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Picard's Approximation Method for Solving a Class of Local Fractional Volterra Integral Equations

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Abstract – In this letter, we fist consider the Picard's successive approximation method for solving a class of the Volterra integral equations in local fractional integral operator sense. Special attention is devoted to the Picard's successive approximate methodology for handling local fractional Volterra integral equations. An illustrative paradigm is shown the accuracy and reliable results.

Keywords – Picard's successive approximation method; Volterra Integral equation; Local fractional operator; Local fractional Volterra integral equations

1. Introduction

The theory of local fractional calculus is one of useful tools to process the fractal and continuously non-differentiable functions [1-8]. It was successfully applied in local fractional Fokker–Planck equation [1], the fractal heat conduction equation [2, 8], fractal-time dynamical systems [4], fractal elasticity [5], local fractional diffusion equation [8], local fractional Laplace equation [7, 9], local fractional integral equations [10, 11, 12], local fractional differential equations [7-13, 14, 15], fractal wave equation [7, 9, 16].

In this letter, by using the Picard's successive approximation method, we consider analysis solution to the non-homogeneous local fractional Volterra integral equation of the second kind [10, 13]. This paper is organized as follows: In section 2, we investigate local fractional integrals and its fractal geometrical explanation. In section 3, the Picard's successive approximation method is proposed based on local fractional integrals. An illustrative example is shown in section 4. Conclusions are in Section 5.

2. Local Fractal Integrals and Fractal Geometrical Explanation

2.1. Local fractional continuity of functions

Definition 1 If there is the relation [6, 7, 10-13]

$$\left|f(x)-f(x_0)\right|<\varepsilon^{\alpha} \tag{2.1}$$

with $\left|x-x_0\right|<\delta$, for $\varepsilon,\delta>0$ and $\varepsilon,\delta\in R$.

Now $f(x)$ is called local fractional continuous at $x=x_0$, denoted by $\lim_{x\to x_0} f(x)=f(x_0)$. Then $f(x)$ is called local fractional continuous on the interval (a,b), denoted by [6, 7, 10-13]

$$f(x)\in C_{\alpha}(a,b). \tag{2.2}$$

Definition 2 A function $f(x)$ is called a non-differentiable function of exponent α, $0<\alpha\le 1$, which satisfy Hölder function of exponent α, then for $x,y\in X$ such that [6, 7, 10-13]

$$\left|f(x)-f(y)\right|\le C\left|x-y\right|^{\alpha}. \tag{2.3}$$

Definition 3 A function $f(x)$ is called to be continuous of order α, $0<\alpha\le 1$, or shortly α continuous, when we have the following relation[6, 7, 10-13]

$$f(x)-f(x_0)=o\left((x-x_0)^{\alpha}\right). \tag{2.4}$$

2.2. Local fractional integrals

Definition 4 Setting $f(x)\in C_{\alpha}(a,b)$, local fractional integral of $f(x)$ of order α in the interval $[a,b]$ is defined [6, 7, 10-16]

$$\begin{aligned}
&{}_aI_b^{(\alpha)}f(x)\\
&=\frac{1}{\Gamma(1+\alpha)}\int_a^b f(t)(dt)^{\alpha}\\
&=\frac{1}{\Gamma(1+\alpha)}\lim_{\Delta t\to 0}\sum_{j=0}^{j=N-1} f(t_j)(\Delta t_j)^{\alpha},
\end{aligned} \tag{2.5}$$

where $\Delta t_j=t_{j+1}-t_j$, $\Delta t=\max\{\Delta t_1,\Delta t_2,\Delta t_j,\ldots\}$ and $[t_j,t_{j+1}]$, $j=0,\ldots,N-1$, $t_0=a$, $t_N=b$, is a partition of the interval $[a,b]$. For any $x\in(a,b)$, there exists [12, 13]

$$ {}_aI_x^{(\alpha)} f(x), \tag{2.6}$$

denoted by

$$ f(x) \in I_x^{(\alpha)}(a,b). \tag{2.7}$$

Here, the following results are valid:

(1) If $f(x) \in I_x^{(\alpha)}(a,b)$, one deduce to [12, 13]

$$ f(x) \in C_\alpha(a,b). \tag{2.8}$$

(2) If $a = b$, then we have [12, 13]

$$ {}_aI_a^{(\alpha)} f(x) = 0 . \tag{2.9}$$

(3) If $a < b$, then we have [12, 13]

$$ {}_aI_b^{(\alpha)} f(x) = -{}_bI_a^{(\alpha)} f(x). \tag{2.10}$$

(4) If there is the fractal dimension $\alpha = 0$, then we have [12, 13]

$$ {}_aI_a^{(0)} f(x) = f(x) . \tag{2.11}$$

(5) For any $f(x) \in C_\alpha(a,b), 0 < \alpha \le 1$, we have local fractional multiple integrals, which is written as[12]

$$ {}_{x_0}I_x^{(k\alpha)} f(x) = \overbrace{{}_{x_0}I_x^{(\alpha)} \cdots {}_{x_0}I_x^{(\alpha)}}^{k\ times} f(x) . \tag{2.12}$$

(6) If $\psi(x,y) \in C_\alpha(a,b) \times C_\alpha(c,d)$, then [12]

$$ {}_aI_b^{(\alpha)} {}_cI_d^{(\alpha)} \psi(x,y) = {}_cI_d^{(\alpha)} {}_aI_b^{(\alpha)} \psi(x,y). \tag{2.13}$$

(7) The sine sub-function can be written as [6, 7]

$$ \sin_\alpha x^\alpha := \sum_{k=0}^{\infty} (-1)^k \frac{x^{\alpha(2k+1)}}{\Gamma[1+\alpha(2k+1)]}, \ 0 < \alpha \le 1 . \tag{2.14}$$

2.3. Fractal geometrical explanation

Definition 5 Let a be an arbitrary but fixed real number. The integral staircase function $S_F^\alpha(x)$ of order α for a set F is given by [4, 11, 12]

$$ S_F^\alpha(x) = \begin{cases} \gamma^\alpha[F,a,x], if\ x \ge a; \\ -\gamma^\alpha[F,x,a], if\ x < a. \end{cases} \tag{2.15}$$

Then we have the following results:

(8) The fractal mass function $\gamma^\alpha[F,a,b]$ can be written as [11, 12]

$$ \gamma^\alpha[F,a,b] = \frac{1}{\Gamma(1+\alpha)} H^\alpha(F \cap (a,b)) = \frac{(b-a)^\alpha}{\Gamma(1+\alpha)}. \tag{2.16}$$

(9) We have [11, 12]

$$ S_F^\alpha(y) - S_F^\alpha(x) = \gamma^\alpha[F,x,y] = \frac{(y-x)^\alpha}{\Gamma(1+\alpha)}. \tag{2.17}$$

(10) If $a < b < c$, we have [4]

$$ \gamma^\alpha[F,a,b] + \gamma^\alpha[F,b,c] = \gamma^\alpha[F,a,c]. \tag{2.18}$$

Remark 1 From formula (2.16) we can write [11,12]

$$ \gamma^\alpha[F,a,b] = \frac{(b-a)^\alpha}{\Gamma(1+\alpha)}. \tag{2.19}$$

Remark 2 From formula (2.17) we deduce to $(b-a)^\alpha + (c-b)^\alpha = (c-a)^\alpha$. Hence, we can understand it by fractal geometry [12]:

$$ H^\alpha(F \cap (b-a)) + H^\alpha(F \cap (c-b)) = H^\alpha(F \cap (c-a)). \tag{2.20}$$

3. Picard's Successive Approximation Method

In this method, we set

$$ u_0(x) = f(x). \tag{3.1}$$

We give the first approximation $u_1(x)$ by

$$ u_1(x) = f(x) + \frac{\lambda^\alpha}{\Gamma(1+\alpha)} \int_0^x K(x,t) f(t) (dt)^\alpha . \tag{3.2}$$

Here, we find that $u_1(x)$ is local fractional continuous if $f(x)$, $K(x,t)$, and $u_0(x)$ are local fractional continuous.

The second approximation $u_2(x)$ can be obtained similarly by replacing $u_0(x)$ by $u_1(x)$ obtained above. And we find that

$$ u_2(x) = f(x) + \frac{\lambda^\alpha}{\Gamma(1+\alpha)} \int_0^x K(x,t) u_1(x) (dt)^\alpha . \tag{3.3}$$

Continuing in this manner, we have an infinite sequence of functions

$$ u_0(x), u_1(x), u_2(x), \cdots, u_n(x), \cdots \tag{3.4}$$

which satisfies the recurrence equations

$$ u_n(x) = f(x) + \frac{\lambda^\alpha}{\Gamma(1+\alpha)} \int_0^x K(x,t) u_{n-1}(x) (dt)^\alpha , \tag{3.5}$$

for $n = 1, 2, 3, \cdots$ and $u_0(x)$ is equivalent to any selected function, which is local fractional continuous. Hence, we have successive approximation as follows:

$$ u_1(x) = f(x) + \frac{\lambda^\alpha}{\Gamma(1+\alpha)} \int_0^x K(x,t) f(t) (dt)^\alpha , \tag{3.6}$$

$$ u_2(x) = f(x) + \frac{\lambda^\alpha}{\Gamma(1+\alpha)} \int_0^x K(x,t) u_1(x) (dt)^\alpha , \tag{3.7}$$

$$u_3(x)=f(x)+\frac{\lambda^\alpha}{\Gamma(1+\alpha)}\int_0^x K(x,t)u_2(x)(dt)^\alpha, \tag{3.8}$$

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$$u_{n-1}(x)=f(x)+\frac{\lambda^\alpha}{\Gamma(1+\alpha)}\int_0^x K(x,t)u_{n-2}(x)(dt)^\alpha, \tag{3.9}$$

$$u_n(x)=f(x)+\frac{\lambda^\alpha}{\Gamma(1+\alpha)}\int_0^x K(x,t)u_{n-1}(x)(dt)^\alpha. \tag{3.10}$$

Thus, at the limit, the solution $u(x)$ is written as

$$u(x)=\lim_{n\to\infty}u_n(x). \tag{3.11}$$

4. An Illustrative Paradigm

Solve the following linear Volterra integral equation

$$u(x)=\frac{x^\alpha}{\Gamma(1+\alpha)}+\frac{1}{\Gamma(1+\alpha)}\int_0^x\frac{(x-t)^\alpha}{\Gamma(1+\alpha)}u(t)(dt)^\alpha. \tag{4.1}$$

Let us set $u_0(x)=\frac{x^\alpha}{\Gamma(1+\alpha)}$, then the first approximation can be written as

$$\begin{aligned}u_1(x)&=\frac{x^\alpha}{\Gamma(1+\alpha)}+\frac{1}{\Gamma(1+\alpha)}\int_0^x\frac{(x-t)^\alpha}{\Gamma(1+\alpha)}u_0(t)(dt)^\alpha\\&=\frac{x^\alpha}{\Gamma(1+\alpha)}+\frac{1}{\Gamma(1+\alpha)}\int_0^x\frac{(x-t)^\alpha}{\Gamma(1+\alpha)}\frac{t^\alpha}{\Gamma(1+\alpha)}(dt)^\alpha\\&=\frac{x^\alpha}{\Gamma(1+\alpha)}-\frac{1}{\Gamma(1+\alpha)}\int_0^x\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}(dt)^\alpha\\&=\frac{x^\alpha}{\Gamma(1+\alpha)}-\frac{t^{3\alpha}}{\Gamma(1+3\alpha)}.\end{aligned} \tag{4.2}$$

The second approximation can be calculated in the similar way, which is

$$\begin{aligned}&u_2(x)\\&=\frac{x^\alpha}{\Gamma(1+\alpha)}+\frac{1}{\Gamma(1+\alpha)}\int_0^x\frac{(x-t)^\alpha}{\Gamma(1+\alpha)}u_1(t)(dt)^\alpha\\&=\frac{x^\alpha}{\Gamma(1+\alpha)}+\frac{1}{\Gamma(1+\alpha)}\int_0^x\frac{(x-t)^\alpha}{\Gamma(1+\alpha)}\left(\frac{x^\alpha}{\Gamma(1+\alpha)}-\frac{t^{3\alpha}}{\Gamma(1+3\alpha)}\right)(dt)^\alpha\\&=\frac{x^\alpha}{\Gamma(1+\alpha)}-\frac{x^{3\alpha}}{\Gamma(1+3\alpha)}-\\&\frac{1}{\Gamma(1+\alpha)}\int_0^x\frac{(x-t)^\alpha}{\Gamma(1+\alpha)}\frac{t^{3\alpha}}{\Gamma(1+3\alpha)}(dt)^\alpha\\&=\frac{x^\alpha}{\Gamma(1+\alpha)}-\frac{x^{3\alpha}}{\Gamma(1+3\alpha)}+\frac{1}{\Gamma(1+\alpha)}\int_0^x\frac{t^{4\alpha}}{\Gamma(1+4\alpha)}(dt)^\alpha\\&=\frac{x^\alpha}{\Gamma(1+\alpha)}-\frac{x^{3\alpha}}{\Gamma(1+3\alpha)}+\frac{x^{5\alpha}}{\Gamma(1+5\alpha)}.\end{aligned} \tag{4.3}$$

Proceeding in this way, we can obtain that

$$\begin{aligned}&u_n(x)\\&=\frac{x^\alpha}{\Gamma(1+\alpha)}+\frac{1}{\Gamma(1+\alpha)}\int_0^x\frac{(x-t)^\alpha}{\Gamma(1+\alpha)}u_{n-1}(x)(dt)^\alpha\\&=\sum_{n=0}^{n}(-1)^n\frac{x^{(2n+1)\alpha}}{\Gamma(1+(2n+1)\alpha)}.\end{aligned} \tag{4.4}$$

The final solution is

$$\begin{aligned}u(x)&=\lim_{n\to\infty}u_n(x)\\&=\lim_{n\to\infty}\left(\sum_{n=0}^{n}(-1)^n\frac{x^{(2n+1)\alpha}}{\Gamma(1+(2n+1)\alpha)}\right)\\&=\sin_\alpha x^\alpha.\end{aligned} \tag{4.5}$$

5 Conclusions

We investigated the local fractional integrals and its fractal geometrical explanation. Based on the local fractional integral operator, we derive the Picard's successive approximation method for solving a class of local fractional integral equation. Special attention is put on the approximation methodology for handling local fractional integral equations in a way for accessible to applied scientists and engineers. We give an illustrative paradigm to elaborate the accuracy and reliable results.

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Vitae

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