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Fast Yang-Fourier Transforms in Fractal Space

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Abstract –The Yang-Fourier transform (YFT) in fractal space is a generation of Fourier transform based on the local fractional calculus. The discrete Yang-Fourier transform (DYFT) is a specific kind of the approximation of discrete transform based on the Yang-Fourier transform in fractal space. In the present letter we point out a new fractal model for the algorithm for fast Yang-Fourier transforms of discrete Yang-Fourier transforms. It is shown that the classical fast Fourier transforms is a special example in fractal dimension $\alpha = 1$.

Keywords –Yang-Fourier transforms; Fast Yang-Fourier transforms; Discrete Yang-Fourier transforms; Fractal space; Local fractional calculus

1. Introduction

Local fractional calculus (fractal calculus) has become a hot topic in both mathematics and engineering [1-15]. Here we give the definition of local fractional derivative [14-19]

$$f^{(\alpha)}(x_0) = \frac{d^\alpha f(x)}{dx^\alpha}\Big|_{x=x_0} = \lim_{x\to x_0}\frac{\Delta^\alpha\left(f(x)-f(x_0)\right)}{(x-x_0)^\alpha} \quad (1.1)$$

with $\Delta^\alpha\left(f(x)-f(x_0)\right) \cong \Gamma(1+\alpha)\Delta\left(f(x)-f(x_0)\right)$ and the definition of local fractional integral [14-19, 27]

$$\frac{1}{\Gamma(1+\alpha)}\int_a^b f(t)(dt)^\alpha = \frac{1}{\Gamma(1+\alpha)}\lim_{\Delta t\to 0}\sum_{j=0}^{j=N-1} f(t_j)(\Delta t_j)^\alpha \quad (1.2)$$

with $\Delta t_j = t_{j+1} - t_j$ and $\Delta t = \max\{\Delta t_1, \Delta t_2, \Delta t_j, \ldots\}$, where for $j = 0,\ldots,N-1$, $[t_j, t_{j+1}]$ is a partition of the interval $[a,b]$ and $t_0 = a, t_N = b$.

Recently, both Yang-Fourier transform (also local fractional Fourier transform) was shown by [14-15, 17, 21-22, 24]

$$\begin{aligned} &F_\alpha\{f(x)\} \\ &= f_\omega^{F,\alpha}(\omega) \\ &= \frac{1}{\Gamma(1+\alpha)}\int_{-\infty}^{\infty} E_\alpha\left(-i^\alpha\omega^\alpha x^\alpha\right) f(x)(dx)^\alpha \end{aligned} \quad (1.3)$$

and the inverse representation was in the form [14-15, 21-22, 24]

$$\begin{aligned} &f(x) \\ &= F_\alpha^{-1}\left(f_\omega^{F,\alpha}(\omega)\right) \\ &= \frac{1}{(2\pi)^\alpha}\int_{-\infty}^{\infty} E_\alpha\left(i^\alpha\omega^\alpha x^\alpha\right) f_\omega^{F,\alpha}(\omega)(d\omega)^\alpha \end{aligned} \quad . \quad (1.4)$$

Furthermore, both Yang-Laplace transform (also local fractional Laplace transform), [14-15, 18, 25, 26]

$$\begin{aligned} &L_\alpha\{f(x)\} \\ &= f_s^{L,\alpha}(s) \\ &= \frac{1}{\Gamma(1+\alpha)}\int_0^\infty E_\alpha\left(-s^\alpha x^\alpha\right) f(x)(dx)^\alpha, 0<\alpha\le 1 \end{aligned} \quad (1.5)$$

and inversion [14-15, 25, 26]

$$\begin{aligned} &L_\alpha^{-1}\left(f_s^{L,\alpha}(s)\right) \\ &= f(t) \\ &= \frac{1}{(2\pi)^\alpha}\int_{\beta-i\infty}^{\beta+i\infty} E_\alpha\left(s^\alpha x^\alpha\right) f_s^{L,\alpha}(s)(ds)^\alpha, \end{aligned} \quad (1.6)$$

were introduced. Moreover, the discrete Yang-Fourier transform (shortly called DYFT) was given in the form [20, 23]

$$F(k) = \sum_{n=0}^{N-1} f(n) W_{N,\alpha}{}^{-nk} \quad (1.7)$$

and inversion was read as [20, 23]

$$f(n) = \frac{1}{\Gamma(1+\alpha)}\frac{1}{N^\alpha}\sum_{k=0}^{N-1} F(k) W_{N,\alpha}{}^{kn} \quad (1.8)$$

with $W_{N,\alpha}{}^{kn} = E_\alpha\left(\frac{i^\alpha n^\alpha k^\alpha (2\pi)^\alpha}{N^\alpha}\right)$. Here, aim of this letter is to suggest a new model for the fast Yang-Fourier transforms based on the discrete Yang-Fourier transforms.

This letter is organized as follows: In section 2, the fast Yang-Fourier transform of discrete Yang-Fourier transform is given. In section 3, the fast Yang-Fourier transform of inverse discrete Yang-Fourier transform is considered. Conclusions are shown in section 4.

2. Fast Yang-Fourier transform of discrete Yang-Fourier transform

In this section we start with the fast Yang-Fourier transform of Yang-Fourier transform. The relations

$$\left[F_N\right]^{\alpha}_{-n,k+1}=\frac{1}{N^{\alpha}}W_{N,\alpha}{}^{-(k+1)n}$$
$$=\frac{1}{N^{\alpha}}W_{N,\alpha}{}^{-kn}W_{N,\alpha}{}^{-n}=\left[F_N\right]^{\alpha}_{-n,k}W_{N,\alpha}{}^{-n} \quad (2.1)$$

and

$$\left[F_N\right]^{\alpha}_{n,k+1}=\frac{1}{N^{\alpha}}W_{N,\alpha}{}^{(k+1)n}$$
$$=\frac{1}{N^{\alpha}}W_{N,\alpha}{}^{kn}W_{N,\alpha}{}^{n}=\left[F_N\right]^{\alpha}_{n,k}W_{N,\alpha}{}^{n} \quad (2.2)$$

are the component formulas for the Yang-Fourier transform.

Suppose that $\{V_0,V_1,V_2,...,V_{N-1}\}$ is the N_{th} order discrete Yang-Fourier transforms of $\{v_0,v_1,v_2,...,v_{N-1}\}$. Starting with the component formulas for the discrete Yang-Fourier transform, we obtain that, for $n=0,1,2,...,N-1$,

$$
\begin{aligned}
&V_n\\
&=\sum_{k=0}^{N-1}W_{N,\alpha}{}^{-(k+1)n}v_k\\
&=\sum_{\substack{k=0\\k-even}}^{N-1}W_{N,\alpha}{}^{-(k+1)n}v_k+\sum_{\substack{k=0\\k-odd}}^{N-1}W_{N,\alpha}{}^{-(k+1)n}v_k\\
&=\frac{1}{2^{\alpha}}\left(\sum_{j=0}^{M-1}W_{2M,\alpha}{}^{-n(2j)}v_{2j}+\sum_{j=0}^{M-1}W_{2M,\alpha}{}^{-n(2j+1)}v_{2j+1}\right)\\
&=\frac{1}{2^{\alpha}}\left(\sum_{j=0}^{M-1}W_{2M,\alpha}{}^{-n(2j)}v_{2j}+W_{M,\alpha}{}^{\frac{n}{2}}\sum_{j=0}^{M-1}W_{2M,\alpha}{}^{-n(2j)}v_{2j+1}\right).
\end{aligned}
$$

and we have the following relation

$$\left[F_{NV}\right]^{\alpha}_{n}=\frac{1}{2^{\alpha}}\left(\left[F_{NV_E}\right]^{\alpha}_{n}+W_{M,\alpha}{}^{-\frac{n}{2}}\left[F_{NV_0}\right]^{\alpha}_{n}\right), \quad (2.3)$$

where V is the sequence vector corresponding to $\{V_0,V_1,V_2,...,V_{N-1}\}$, V_E is the $M-th$ order sequence of even-index v_k's $\{V_0,V_2,...,V_{N-2}\}$ and V_O is the $M-th$ order sequence of odd-index v_k's $\{V_1,V_3,...,V_{N-1}\}$.

Here we can deduce that

$$
\begin{aligned}
&W_{M,\alpha}{}^{-(M+l)}\\
&=E_{\alpha}\left(-i^{\alpha}\left(\frac{2\pi}{M}\right)^{\alpha}\left(M+l\right)^{\alpha}\right)\\
&=E_{\alpha}\left(-i^{\alpha}\left(\frac{2\pi}{M}\right)^{\alpha}l^{\alpha}\right)\\
&=W_{M,\alpha}{}^{-l}
\end{aligned}
\quad (2.4)
$$

and

$$
\begin{aligned}
&W_{M,\alpha}{}^{-\frac{M+l}{2}}\\
&=E_{\alpha}\left(-i^{\alpha}\left(\frac{\pi}{M}\right)^{\alpha}\left(M+l\right)^{\alpha}\right)\\
&=-E_{\alpha}\left(-i^{\alpha}\left(\frac{\pi}{M}\right)^{\alpha}l^{\alpha}\right)\\
&=W_{M,\alpha}{}^{-\frac{l}{2}}
\end{aligned}
. \quad (2.5)
$$

Hence for $l=0,1,2,...,m-1$,

$$
\begin{aligned}
&V_l\\
&=\frac{1}{2^{\alpha}}\left(\sum_{j=0}^{M-1}W_{M,\alpha}{}^{-lj}v_{2j}+W_{M,\alpha}{}^{-\left(\frac{l}{2}\right)j}\sum_{j=0}^{M-1}W_{M,\alpha}{}^{-lj}v_{2j+1}\right) \quad (2.6)\\
&=\frac{1}{2^{\alpha}}\left(\left[F_{MV_E^{-1}}\right]^{\alpha}_{l}+W_{M,\alpha}{}^{-\left(\frac{l}{2}\right)j}\left[F_{MV_0^{-1}}\right]^{\alpha}_{l}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
&V_{M+l}\\
&=\frac{1}{2^{\alpha}}\left(\sum_{j=0}^{M-1}W_{M,\alpha}{}^{-lj}v_{2j}-W_{M,\alpha}{}^{\left(\frac{l}{2}\right)j}\sum_{j=0}^{M-1}W_{M,\alpha}{}^{-lj}v_{2j+1}\right). \quad (2.7\\
&=\frac{1}{2^{\alpha}}\left(\left[F_{MV_E^{-1}}\right]^{\alpha}_{l}-W_{M,\alpha}{}^{-\left(\frac{l}{2}\right)j}\left[F_{MV_0^{-1}}\right]^{\alpha}_{l}\right)
\end{aligned}
$$
)

Here, formulas (2.6) and (2.7) contain common elements that can be computed once for each l and then used to compute both V_l and V_{M+l}. Hence we can obtain the total number of computations to find all the V_n's. That is to say, this process of increasing levels to our algorithm can be continued to the K^{th} level provided to $N=2^K N_0$ for some integer N_0. Moreover, that integer, $N_0=2^{-K}N$ will also be the order of the discrete Yang-Fourier transforms and inverse discrete Yang-Fourier transforms. If $N=2^K$, it is this final K^{th} level algorithm, fully implemented and refined, that is called a fast Yang-Fourier transform of the discrete Yang-Fourier transforms.

3. Fast Yang-Fourier transform of inverse discrete Yang-Fourier transform

In this section we start with the fast Yang-Fourier transform of inverse Yang-Fourier transform. Similarly, suppose that $\{V_0^{-1},V_1^{-1},...,V_{N-1}^{-1}\}$ is the N_{th} order discrete Yang-Fourier transforms of $\{v_0^{-1},v_1^{-1},...,v_{N-1}^{-1}\}$, starting with the component formulas for the inverse discrete

Yang-Fourier transform, we obtain that, for $n=0,1,2,...,N-1,$

$$
\begin{aligned}
&V_n^{-1}\\
&=\frac{1}{\Gamma(1+\alpha)}\frac{1}{N^{\alpha}}\sum_{k=0}^{N-1}W_{N,\alpha}^{(k+1)n}v_k^{-1}\\
&=\frac{1}{\Gamma(1+\alpha)}\frac{1}{N^{\alpha}}\left(\sum_{\substack{k=0\\k-even}}^{N-1}W_{N,\alpha}^{(k+1)n}v_k^{-1}+\sum_{\substack{k=0\\k-odd}}^{N-1}W_{N,\alpha}^{(k+1)n}v_k^{-1}\right)\\
&=\frac{1}{\Gamma(1+\alpha)}\frac{1}{(2M)^{\alpha}}\left(\sum_{j=0}^{M-1}W_{2M,\alpha}^{n(2j)}v_{2j}^{-1}+\sum_{j=0}^{M-1}W_{2M,\alpha}^{n(2j+1)}v_{2j+1}^{-1}\right)\\
&=\frac{1}{\Gamma(1+\alpha)}\frac{1}{(2M)^{\alpha}}\left(\sum_{j=0}^{M-1}W_{2M,\alpha}^{n(2j)}v_{2j}^{-1}+W_{M,\alpha}^{\frac{n}{2}}\sum_{j=0}^{M-1}W_{2M,\alpha}^{n(2j)}v_{2j+1}^{-1}\right).
\end{aligned}
\tag{3.1}
$$

and we have the following relation

$$
\left[F_{NV}\right]_n^{\alpha}=\frac{1}{\Gamma(1+\alpha)}\frac{1}{(2M)^{\alpha}}\left(\left[F_{NV_E^{-1}}\right]_n^{\alpha}+W_{M,\alpha}^{\frac{n}{2}}\left[F_{NV_O^{-1}}\right]_n^{\alpha}\right),
\tag{3.2}
$$

where V^{-1} is the sequence vector corresponding to $\left\{V_0^{-1},V_1^{-1},V_2^{-1},...,V_{N-1}^{-1}\right\}$, V_E^{-1} is the $M-th$ order sequence of even-index $v_k^{-1}{}'s$ $\left\{V_0^{-1},V_2^{-1},...,V_{N-2}^{-1}\right\}$ and V_O^{-1} is the $M-th$ order sequence of odd-index $v_k^{-1}{}'s$ $\left\{V_1^{-1},V_3^{-1},...,V_{N-1}^{-1}\right\}$.

Here we can deduce that

$$
\begin{aligned}
&W_{M,\alpha}^{M+l}\\
&=E_{\alpha}\left(i^{\alpha}\left(\frac{2\pi}{M}\right)^{\alpha}(M+l)^{\alpha}\right)\\
&=E_{\alpha}\left(i^{\alpha}\left(\frac{2\pi}{M}\right)^{\alpha}l^{\alpha}\right)\\
&=W_{M,\alpha}^{l}
\end{aligned}
\tag{3.3}
$$

and

$$
\begin{aligned}
&W_{M,\alpha}^{\frac{M+l}{2}}\\
&=E_{\alpha}\left(i^{\alpha}\left(\frac{\pi}{M}\right)^{\alpha}(M+l)^{\alpha}\right)\\
&=-E_{\alpha}\left(i^{\alpha}\left(\frac{\pi}{M}\right)^{\alpha}l^{\alpha}\right)\\
&=W_{M,\alpha}^{\frac{l}{2}}
\end{aligned}
\tag{3.4}
$$

Hence for $l=0,1,2,...,m-1$,

$$
\begin{aligned}
&V_l^{-1}\\
&=\frac{1}{\Gamma(1+\alpha)}\frac{1}{(2M)^{\alpha}}\left(\sum_{j=0}^{M-1}W_{M,\alpha}^{lj}v_{2j}+W_{M,\alpha}^{\left(\frac{l}{2}\right)j}\sum_{j=0}^{M-1}W_{M,\alpha}^{lj}v_{2j+1}\right)\\
&=\frac{1}{\Gamma(1+\alpha)}\frac{1}{2^{\alpha}}\left(\left[F_{MV_E}\right]_l^{\alpha}+W_{M,\alpha}^{\left(\frac{l}{2}\right)j}\left[F_{MV_0}\right]_l^{\alpha}\right)
\end{aligned}
\tag{3.5}
$$

and

$$
\begin{aligned}
&V_{M+l}^{-1}\\
&=\frac{1}{\Gamma(1+\alpha)}\frac{1}{(2M)^{\alpha}}\left(\sum_{j=0}^{M-1}W_{M,\alpha}^{lj}v_{2j}-W_{M,\alpha}^{\left(\frac{l}{2}\right)j}\sum_{j=0}^{M-1}W_{M,\alpha}^{lj}v_{2j+1}\right)\\
&=\frac{1}{\Gamma(1+\alpha)}\frac{1}{2^{\alpha}}\left(\left[F_{MV_E}\right]_l^{\alpha}-W_{M,\alpha}^{\left(\frac{l}{2}\right)j}\left[F_{MV_0}\right]_l^{\alpha}\right)
\end{aligned}
. \tag{3.6}
$$

It is shown that, formulas (2.12) and (2.13) contain common elements that can also be computed once for each l and then used to compute both V_l^{-1} and V_{M+l}^{-1}. These can also yield the total number of computations to find all the $V_n^{-1}{}'s$. That is to say, this process of increasing levels to our algorithm of inverse discrete Yang-Fourier transforms is similar to that of the discrete Yang-Fourier transforms. Taking into account the relation $N=2^K$, it is also this final K^{th} level algorithm, fully implemented and refined, that is called a fast Yang-Fourier transform of the inverse discrete Yang-Fourier transforms.

3. Conclusions

In the present letter we suggest the fast algorithm for the discrete Yang-Fourier transform (DYFT), which is a specific kind of the approximation of discrete transform based on the Yang-Fourier transform in fractal space[20, 23]. Here, we call the fast Yang-Fourier transform. Moreover, it is shown that the classical fast Fourier transforms is a special example in fractal dimension $\alpha=1$. Based on the fast Yang-Fourier transform, we may structure a new algorithm for the generalized Fourier transforms in fractal space.

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