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The numerical solution of third-order boundary value problems by the modified decomposition method

Yahya Qaid Hasan
Department of Mathematics, Faculty of Applied Science, Thamar University
Thamar, 87246, Yemen
E-mail:yahya217@yahoo.com

Abstract- In this paper, an efficient modification of Adomian decomposition method (ADM) is introduced for solving third-order boundary value problems . A modified form of the (ADM) is applied to construct the numerical solution for such problems . The scheme is tested on one linear and two nonlinear problems. The obtained results demonstrate efficiency of the modified method.

Keywords- Adomian decomposition method; Three-point boundary value problem.
MSC: 65L05

1 Introduction

There are few studies with regard to numerical solutions of higher-order boundary value problems in literature[5,6-8,16]. The proofs are made the existence and uniqueness of these kind of problems by [4]. Several types of boundary value problems solved by this method which had Dirichlet, Neumann or Robin conditions through many works by Adomian [3,1],Adomian and Rach[2]. Recently, Deeba et al[7]used Adomian method for obtaining analytical and numerical solutions of Breatu equation. Wazwaz [10,11] employed Adomian method to solve boundary value problems with Dirichlet and Neumann conditions, and Wazwaz[12] presented a reliable algorithm for obtaining positive solutions for non linear boundary value problems. Wazwaz [13] has further justified the validity of using the decomposition method where mixed boundary conditions were used to obtain blow-up solutions. Wazwaz [14] presented numerical results of fifth order boundary value problems for using the decomposition method by Wazwaz and a comparison between the errors obtained by using the sixth-degree B-spline method. The results show that the decomposition method was more accurate and easy than B-spline method. In this paper, a new modification of the (ADM) is proposed to overcome difficulties occurred in the standard (ADM) for solving three-point boundary value problems, namely, the modified ADM (MADM). Main idea of the MADM is to create a canonical form containing all boundary conditions so that the zeroth component is explicitly determined without additional calculations and all other components are also easily determined.

2 Analysis of the method

Consider the third-order boundary value problems(BVPs) of the form

$$y^{'''}(x) = f(x, y), \tag{1}$$

with boundary conditions

$$y(a) = A, y^{'}(0) = B, y^{'}(b) = C, \tag{2}$$

or

$$y(0) = A, y^{'}(0) = B, y(a) = C, \tag{3}$$

or

$$y(0) = A, y(a) = B, y^{''}(b) = C, \tag{4}$$

where $f(x, y)$ is given continuous, linear or nonlinear function, and A, B, C, a, b are real finite constants. In an operator form,(1) can be written as

$$Ly = f(x, y), \tag{5}$$

where the differential operators L are given as

$$L(.) = x^{-1}\frac{d}{dx}x^2\frac{d}{dx}x^{-1}\frac{d}{dx}(.), \tag{6}$$

$$L(.) = x^{-1}\frac{d^2}{dx^2}x^3\frac{d}{dx}x^{-2}(.), \tag{7}$$

$$L(.) = \frac{d}{dx}x^{-1}\frac{d}{dx}x^2\frac{d}{dx}x^{-1}(.). \tag{8}$$

The inverse operators L^{-1} are defined respectively as

$$L^{-1}(.) = \int_a^x x \int_b^x x^{-2} \int_0^x x(.)dxdxdx, \tag{9}$$

$$L^{-1}(.) = x^2 \int_a^x x^{-3} \int_b^x \int_0^x x(.)dxdxdx, \tag{10}$$

$$L^{-1}(.) = x \int_a^x x^{-2} \int_0^x x \int_b^x (.)dxdxdx. \tag{11}$$

Operating with inverse operator(9),(10),(11) on (5) and using the boundary conditions(2),(3),(4) respectively, yields

$$y(x) = A - aB - \frac{Ca^2}{2b} + \frac{Ba^2}{2b} + Bx$$

$$+\frac{C-B}{2b}x^2 + L^{-1}(f(x, y)), \tag{12}$$

$$y(x) = A+Bx+(\frac{C}{a^2}-\frac{B}{a}-\frac{A}{a^2})x^2+L^{-1}(f(x, y)), \tag{13}$$

$$y(x) = A+(\frac{B}{a}-\frac{aC}{2}-\frac{A}{a})x+\frac{C}{2}x^2+L^{-1}(f(x, y)). \tag{14}$$

The Adomian decomposition method defines the solution $y(x)$ as the decomposition series

$$y(x) = \sum_{n=0}^{\infty} y_n(x), \tag{15}$$

where the components $y_n(x)$ will be determined recurrently. The nonlinear $f(x, y)$ is expressed as an infinite series of polynomials

$$f(x, y) = \sum_{n=0}^{\infty} A_n, \tag{16}$$

where A_n are Adomian polynomials that can be constructed by algorithm derived in [15]. Through using Adomian decomposition method, the component $y_n(x)$ can be determined respectively as

$$y_0 = A - aB - \frac{Ca^2}{2b} + \frac{Ba^2}{2b} + Bx + \frac{C-B}{2b}x^2,$$

$$y_{n+1} = L^{-1}A_n, n \geq 0, \tag{17}$$

$$y_0 = A + Bx + (\frac{C}{a^2} - \frac{B}{a} - \frac{A}{a^2})x^2,$$

$$y_{n+1} = L^{-1}A_n, n \geq 0, \tag{18}$$

$$y_0 = A + (\frac{B}{a} - \frac{aC}{2} - \frac{A}{a})x + \frac{C}{2}x^2,$$

$$y_{n+1} = L^{-1}A_n.n \geq 0, \tag{19}$$

After determination of specific number of the components y_k, the approximation

$$\Phi_n = \sum_{k=0}^{n} y_k$$

can be used to approximate solution of BVP.

3 Numerical examples

Example 1. Consider the linear BVP

$$y^{'''}(x) = y(x) - 3e^x, \quad 0 < x < 1, \tag{20}$$

$$y^{'}(0) = 0, y(1) = 0, y(0) = 1.$$

The exact solution is $y(x) = (1 - x)e^x$.

In an operator form(7), Eq.(20) becomes

$$Ly = y - 3e^x. \tag{21}$$

Applying the inverse operator (10)on (21) yields

$$y = 4 + 3x + (3e - 7)x^2 - 3e^x + L^{-1}y,$$

Using Adomian decomposition for $y(x)$ as given in(15)we obtain

$$\sum_{n=0}^{\infty} y_n(x) = 4+3x+(3e-7)x^2-3e^x+L^{-1}\sum_{n=0}^{\infty} y_n.$$

The components $y_n(x)$ can be recursively determined by using the relation

$$y_0 = 4+3x+(3e-7)x^2-3e^x,$$

$$y_{n+1} = L^{-1}y_n, \quad n \geq 0.$$

This in turn given

$$y_0 = 4+3x+(3e-7)x^2-3e^x,$$

$$y_1 = 3+3x+(\frac{59e}{20}-\frac{267}{40})x^2+\frac{2x^3}{3}$$

$$+\frac{x^4}{8}+(\frac{e}{20}-\frac{7}{60})x^5-3e^x,$$

$$\vdots$$

The numerical solution, Table 1, is very useful with only firs four components.

Example 2. Let us consider the following nonlinear BVP:

$$y^{'''}(x) = e^{-x}y^2(x), \quad 0 < x < 1, \tag{22}$$

$$y^{'}(0) = 1, y(0) = 1, y(1) = e.$$

The exact solution is $y(x) = e^x$.

In an operator form(6), Eq.(22) becomes

$$Ly = e^{-x}y^2(x). \tag{23}$$

Applying the inverse operator (9)on (23) yields

$$y = 1+x+\frac{(e-1)}{2}x^2+L^{-1}e^{-x}y^2,$$

Using Adomian decomposition for $y^2(x)$ as given in(15)we obtain

$$\sum_{n=0}^{\infty} y_n(x) = 1+x+\frac{(e-1)}{2}x^2+L^{-1}\sum_{n=0}^{\infty} e^{-x}A_n.$$

The components $y_n(x)$ can be recursively determined by using the relation

$$y_0 = 1+x+\frac{(e-1)}{2}x^2,$$

$$y_{n+1} = L^{-1}e^{-x}A_n, \quad n \geq 0.$$

Using Taylor series of e^{-x} with order 9.

Numerical results calculated with first three components are listed in Table 2.

Example 3. Consider the nonlinear BVP

$$y^{'''}(x) = -e^x y^2, \quad 0 < x < 1, \tag{24}$$

subject to the boundary conditions

$$y(0) = 1, y^{''}(0) = 1, y(1) = \frac{1}{e}.$$

The exact solution is $y(x) = e^{-x}$.

In an operator form(8), Eq.(24) becomes

$$Ly = -e^x y^2(x). \tag{25}$$

Applying the inverse operator (11)on (25) yields

$$y = 1+x(\frac{1}{e}-\frac{3}{2})+\frac{1}{2}x^2-L^{-1}e^x y^2,$$

Using Adomian decomposition for $y^2(x)$ as given in(15)we obtain

$$\sum_{n=0}^{\infty} y_n(x) = 1+x(\frac{1}{e}-\frac{3}{2})+\frac{1}{2}x^2-L^{-1}\sum_{n=0}^{\infty} e^x A_n.$$

The components $y_n(x)$ can be recursively determined by using the relation

$$y_0 = 1+x(\frac{1}{e}-\frac{3}{2})+\frac{1}{2}x^2,$$

$$y_{n+1} = -L^{-1}e^x A_n, \quad n \geq 0.$$

Using Taylor series of e^x with order 9.

Numerical results calculated with first three components are listed in Table 3.

4 Conclusions

ADM has been successful for solving many application problems with simple calculations. However, it has difficulties in dealing with boundary conditions for solving three-point boundary problems. Many approaches have been presented to overcome these difficulties. However, they require additional computational work since all boundary conditions are not included in the canonical form. Our fundamental goal is to create the canonical form containing all boundary conditions. This goal has been achieved in the new modified ADM. The MADM does not require us to calculate the unknown constant which is usually a derivative at the boundary. From the results in illustrative examples, it is concluded that MADM is very effective algorithm which provides promising results with simple calculations.

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