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Note on some characters for fractional (*g*,*f*,*n*)-critical graphs

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Abstract –Many problems in intelligent transportation systems can be modeled by fractional factor. A graph *G* is called a fractional (*g*, *f* ,*n*)-critical graph, if any *n* vertices is removed from *G*, then the remaining graph remain has a fractional (*g*,*f*)-factor. In this paper, several characters for fractional (*g*, *f* ,*n*)-critical graphs are given.

Keywords –Factor; Fractional (*g*, *f*)-factor; Fractional (*g*, *f* ,*n*)-critical graph

1. Introduction

Let $G=(V(G),E(G))$ be a graph with vertex set $V(G)$ and edge set $E(G)$. Let $d_G(x)$ be the degree of x in G and $\delta(G)$ be the minimum vertex degree of G. For a vertex set $S\subseteq V(G)$, the subgraph of G induced by S is denoted by $G[S]$, $i(G-S)$ and $c(G-S)$ are used for the number of isolated vertices and the number of components in $G-S$, respectively. A subset I of $V(G)$ is an independent set if no two vertices of I are adjacent in G and a set C of $V(G)$ is a covering set if every edge of G is incident to a vertex in C. For any two subsets S, $T\subseteq V(G)$, we denote $E(S,T)$ ={ $uv\in E(G): u\in S, v\in T$ }. The notation and terminology used but undefined in this paper can be found in [1].

Suppose that g and f are two integer-valued functions on $V(G)$ such that $0\le g(x)\le f(x)$ for all $x\in V(G)$. A fractional (g,f)-factor is a function h that assigns to each edge of a graph G a number in [0,1] so that for each vertex x we have $g(x) \le d_G^h(x) \le f(x)$, where $d_G^h(x)=\sum_{e\in E(x)} h(e)$ is called the fractional degree of x in G. If $g(x)=a$ and $f(x)=b$ for all $x\in V(G)$, then a fractional (g,f)-factor is just a fractional $[a,b]$-factor. If $g(x)=f(x)=k$ ($k\ge 1$ is an integer) for all $x\in V(G)$, then a fractional (g,f)-factor is just a fractional k-factor. A graph G is called a fractional (g, f ,n)-critical graph, if any n vertices is removed from G, then the remaining graph remain has a fractional (g,f)-factor. A graph G is called a fractional (g, f)-deleted graph, if any edge e is removed from G, then the remaining graph remain has a fractional (g,f)-factor. If $g(x)=a$ and $f(x)=b$ for all $x\in V(G)$, then fractional (g,f,n)-critical graph and fractional (g,f)-deleted graph are just a fractional $[a,b,n]$-critical graph and $[a,b]$-deleted graph, respectively. If $g(x)=f(x)=k$ ($k\ge 1$ is an integer) for all $x\in V(G)$, then fractional (g,f,n)-critical graph and fractional (g,f)-deleted graph are just a fractional $[k,n]$-critical graph and k-deleted graph, respectively.

Many networks problems in the real-world can be modeled by fractional factor. In such a network, an important example of is an intelligent transportation systems with vertices and edges modeling cities and communication channels, respectively. Other examples are the railroad network with vertices and edges representing railroad stations and railways between two stations, respectively.

Heinrich et al. [2] gave a necessary and sufficient condition of ($g<f$)-factors given by

Theorem 1. [2] Let $g(x)$ and $f(x)$ be non-negative integral-valued functions defined on $V(G)$. If either one of the following conditions holds

(i) $g(x) < f(x)$ for every vertex $x\subseteq V(G)$;

(ii) G is bipartite;

then G has a (g, f)-factor if and only if for any set S of $V(G)$

$$g(T)-d_{G-S}(T)\le f(S)$$

Where

$$T=\{x: x\in V(G)-S, d_{G-S}(x)\le g(x)\}.$$

In the above theorem, to confirm a graph possessing (g, f)-factors, we need only to verify the much simpler inequality above for every vertex set S, in contrast with the verification of a more complex inequality for all possible pair of disjoint vertex sets (S ,T) in Lovasz's original characterization of general (g, f)-factor. This simpler criterion enables us to deal with factor problems with additional properties.

Let $g(x)=a<b=f(x)$ in Theorem 1, it yields a necessary and sufficient condition for existence of $[a, b]$-factors.

Theorem 2. [3] Let G be a graph and let $a<b$ be two positive integers. Then G has an $[a, b]$-factor if and only if for any $S\subseteq V(G)$,

$$a|T|-d_{G-S}(T)\le b|S|$$

holds, where

$$T=\{x:x\in V(G)-S,d_{G-S}(x)\le a\text{-}1\}.$$

Then G has $[a, b]$-factors.

As for fractional (g,f)-factor, Liu and Zhang [4] obtained the necessary and sufficient condition for existence of fractional (g,f)-factors.

Theorem 3. [4] Let $g(x)$ and $f(x)$ be non-negative integral-valued functions defined on $V(G)$ satisfying $g(x)\le f(x)$ for all $x\in V(G)$. G has a fractional (g,f)-factor if and only if for any subset S of $V(G)$

$$g(T)-d_{G-S}(T)\le f(S).$$

Where

$$T=\{x:x\in V(G)-S,d_{G-S}(x)\le g(x)\}.$$

Liu [5] gave the necessary and sufficient condition for fractional (g,f,n)-critical graphs.

Theorem 4. [5] Let $g(x)$ and $f(x)$ be non-negative integral-valued functions defined on $V(G)$ satisfying $g(x)\le f(x)$ for all $x\in V(G)$. Let n be a positive integer. G is a fractional (g,f,n)-critical graph if and only if for any subset S of $V(G)$ with $|S|\ge n$,

$$g(T)-d_{G-S}(T)\le f(S)-f_n(S)$$

Where

$$T=\{x:x\in V(G)-S,d_{G-S}(x)\le g(x)\},$$

and

$$f_n(S)=\max\{f(U):U\subseteq S,|U|=n\}.$$

In this paper, we obtain some results for fractional (g,f,n)-critical graph.

2. Main results and proofing

Throughout the paper, we always assume that a,b and n are positive integers satisfying $1\le a<b$. So we will not reiterate these condition again in the theorems or proofs. The main results in this paper are given below.

Theorem 5. Let G be a graph and $g(x)$, $f(x)$ be two non-negative integral-valued functions defined on $V(G)$ satisfying $a\le g(x)<f(x)\le b$ for all $x\in V(G)$. $\delta(G)\ge b+n$. If, for any arbitrary n-subset $V'\subset V(G)$, $G-V'$ has fractional (g,f)-factors, then, for any $(n\text{-}1)$ -subset $V''\subset V(G)$, $G-V''$ has fractional (g,f)-factors as well.

The second result present a different type of sufficient condition for the existence of fractional (g,f)-factors excluding any edge of $E(G)$.

Theorem 6. Let G be a graph and $g(x)$, $f(x)$ be two non-negative integral-valued functions defined on $V(G)$ satisfying $a\le g(x)<f(x)\le b$ for all $x\in V(G)$. $\delta(G)\ge b+2$. If $G-\{x,y\}$ has fractional (g,f)-factors for every pair of vertices $x,y\in V(G)$ then $G-e$ has fractional (g,f)-factors for any given edge $e\in E(G)$.

Proof Theorem 5. We verify the theorem for the case of $n=1$ first, i.e., the following claim:

Claim. If $G-x$ has fractional (g,f)-factors for any $x\in V(G)$, then G has fractional (g,f)-factors.

Otherwise, G has no fractional (g,f)-factors and thus, by Theorem 3, there exists $S\subset V(G)$ such that $b|T|-d_{G-S}(T)>a|S|$, Where

$$T=\{x:x\in V(G)-S,d_{G-S}(x)\le b\text{-}1\}.$$

Choose a vertex v from S, let $S'=S-\{v\}$, then

$$(G-\{v\})\text{-}S'=G-S,$$

and

$$\{x:x\in V(G-v)\text{-}S',d_{G-S}\text{-}S'(x)\le b\text{-}1\}=T.$$

Therefore we have

$$b|T|-d_{G-S}(T)\le a|S'|=a|S|\text{-}a<a|S|$$

since G-v has fractional (g,f)-factors, a contradiction since $b|T|-d_{G-S}(T)>a|S|$. Hence, G has fractional (g,f)-factors.

Applying the above claim and using induction arguments, we can see that $G-V''$ has fractional (g,f)-factors for any $n-1$ subset V'' if $G-V'$ has fractional (g,f)-factors for any n-subset V'.

Next we present a characterization for fractional (g,f)-factors excluding an edge. As an application, Theorem 6 can be easily derived from it. In fact, the lemma itself is of interest.

Lemma 1. [6] Let G be a graph and $e=uv$ be any edge of G. $g(x)$, $f(x)$ be two non-negative integral-valued functions defined on $V(G)$ satisfying $a\le g(x)<f(x)\le b$ for all $x\in V(G)$. Then G has fractional (g,f)-factors excluding the edge e if and only if

$$f(S)-g(T)+d_{G-S}(T)\ge\rho(S) \tag{1}$$

holds for any $S\subset V(G)$, where $G'=G-e$, $T'=\{x:x\in V(G)-S,d_{G'-S}(x)\le g(x)\}$, and

$$\rho(S)=\begin{cases}2,\ \{u,v\}\subseteq T\\ 1,\ \text{one of}\{u,v\}\text{lies in } T' \text{ and the other is in } G\text{-}(S\cup T')\\ 0,\ \text{otherwise.}\end{cases}$$

Proof of Theorem 6. Let S be any subset of $V(G)$.

If $S=\phi$ then $T=\phi$ and

$$f(S)-g(T)+d_{G-S}(T)=0=\rho(S).$$

If $|S|=1$, then $|T|=0$ (since $\delta(G)\ge b+2$) and thus

$$f(S)-g(T)+d_{G-S}(T)=a|S|=a\ge 1.$$

If $|S| \geq 2$, then there exist vertices $x,y \in S$. Let $V' = \{x,y\}$ in Lemma 1, since $G - \{x,y\}$ has fractional (*g*,*f*)-factors, then we have

$$f(S) - g(T) + d_{G-S}(T) \geq 2a \geq 2 .$$

Therefore, we conclude (1) for any $S \subset V(G)$. By Lemma 1, $G - e$ has fractional (*g*,*f*)-factors.

3. Conclusions

In this paper, two characters for fractional (*g*, *f* ,*n*)-critical graphs are given. The first character shows that under the condition $\delta(G) \geq b + n$. If *G* is a fractional (*g*, *f* ,*n*)-critical graph, then *G* is a fractional (*g*, *f* ,*n*-1)-critical graph as well. The second character shows that under the condition $\delta(G) \geq b + n$. If *G* is a fractional (*g*, *f* ,2)-critical graph, then *G* is a fractional (*g*,*f*)-deleted graph.

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Vitae

Wei Gao, male, was born in the city of Shaoxing, Zhejiang Province, China on Feb.13, 1981. He got two bachelor degrees on computer science from Zhejiang industrial university in 2004 and mathematics education form College of Zhejiang education in 2006. Then, he enrolled in department of computer science and information technology, Yunnan normal university, and got Master degree there in 2009. He is PhD student in department of Mathematics, Soochow University, from September 2009 to June 2012. Then, he got PHD degree in June 2012.

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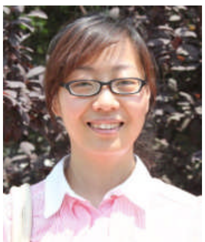

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