_i+\lambda\varsigma,b+\beta+\sum_{i=1}^r t_ik_i+\mu\varsigma\right)\times$$
$$\left[\psi\left(a+\alpha+\textstyle\sum_{i=1}^r s_ik_i+\lambda\varsigma\right)+\psi\left(b+\beta+\sum_{i=1}^r t_ik_i+\mu\varsigma\right)\right.$$
$$\left.-2\psi\left(a+b+\alpha+\beta+\textstyle\sum_{i=1}^r(s_i+t_i)k_i+(\lambda+\mu)\varsigma\right)\right] \tag{2.3}$$

Fourth Integral:

$$\int_0^1\int_0^1\left[\frac{(1-x)y}{1-xy}\right]^a\left[\frac{1-y}{1-xy}\right]^b\left[\frac{(1-xy)^2}{(1-x)(1-y)}\right]$$
$$\frac{y^\alpha}{(1-xy)^{\alpha+\beta}}F_4\left[\alpha,\beta;\beta,\alpha;\frac{-x}{(1-x)(1-y)},\frac{-y}{(1-x)(1-y)}\right]$$
$$\prod_{i=1}^r S_{n_i}^{m_i}\left[C_i\left(\frac{(1-x)y}{1-xy}\right)^{s_i}\left(\frac{1-y}{1-xy}\right)^{t_i}\right]$$
$$\bar{H}_{P,Q}^{M,N}\left[z\left(\frac{(1-x)y}{1-xy}\right)^\lambda\left(\frac{1-y}{1-xy}\right)^\mu\right]\log\left[\frac{(1-x)y}{(1-y)}\right]dxdy$$
$$=\prod_{i=1}^r\sum_{k_i=0}^{[n_i/m_i]}\frac{(-n_i)_{m_ik_i}}{k_i!}A_{n_i,k_i}C_i^{k_i}\sum_{t=0}^\infty\sum_{h=1}^m\bar{f}(\varsigma)z^\varsigma$$
$$B\left(a+\alpha+\textstyle\sum_{i=1}^r s_ik_i+\lambda\varsigma,b+\beta+\sum_{i=1}^r t_ik_i+\mu\varsigma\right)\times$$
$$\left[\psi\left(a+\alpha+\textstyle\sum_{i=1}^r s_ik_i+\lambda\varsigma\right)-\psi\left(b+\beta+\sum_{i=1}^r t_ik_i+\mu\varsigma\right)\right] \tag{2.4}$$

The above integrals are convergent under the conditions (1.3), (1.7) and

i) a, b, s_i, t_i, α, β, λ, μ are all positive.

j) $\mathrm{Re}\left[a+\lambda\left(\frac{b_j}{\beta_j}\right)\right]>0, 1\le j\le M$

k) $\mathrm{Re}\left[b+\mu\left(\frac{b_j}{\beta_j}\right)\right]>0, 1\le j\le M$

The following important and interesting result will be required to establish the above integrals:

$$\int_0^1\int_0^1\left[\frac{(1-x)y}{1-xy}\right]^a\left[\frac{1-y}{1-xy}\right]^b\left[\frac{(1-xy)^2}{(1-x)(1-y)}\right]$$
$$F_4\left[\alpha,\beta;\beta,\alpha;\frac{-x}{(1-x)(1-y)},\frac{-y}{(1-x)(1-y)}\right]$$
$$\frac{y^\alpha}{(1-xy)^{\alpha+\beta}}\prod_{i=1}^r S_{n_i}^{m_i}\left[C_i\left(\frac{(1-x)y}{1-xy}\right)^{s_i}\left(\frac{1-y}{1-xy}\right)^{t_i}\right]$$
$$\bar{H}_{P,Q}^{M,N}\left[z\left(\frac{(1-x)y}{1-xy}\right)^\lambda\left(\frac{1-y}{1-xy}\right)^\mu\right]dxdy$$
$$=\prod_{i=1}^r\sum_{k_i=0}^{[n_i/m_i]}\frac{(-n_i)_{m_ik_i}}{k_i!}A_{n_i,k_i}C_i^{k_i}\bar{H}_{P+2,Q+1}^{M,N+2}\left[z\left|\begin{matrix}T_1\\T_2\end{matrix}\right.\right] \tag{2.5}$$

$$=\prod_{i=1}^r\sum_{k_i=0}^{[n_i/m_i]}\frac{(-n_i)_{m_ik_i}}{k_i!}A_{n_i,k_i}C_i^{k_i}$$
$$\sum_{t=0}^\infty\sum_{h=0}^\mu\bar{f}(\varsigma)z^\varsigma B\left(a+\alpha+\textstyle\sum_{i=1}^r s_ik_i+\lambda\varsigma,b+\beta+\sum_{i=1}^r t_ik_i+\mu\varsigma\right) \tag{2.6}$$

where

$$T_1=\left(1-a-\alpha-\textstyle\sum_{i=1}^r s_ik_i,\lambda;1\right),$$
$$\left(1-b-\beta-\textstyle\sum_{i=1}^r t_ik_i,\mu;1\right),\left(a_j,\alpha_j;A_j\right)_{1,N},\left(a_j,\alpha_j\right)_{N+1,P}$$
$$T_2=\left(b_j,\beta_j\right)_{1,M},\left(b_j,\beta_j;B_j\right)_{M+1,Q}$$
$$,\left(1-a-b-\alpha-\beta-\textstyle\sum_{i=1}^r(s_i+t_i)k_i,\lambda+\mu;1\right)$$

Proof: To evaluate the above the integral, first we express $S_{n_i}^{m_i}[x]$ in its series form with the help of Eq. (1.10), $\bar{H}$ -function in Mellin-Barnes contour integral form from Eq. (1.1) and using the relation from Eq. (1.11), then changing the order of integration and summation we have the RHS as follows (let say I):

$$I=\prod_{i=1}^r\sum_{k_i=0}^{[n_i/m_i]}\frac{(-n_i)_{m_ik_i}}{k_i!}A_{n_i,k_i}C_i^{k_i}$$
$$\frac{1}{2\pi i}\int_L\bar{\phi}(\varsigma)z^\varsigma\left[\int_0^1\int_0^1\frac{(1-x)^{a+\alpha+\sum_{i=1}^r s_ik_i+\lambda\varsigma-1}\,y^{a+\alpha+\sum_{i=1}^r s_ik_i+\lambda\varsigma}(1-y)^{b+\beta+\sum_{i=1}^r t_ik_i+\mu\varsigma-1}}{(1-xy)^{a+b+\alpha+\beta+\sum_{i=1}^r(s_i+t_i)k_i+(\lambda+\mu)\varsigma-1}}dxdy\right]d\varsigma$$

Now apply the formulae given by Eq. (1.12), we have:

$$I = \prod_{i=1}^{r} \sum_{k_i=0}^{[n_i/m_i]} \frac{(-n_i)_{m_i k_i}}{k_i!} A_{n_i,k_i} C_i^{k_i}$$

$$\frac{1}{2\pi i}\int_L \bar{\phi}(\varsigma) z^{\varsigma} B\left(a+\alpha+\sum_{i=1}^{r} s_i k_i + \lambda\varsigma, b+\beta+\sum_{i=1}^{r} t_i k_i + \mu\varsigma\right) d\varsigma$$

After little simplification, we get the right hand side of Eq. (2.5). (Eq. (2.6) can be obtained by changing the , $\bar{H}$ -function into series form given in Eq. (1.4)).

Method of Proof: To prove First Integral , by taking the partial derivative of both sides of (2.6) with respect to a. Second Integral is similarly established by taking the partial derivative of both sides of (2.6) with respect to b. To establish Third & Forth Integrals, we use the First & Second Integrals , first adding the First & Second Integrals , than we get the Third Integral. Forth Integral is similarly established by subtracting Second Integral from First Integral.

3. Special Cases

(i) By applying the our results given in Eqn. (2.5) to the case of Hermite Polynomials [15] by setting $S_{n_1}^{2}(x) \to x^{n_1/2} H_{n_1}\left[\frac{1}{2\sqrt{x}}\right]$ in which $r=1, m_1=2; A_{n_1,k_1}=(-1)^{k_1}$, we have the following interesting results which is believe to be new :

$$\int_0^1\int_0^1 \left[\frac{(1-x)y}{1-xy}\right]^a \left[\frac{1-y}{1-xy}\right]^b \left[\frac{(1-xy)^2}{(1-x)(1-y)}\right] \frac{y^{\alpha}}{(1-xy)^{\alpha+\beta}} F_4\left[\alpha,\beta;\beta,\alpha;\frac{-x}{(1-x)(1-y)},\frac{-y}{(1-x)(1-y)}\right]$$

$$\left[C_1\left(\frac{(1-x)y}{1-xy}\right)^{s_1}\left(\frac{1-y}{1-xy}\right)^{t_1}\right]^{n_1/2} H_{n_1}\left[\frac{1}{2}\left(C_1\left(\frac{(1-x)y}{1-xy}\right)^{s_1}\left(\frac{1-y}{1-xy}\right)^{t_1}\right)^{-\frac{1}{2}}\right]$$

$$\bar{H}_{P,Q}^{M,N}\left[z\left(\frac{(1-x)y}{1-xy}\right)^{\lambda}\left(\frac{1-y}{1-xy}\right)^{\mu}\right]dxdy \tag{3.1}$$

$$= \sum_{k_1=0}^{[n_1/2]} \frac{(-n_1)_{m_1 k_1}}{k_1!}(-1)^{k_1} C_1^{k_1} \bar{H}_{P+2,Q+1}^{M,N+2}\left[z\left|\begin{matrix}T_1'\\T_2'\end{matrix}\right.\right]$$

where

$$T_1' = \left(1-a-\alpha-s_1k_1,\lambda;1\right),\left(1-b-\beta-t_1k_1,\mu;1\right),\left(a_j,\alpha_j;A_j\right)_{1,N},\left(a_j,\alpha_j\right)_{N+1,P}$$

$$T_2' = \left(b_j,\beta_j\right)_{1,M},\left(b_j,\beta_j;B_j\right)_{M+1,Q},\left(1-a-b-\alpha-\beta-\left(s_1+t_1\right)k_1,\lambda+\mu;1\right)$$

The conditions of convergence of Eq. (3.1) can be easily follow from those of Eq. (2.5).

(ii) By applying our result given in Eq. (2.5) to the case of Lagurre Polynomials [15] by setting $S_{n_1}^{1}(x) \to L_{n_1}^{(\alpha)}[x]$ in which $r=1, m_1=1; A_{n_1,k_1}=\binom{n_1+\alpha'}{n_1}\frac{1}{(\alpha'+1)_{k_1}}$ We have the following interesting result:

$$\int_0^1\int_0^1 \left[\frac{(1-x)y}{1-xy}\right]^a \left[\frac{1-y}{1-xy}\right]^b \left[\frac{(1-xy)^2}{(1-x)(1-y)}\right] \frac{y^{\alpha}}{(1-xy)^{\alpha+\beta}} F_4\left[\alpha,\beta;\beta,\alpha;\frac{-x}{(1-x)(1-y)},\frac{-y}{(1-x)(1-y)}\right]$$

$$L_{n_1}^{(\alpha')}\left[C_1\left(\frac{(1-x)y}{1-xy}\right)^{s_1}\left(\frac{1-y}{1-xy}\right)^{t_1}\right] \bar{H}_{P,Q}^{M,N}\left[z\left(\frac{(1-x)y}{1-xy}\right)^{\lambda}\left(\frac{1-y}{1-xy}\right)^{\mu}\right]dxdy$$

$$= \sum_{k_1=0}^{n_1} \frac{(-n_1)_{k_1}}{k_1!}\binom{n_1+\alpha'}{n_1}\frac{1}{(\alpha'+1)_{k_1}} C_1^{k_1} \bar{H}_{P+2,Q+1}^{M,N+2}\left[z\left|\begin{matrix}T_1''\\T_2''\end{matrix}\right.\right] \tag{3.2}$$

where

$$T_1'' = \left(1-a-\alpha-s_1k_1,\lambda;1\right),\left(1-b-\beta-t_1k_1,\mu;1\right),\left(a_j,\alpha_j;A_j\right)_{1,N},\left(a_j,\alpha_j\right)_{N+1,P}$$

$$T_2'' = \left(b_j,\beta_j\right)_{1,M},\left(b_j,\beta_j;B_j\right)_{M+1,Q},\left(1-a-b-\alpha-\beta-\left(s_1+t_1\right)k_1,\lambda+\mu;1\right)$$

The conditions of convergence of Eq. (3.2) can be easily followed from those of Eq. (2.5).

(iii*) If we put* $A_j = B_j = 1$, $\bar{H}$ -function reduces to Fox H-function [7, p. 10, Eqn. (2.1.1)], then the Eqn. (2.5) take the following form:

$$\int_0^1\int_0^1 \left[\frac{(1-x)y}{1-xy}\right]^a \left[\frac{1-y}{1-xy}\right]^b \left[\frac{(1-xy)^2}{(1-x)(1-y)}\right] \frac{y^{\alpha}}{(1-xy)^{\alpha+\beta}} F_4\left[\alpha,\beta;\beta,\alpha;\frac{-x}{(1-x)(1-y)},\frac{-y}{(1-x)(1-y)}\right]$$

$$\prod_{i=1}^{r} S_{n_i}^{m_i}\left[C_i\left(\frac{(1-x)y}{1-xy}\right)^{s_i}\left(\frac{1-y}{1-xy}\right)^{t_i}\right] H_{P,Q}^{M,N}\left[z\left(\frac{(1-x)y}{1-xy}\right)^{\lambda}\left(\frac{1-y}{1-xy}\right)^{\mu}\right]dxdy$$

$$= \prod_{i=1}^{r} \sum_{k_i=0}^{[n_i/m_i]} \frac{(-n_i)_{m_i k_i}}{k_i!} A_{n_i,k_i} C_i^{k_i} H_{P+2,Q+1}^{M,N+2}\left[z\left|\begin{matrix}T_1'''\\T_2'''\end{matrix}\right.\right] \tag{3.3}$$

where

$$T_1''' = \left(1-a-\alpha-\sum_{i=1}^{r} s_i k_i, \lambda; 1\right),$$
$$\left(1-b-\beta-\sum_{i=1}^{r} t_i k_i, \mu; 1\right), \left(a_j, \alpha_j; 1\right)_{1,N}, \left(a_j, \alpha_j\right)_{N+1,P}$$
$$T_2''' = \left(b_j, \beta_j\right)_{1,M}, \left(b_j, \beta_j; 1\right)_{M+1,Q}$$
$$, \left(1-a-b-\alpha-\beta-\sum_{i=1}^{r}\left(s_i+t_i\right)k_i, \lambda+\mu; 1\right)$$

The conditions of convergence of Eq. (3.3) can be easily obtained from those of Eq. (2.5).

(iv) *If* we put $n=p$, $m=1$, $q=q+1$, $b_1=0, \beta_1=1,\ a_j=1-a_j,\ b_j=1-b_j$, *then the* $\overline{H}$ -*f*unction reduces to generalized wright hypergeometric function [16] i. the Eq. (2.5) take the following form after little simplification*:*

$$\int_0^1\int_0^1 \left[\frac{(1-x)y}{1-xy}\right]^a \left[\frac{1-y}{1-xy}\right]^b \left[\frac{(1-xy)^2}{(1-x)(1-y)}\right]$$
$$\frac{y^{\alpha}}{(1-xy)^{\alpha+\beta}} F_4\left[\alpha,\beta;\beta,\alpha; \frac{-x}{(1-x)(1-y)}, \frac{-y}{(1-x)(1-y)}\right]$$
$$\prod_{i=1}^{r} S_{n_i}^{m_i}\left[C_i\left(\frac{(1-x)y}{1-xy}\right)^{s_i}\left(\frac{1-y}{1-xy}\right)^{t_i}\right]$$
$${}_P\overline{\Psi}_Q\left[\begin{matrix}\left(a_j,\alpha_j;A_j\right)_{1,P} \\ \left(b_j,\beta_j;B_j\right)_{1,Q}\end{matrix}; -z\left(\frac{(1-x)y}{1-xy}\right)^{\lambda}\left(\frac{1-y}{1-xy}\right)^{\mu}\right]dxdy$$
$$= \prod_{i=1}^{r}\sum_{k_i=0}^{[n_i/m_i]} \frac{(-n_i)_{m_i k_i}}{k_i!} A_{n_i,k_i} C_i^{k_i}\ {}_{P+2}\overline{\Psi}_{Q+1}\left[\begin{matrix}T_1^* \\ T_2^*\end{matrix}; -z\right] \tag{3.4}$$

where

$$T_1^* = \left(1-a-\alpha-\sum_{i=1}^{r} s_i k_i, \lambda; 1\right),$$
$$\left(1-b-\beta-\sum_{i=1}^{r} t_i k_i, \mu; 1\right), \left(a_j, \alpha_j; 1\right)_{1,P}$$
$$T_2^* = \left(b_j, \beta_j; 1\right)_{1,Q}$$
$$, \left(1-a-b-\alpha-\beta-\sum_{i=1}^{r}\left(s_i+t_i\right)k_i, \lambda+\mu; 1\right)$$

4. Conclusions

The results obtained in this article are useful in deriving certain unified integrals involving the hypergeoemtric and generalized hypergeometric functions. Due to presence of Apple series in our main integrals, the integrals are very useful from the applications point of view. On specializing the parameters, we can easily obtain the known and new integrals.

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Vitae

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