Vol. 2, No. 2, 2012, ISSN 2167-6380

www.worldsciencepublisher.org

On a Unified Integral Involving the Product of Srivastava's Polynomials and Generalized Mellin-Barnes Type of Contour Integral

Praveen Agarwal

Department of Mathematics, Anand International College of Engineering, Jaipur-303012, India
E-mail: goyal.praveen2011@gmail.com

Abstract – The aim of the present paper is to evaluate a new unified integral whose integrand contains products of the $\overline{H}$ -function, Srivastava polynomials and generalized hypergeometric functions having general arguments. The integral is unified in nature and act as key integral formulae from which we can obtain as their special cases. For the sake of illustration, we record here some special cases of our main integral which are also new and of interest by themselves.

Mathematics Subject Classification2010: 33C20, 33C60, 33C70.

Key words: $\overline{H}$ -function, Srivastava Polynomials, Generalized hypergeometric function.

1. Introduction

The Srivastava polynomials introduced by Srivastava defined as follows [26, p.1, Eq. (1)]:

$$S_n^m[x] = \sum_{k=0}^{[n/m]} \frac{(-n)_{mk}}{k!} A_{n,k} x^k \text{ , (n=0, 1, 2…)} \tag{1.1}$$

where m is an arbitrary positive integers and the coefficients $A_{n,k}(n,k \geq 0)$ are arbitrary constants, real or complex. On suitably specializing the coefficients $A_{n,k}$, $S_n^m[x]$ yields a number of known polynomials as its special cases. These include, among others, the Hermite polynomials, the Jacobi polynomials, the Lagurre polynomials, the Bessel's polynomials and several others [27, pp. 158-161].

A lot of research work has recently come up on the study and development of a function that is more general than the Fox H-function, popularly known as $\overline{H}$ - function. It was introduced by Inayat-Hussain [19, 20] and now stands on a fairly firm footing through the research contributions of various authors [1-3, 7-9, 15-16, 18-20, 24-25].
The $\overline{H}$ -function will be defined and represented as follows [19]:

$$\overline{H}_{P,Q}^{M,N}[z] \equiv \overline{H}_{P,Q}^{M,N}\left[z \middle| \begin{matrix} (a_j,\alpha_j;A_j)_{1,N}, (a_j,\alpha_j)_{N+1,P} \\ (b_j,\beta_j)_{1,M}, (b_j,\beta_j;B_j)_{M+1,Q} \end{matrix} \right]$$

$$= \frac{1}{2\pi i}\int_L z^{\xi}\overline{\phi}(\xi)d\xi. \ (z \neq 0) \tag{1.2}$$

$$\overline{\phi}(\xi) = \frac{\prod_{j=1}^{M}\Gamma(b_j-\beta_j\xi)\prod_{j=1}^{N}\left\{\Gamma(1-a_j+\alpha_j\xi)\right\}^{A_j}}{\prod_{j=M+1}^{Q}\left\{\Gamma(1-b_j+\beta_j\xi)\right\}^{B_j}\prod_{j=N+1}^{P}\Gamma(a_j-\alpha_j\xi)} \tag{1.3}$$

Buschman and Srivastava [16] has proved that the integral on the right hand of (1.1) is absolutely convergent when $\Omega > 0$ and $|\arg z| < \frac{1}{2}\pi\Omega$, where

$$\Omega = \sum_{j=1}^{M}\beta_j + \sum_{j=1}^{N}a_j\alpha_j - \sum_{j=M+1}^{Q}b_j\beta_j - \sum_{j=N+1}^{P}\alpha_j > 0 \tag{1.4}$$

here, and throughout the paper $a_j\ (j=1,...,P)$ and b_j $(j=1,...,Q)$ are complex parameters, $\alpha_j \geq 0$ $(j=1,...,P)$, $\beta_j \geq 0\ (j=1,...,Q)$ (not all zero simultaneously) and the exponents $A_j\ (j=1,...,N)$ and $B_j\ (j=M+1,...,Q)$ can take on non-negative values. For further details of $\overline{H}$ -function one can refer the original paper of Buschman and Srivastava [16].

Generalized hypergeometric function is defined as follows [18]:

$$
{}_P\overline{F}_Q\left[\begin{matrix}\left(a_j,\alpha_j;A_j\right)_{1,P};\\ \left(b_j,\beta_j;B_j\right)_{1,Q};\end{matrix}\; z\right]
= \frac{\prod\limits_{j=1}^{Q}\left\{\Gamma(b_j)\right\}^{B_j}}{\prod\limits_{j=1}^{P}\left\{\Gamma(a_j)\right\}^{A_j}}\,\overline{H}^{1,P}_{P,Q+1}\left[-z\left|\begin{matrix}\left(1-a_j,\alpha_j;A_j\right)_{1,P}\\ (0,1),\left(1-b_j,\beta_j;B_j\right)_{1,Q}\end{matrix}\right.\right] \qquad \textbf{(1.5)}
$$

The function ${}_p\overline{F}_q$ reduce to well-known ${}_pF_q$ for $A_j=1(j=1,...,p), B_j=1(j=1,...,q)$ in it. For further details one can refer to [23].

This paper was probably the first where the explicit solution of unified integral in terms of the $\overline{H}$ -function. Such investigations now are of a great interest in connection with applications; for example, see in this connection [1-14].

2. Main Results

$$
\int_0^\infty x^{l-1}\left[x+\alpha+\left(x^2+2\alpha x\right)^{1/2}\right]^{-m}\times
{}_R\overline{F}_S\left[\begin{matrix}\left(g_j,\alpha_j;G_j\right)_{1,R};\\ \left(h_j,\beta_j;H_j\right)_{1,S};\end{matrix}\; a\left(x+\alpha+\left(x^2+2\alpha x\right)^{1/2}\right)^{-n}\right]
$$

$$
S_V^U\left[\left\{x+\alpha+\left(x^2+2\alpha x\right)^{1/2}\right\}^{-p}\right]\times
S_{V'}^{U'}\left[\left\{x+\alpha+\left(x^2+2\alpha x\right)^{1/2}\right\}^{-p'}\right]\times
\overline{H}^{M,N}_{P,Q}\left[z\left\{x+\alpha+\left(x^2+2\alpha x\right)^{1/2}\right\}^{-q}\right]dx
$$

$$
=2\alpha^{-m}\left(\frac{1}{2}\alpha\right)^{l}\Gamma(2l)
\sum_{k=0}^{[V/U]}\sum_{k'=0}^{[V'/U']}\sum_{r=0}^{\infty}\frac{\prod\limits_{j=1}^{R}\left\{\left(g_j\right)_r\right\}^{Gj}}{\prod\limits_{j=1}^{S}\left\{\left(h_j\right)_r\right\}^{H_j}}\frac{a^r}{r!}\frac{(-V)_{Uk}}{k!}\frac{(-V')_{U'k'}}{k'!}
$$

$$
A(V,k)\,A'(V',k')(\alpha)^{-nr-kp-k'p'}
\overline{H}^{M,N+2}_{P+2,Q+2}\left[z\alpha^{-q}\left|\begin{matrix}(-m-nr-kp-k'p',q;1),\\ (b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q},\end{matrix}\right.\right.
$$

$$
\left(1+l--m-nr-kp-k'p',q;1\right),\ \left(1-m-nr-kp-k'p',q;1\right),\ (a_j,\alpha_j;A_j)_{1,N},\ (a_j,\alpha_j)_{N+1,P}\ \left.\left(-m-nr-kp-k'p'-l,q;1\right)\right] \qquad (2.1)
$$

The conditions of validity of (2.1) are

(i) $q\geq 0, \mathrm{Re}(l,m,n,p,p')>0.$

(ii) $R\leq S; R=S+1, |a\alpha|<1$

(iii) $|\arg z|<\frac{1}{2}\pi\Omega, \Omega>0$ Where Ω is given by (1.4).

(iv) $\mathrm{Re}(l)-\mathrm{Re}(m)-q\min\mathrm{Re}\left(\frac{b_j}{\beta_j}\right)<0$

Proof: To evaluate the integral (2.1) we first express $S_V^U[.]$, $S_{V'}^{U'}[.]$ and ${}_RF_S[.]$ in its series form with the help of (1.1) and (1.5) respectively and put the value of $\overline{H}^{M,N}_{P,Q}[z]$ in terms of Mellin-Barnes contour integral by help of (1.2). Interchanging the order of integration and summation (which is permissible under the conditions stated with (2.1)) and evaluate the x-integral with the help of the following result [22]:

$$
\int_0^\infty x^{a-1}\left[x+c+\left(x^2+2cx\right)^{1/2}\right]^{-b}dx
=2bc^{-b}\left(\frac{1}{2}c\right)^{a}\frac{\Gamma(2a)\Gamma(b-a)}{\Gamma(1+b+a)},\quad 0<\mathrm{Re}(a)<b, \qquad \textbf{(2.2)}
$$

Finally, interpreting the ξ contour integral in terms of the $\overline{H}$ -function, we arrive at the right hand side of (2.1).

3. An Important Special Cases of Main Result:

On account of the most general nature of $\overline{H}$ -function, $S_V^U[y], S_{V'}^{U'}[y]$ and ${}_RF_S[.]$ occurring in our main integral given by (2.1), a large number of integrals involving simpler functions of one variable can be easily obtained as their special cases. We however gave here only some special cases by way of illustration:

(i) If we take $A_j(j=1,...,N)$ and $B_j(j=M+1,...,Q)$ are all equal to unity in equation (2.1), then the $\overline{H}$ -function reduce to the Fox H- function [18]. We have an interesting result it is also believed to be new result:

$$\int_0^\infty x^{l-1}\left[x+\alpha+\left(x^2+2\alpha x\right)^{1/2}\right]^{-m}$$
$$_R\overline{F}_S\left[a\left(x+\alpha+\left(x^2+2\alpha x\right)^{1/2}\right)^{-n}\right]$$
$$S_V^U\left[\left\{x+\alpha+\left(x^2+2\alpha x\right)^{1/2}\right\}^{-p}\right]$$
$$S_{V'}^{U'}\left[\left\{x+\alpha+\left(x^2+2\alpha x\right)^{1/2}\right\}^{-p'}\right]$$
$$H_{P,Q}^{M,N}\left[z\left\{x+\alpha+\left(x^2+2\alpha x\right)^{1/2}\right\}^{-q}\right]dx$$
$$=2\alpha^{-m}\left(\frac{1}{2}\alpha\right)^{l}\Gamma(2l)$$
$$\sum_{k=0}^{[V/U]}\sum_{k'=0}^{[V'/U']}\sum_{r=0}^{\infty}\frac{\prod_{j=1}^{R}\left\{\left(g_j\right)_r\right\}^{Gj}}{\prod_{j=1}^{S}\left\{\left(h_j\right)_r\right\}^{H_j}}\frac{a^r}{r!}\frac{(-V)_{Uk}}{k!}\frac{(-V')_{U'k'}}{k'!}$$
$$A(V,k)\,A'(V',k')(\alpha)^{-nr-kp-k'p'}$$
$$H_{P+2,Q+2}^{M,N+2}\left[z\alpha^{-q}\left|\begin{matrix}(-m-nr-kp-k'p',q;1),\\ (b_j,\beta_j)_{1,M},(\mathrm{b}_j,\beta_j;B_j)_{M+1,Q},\end{matrix}\right.\right.$$
$$(1+l--m-nr-kp-k'p',q;1),$$
$$(1-m-nr-kp-k'p',q;1),$$
$$\left.\begin{matrix}(\mathrm{a}_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\ (-m-nr-kp-k'p'-l,q;1)\end{matrix}\right]$$

(3.1)

where conditions of validity of (3.1) easily follow from those given in (2.1).

(ii) If we take $V=0$ and $V'=0$ in (2.1), we arrive at the following integrals which is also believe to be new and sufficiently general in nature:

$$\int_0^\infty x^{l-1}\left[x+\alpha+\left(x^2+2\alpha x\right)^{1/2}\right]^{-m}$$
$$_R\overline{F}_S\left[a\left(x+\alpha+\left(x^2+2\alpha x\right)^{1/2}\right)^{-n}\right]$$
$$\overline{H}_{P,Q}^{M,N}\left[z\left\{x+\alpha+\left(x^2+2\alpha x\right)^{1/2}\right\}^{-q}\right]dx$$
$$=2\alpha^{-m}\left(\frac{1}{2}\alpha\right)^{l}\Gamma(2l)\sum_{r=0}^{\infty}\frac{\prod_{j=1}^{R}\left\{\left(g_j\right)_r\right\}^{Gj}}{\prod_{j=1}^{S}\left\{\left(h_j\right)_r\right\}^{H_j}}\frac{a^r}{r!}(\alpha)^{-nr}$$
$$\overline{H}_{P+2,Q+2}^{M,N+2}\left[z\alpha^{-q}\left|\begin{matrix}(-m-nr,q;1),\\ (b_j,\beta_j)_{1,M},\end{matrix}\right.\right.$$
$$\left.\begin{matrix}(1+l-m-nr,q;1),(\mathrm{a}_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\ (\mathrm{b}_j,\beta_j;B_j)_{M+1,Q},(1-m-nr,q;1),(-m-nr-l,q;1)\end{matrix}\right]$$

(3.2)

conditions of validity of (3.2) easily follow from those given in (2.1)

(iii) If in (3.1) we take $V'=0$ and $_R\overline{F}_S$ are unity then we have a known result [17].

4. Conclusions

The present paper is to evaluate a new unified integral whose function involved in the integral formulae as well as their arguments are quite general in nature and so our findings provide interesting unifications and extensions of a number of (known and new) results.

Acknowledgements

The author is thankful to worthy referee for his useful suggestions.

References

[1] P. Agarwal, On multiple integral relations involving generalized Mellin-Barnes type of contour integral, Tamsui Oxf. J. Math. Sci. 27 (4): (2011), 449-462.

[2] P. Agarwal, New Unified Integral Involving a Srivastava Polynomials and $\overline{H}$ -function , Journal of Fractional Calculus and Applications, 3(3), (2012), 1- 7.

[3] P.Agarwal, Integral Formulae's Involving Two $\overline{H}$ -function and Multivariable Polynomials, Global Journal of Science Frontier Research Mathematics & Decision Sciences 12(4)1, (2012).

[4] P.Agarwal, On A New Unified Integral Involving Hypergeometric Functions, Advances in Computational Mathematics and its Applications 2(1), (2012), 239-242.

[5] P.Agarwal, On New Unified Integrals involving Appell Series, Advances in Mechanical Engineering and its Applications 2(1), (2012),115-120.

[6] P. Agarwal, Fractional integration of the product of two H-functions and a general class of polynomials Asian Journal of Applied Sciences 5(3), (2012),144-153.

[7] P. Agarwal and M. Chand, New Theorems Involving the Generalized Mellin-Barnes Type of Contour Integrals and

General Class of Polynomials, Global Journal of Science Frontier Research Mathematics & Decision Sciences, 12(3)1.0, (2012).

[8] P. Agarwal and M. Chand, Certain Integral Properties of Generalized Contour Integral Associated with Feynman Integrals, International Journal of Physics and Mathematical Sciences,1(1),(2011),1-8.

[9] P. Agarwal and M. Chand, New finite integrals involving product of $\overline{H}$ -function and Srivastava Polynomials, Asian Journal of Mathematics & Statistics,5(4),(2012),142-149.

[10] P. Agarwal and S. Jain, New Finite Double Integral Formulae Involving Polynomials and Functions of General Naturen, Journal of the Applied Mathematics, Statistics and Informatics, 7(2), (2011).

[11] P. Agarwal and S. Jain, New Integral Formulas Involving Polynomials and $\overline{I}$ -function, Journal of the Applied Mathematics, Statistics and Informatics, 8(1), (2012), 79-88.

[12] P.Agarwal and S.Jain, On unified finite integrals involving a multivariable polynomial and a generalized Mellin Barnes type of contour integral having general argument, Nat. Acad. Sci. Lett., 32(8 & 9), (2009).

[13] P. Agarwal, S. Jain and M. Chand, Finite Integrals Involving Jacobi Polynomials and I-function, Theoretical Mathematics & Applications, 1(1),(2011), 115-123.

[14] P. Agarwal, S. Jain and M. Chand, On Integrals Involving Product of $H[x_1,...,x_r]$ and the Generalized Polynomials $S_{n_1,...,n_r}^{m_1,...,m_r}[x]$,International Mathematical Forum, 6(11), (2011), 529 – 539.

[15] B.L.J. Braaksma, Asymptotic expansions and analytic continuations for a class of Barnes-integrals, Compositio Math., 15 (1964), pp. 239-341.

[16] R.G. Buschman and H.M. Svrivastava, The H-function associated with a certain class of Feyman integrals, J. Phys. A : Math. Gen., 23 (1990), pp. 4707-4710.

[17] M. Garg and S. Mittal, On a new unified integral, Proc. Indian Acad. Sci. (Math. Sci.) Vol. 114, No. 2, May 2003, pp. 99–101.

[18] Gupta KC and Soni RC. On a basic integral formula involving the product of the $\overline{H}$ -function and Fox H-function, J. Raj. Acad. Phy. Sci., 2005; 4 (3):157-164.

[19] Inayat-Hussain AA. New properties of hypergeometric series derivable from Feynman integrals: I. Transformation and reeducationformulae, *J. Phys. A: Math. Gen. 1987*; 20:4109-4117.

[20] Inayat-Hussain AA. New properties of hypergeometric series derivable from Feynman integrals: II.A generalization of the H-function, J.Phys.A.Math.Gen.1987; 20:4119-4128

[21] A.M. Mathi and R.K. Saxena, The H-function with Applications in Statistics and Other Disciplines, Wiley Eastern, New Delhi (1978).

[22] F. Oberhettinger, Tables of Mellin transforms (Berlin, Heidelberg, New York: Springer-Verlag), 1974 p. 22.

[23] E. D. Rainville, Special Functions, Chelsea Publ. Co., Bronx, New York, (1971).

[24] A.K. Rathie, A new generalization of generalized hypergeometric functions, Le mathematic he Fasc. II 52(1997), 297-310.

[25] R.K.Saxena, Functional relations involving generalized H-function, Le Mathematiche.LIII (1998)-Fasc.I, pp. 123- 131.

[26] H. M. Srivastava, A contour integral involving Fox's H-function, Indian J. Math., 14(1972), 1-6.

[27] H. M. Srivastava and N. P. Singh, The integration of certain products of the multivariable H-function with a general class of polynomials, Rend. Circ. Mat. Palermo Ser. 2, 32(1983), 157-187.

.

Vitae

Dr Praveen Agarwal, was born in 1979. He obtained a Ph.D. degree in 2006 in Mathematics Department from University of University.

He worked as an Associate Professor in Anand International College of Engineering, Jaipur. His research interest includes Integral transforms, Special functions, Integral equations, Statistical distributions, Fractional calculus, generalized functions, and Fractional deferential equations and Mathematical Physics.

He is the Editor In-Chief of the International Journal of Mathematical Research. He is the editor of American Journal of Mathematics and Statistics, South Asian Journal of Mathematics, Asian Journal of Current Engineering & Math's [AJCEM], World Academy of Science, International Journal of Scientific and Engineering Research (IJSER),International Journal of Soft Computing and Engineering (IJSCE), Asian Journal of Mathematics & Statistics, Trends in Applied Sciences Research, Asian Journal of Scientific Research, Journal of Applied Sciences& Asian Journal of Applied Sciences, Shiv Shakti International Journal in Multidisciplinary and Academic Research (SSIJMAR), World Science Publisher, International Journal of Engineering and Advanced Technology™.

He is the Member of Who's Who in the World 2013 (30th pearl Anniversary Edition), American Biographical Institute's prestigious honor "MAN OF THE YEAR 2012". Of ABI, The Cambridge Certificate for Outstanding Educational Achievement" 2012, of IBC, "2000 Outstanding Intellectuals of the 21st Century 2012" International biographical centre, Cambridge, England, Who's Who in Science and Engineering 2011-2012 (11th Edition), Who's Who in the World 2012 (29th Edition), International Association of Engineers (IAENG),Rajasthan Academy of Physical Sciences, Society for Special Function and their Applications.