

Control of 3D Julia Sets Generated by Commutative Hypercomplex Dynamical Systems

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Abstract – Proper affine transformations are applied to commutative hypercomplex dynamical systems. The whole magnification, minification, stretching or shrinking along the axes and the general affine transformations including sheering and rotational transformation of 3D Julia sets generated by the commutative hypercomplex dynamical systems are achieved. It is also shown that the Julia sets of the transformed dynamical systems preserve the same topological properties. The proposed method overcomes the disadvantage of image resolution degrading issue and provides a simple and reliable approach to generate high resolution image.

Keywords – Computer application; Control of Julia set; Commutative hypercomplex; Dynamical system.

1. Introduction

Gaston Julia, a famous French mathematician investigated the iteration process of a complex function intensively in 1918 and attained the Julia set, a very important and useful concept [1]. The Julia set has been applied widely in physics, biology, mathematics, engineering, etc. and has been attracted much research interest [2–6]. One can generate beautiful Julia sets by iterating dynamical systems $z_{n+1} = f(z_n, c)$ on complex plane for given complex c . Variant c will yield different fractal sets with variant shapes, showing us a new symmetric fractal world and attracting more and more researchers to investigate the inherent characteristics of fractal geometry. It could be argued that Julia sets have been intensively studied, but most of these conclusions are just focused on the drawing of Julia sets and their characteristics of different types of functions [7–11]. As for the control of Julia sets via geometrical transformations, there are few references. As a matter of fact, thanks to the resolution-independent nature of fractal sets, it is important and applicable to control fractal sets on complex plane and hypercomplex plane by means of topological transformations acting on the considered dynamical systems. Sui and Liu firstly proposed the control of Julia sets on plane; they showed the invariant topological property of two-dimensional Julia sets by complicated techniques [12]. In this paper, we will consider the control of commutative hypercomplex dynamical systems. Commutative hypercomplex possesses different algebraic structure from both complex and quaternion. The corresponding fractals are completely different as well [13]. It is worth studying the control issue of the Julia sets of commutative hypercomplex dynamical systems. We will utilize generic topological matrix transformations to act on the commutative hypercomplex dynamical systems. The proposed topologically equivalent algebraic transformations

will render the resolutions of the generated fractal sets invariant and therefore overcome the degraded problem by conventional geometrical transformations. Meanwhile, the proofs presented here are simple and can be easily generalized to generic topological cases, surmounting the tedious techniques used in Ref. [12].

2. Commutative hypercomplex numbers

Commutative hypercomplex numbers have four distinct components. For convenience, we shall use the notation $z = x + yi + rj + sk$, or equivalently, $z = (x, y, r, s)^T$ to represent a commutative hypercomplex number, where $x, y, r, s \in R$. The four imaginary basis vectors satisfy the following relations [13]

$$ii = jj = -1, kk = 1, ij = ji = k,$$

$$ik = ki = -j, kj = jk = -i.$$

Two commutative hypercomplex numbers z_1 and z_2 are then provided the addition and multiplication operation by

$$\begin{aligned} z_1 + z_2 &= (x_1, y_1, r_1, s_1)^T + (x_2, y_2, r_2, s_2)^T \\ &= (x_1 + x_2, y_1 + y_2, r_1 + r_2, s_1 + s_2)^T, \end{aligned}$$

$$\begin{aligned} z_1 \times z_2 &= (x_1x_2 - y_1y_2 - r_1r_2 + s_1s_2, x_1y_2 + x_2y_1 - r_1s_2 - r_2s_1, \\ &\quad x_1r_2 + x_2r_1 - y_1s_2 - y_2s_1, x_1s_2 + x_2s_1 + y_1r_2 + y_2r_1)^T. \end{aligned}$$

It follows from the definition that the multiplication of commutative hypercomplex numbers is different from quaternions. Actually, two commutative hypercomplex numbers are commutative while two quaternions are not.

3. The control of Julia sets derived by commutative hypercomplex dynamical systems

For the sake of simplicity, we rewrite commutative hypercomplex dynamical systems $z_{n+1} = f(z_n, c)$ as $z_{n+1} = f(z_n)$. The Julia sets generated by $z_{n+1} = f(z_n)$ always possess the following characteristics [10]:

(1) If $f(z)$ is a complex polynomial, and $f^{-1}(z)$ is its corresponding inverse function, the Julia set $J(f)$ of $f(z)$ is the closure of its period repelling points; it is a noncountable strict subset, and keeps invariability under $f(z)$ and $f^{-1}(z)$.

(2) If $z \in J(f)$, then $J(f)$ is the closure of $\cup_{k=1}^{\infty} f^{-k}(z)$.

(3) The Julia set is the boundary of every attracting field, and for any positive integer p , $J(f) = J(f^p)$.

We shall make a proper transformation on the commutative hypercomplex dynamical system $z_{n+1} = f(z_n)$, and then prove that the transformed dynamical system possesses the same topological properties for their corresponding Julia sets, therefore realizing the control of Julia sets without changing the characteristics of the Julia sets.

3.1 Wholly magnifying or minifying control of Julia sets

Theorem 1. Given a commutative hypercomplex polynomial $f(z)$, if p belongs to its Julia set, $t \neq 0$ is a real number, then tp belongs to the Julia set of $g(z) = tf(\frac{1}{t}z)$.

Proof. (i) Set $z' = \frac{1}{t}z$, it then follows from $g(z) = tf(\frac{1}{t}z)$ that $f(z') = \frac{1}{t}g(tz')$. Assume that p belongs to Julia set of $f(z)$, that is, p is one n -period repelling point of $f(z)$, $n \geq 1$,

$$\begin{aligned} p &= f^n(p) = f^{n-1}(f(p)) = f^{n-1}\left(\frac{1}{t}g(tp)\right) \\ &= f^{n-2}\left(\frac{1}{t}g^2(tp)\right) = \dots = \frac{1}{t}g^n(tp). \end{aligned}$$

Therefore $tp = g^n(tp)$, implying that tp is also a n -period point of $g(z)$.

(ii) We now show that tp is one n -period repelling point of $g(z)$. If p is one n -period repelling point of $z_{n+1} = f(z_n)$, then $p = f^n(p)$, and $|(f^n(p))'| > 1$. Let $h(z) = f^n(z)$, then $|h'(p)| > 1$.

For a very small commutative hypercomplex number $\varepsilon = \varepsilon_q + \varepsilon_x i + \varepsilon_y j + \varepsilon_z k$, $|\varepsilon|$ is small, we assume that

$$p + \delta = f^n(p + \varepsilon) = h(p + \varepsilon) \approx p + \varepsilon \cdot h'(p),$$

implying that $\delta \approx \varepsilon \cdot h'(p)$ and yielding

$$|\delta| = |\varepsilon \cdot h'(p)| = |\varepsilon| |h'(p)| > |\varepsilon|.$$

Due to $f^n(z) = \frac{1}{t}g^n(tz)$,

$$p + \delta = f^n(p + \varepsilon) = \frac{1}{t}g^n(t(p + \varepsilon)),$$

one has $tp + t\delta = g^n(tp + t\varepsilon)$. Let $u(z) = g^n(z)$,

$$tp + t\delta = g^n(tp + t\varepsilon) = u(tp + t\varepsilon) \approx tp + t\varepsilon u'(tp).$$

We get $t\delta \approx t\varepsilon u'(tp)$, $|\delta| = |\varepsilon| |u'(tp)|$. Thanks to $|\delta| > |\varepsilon|$, we finally obtain

$$|u'(tp)| > 1, \quad |[g^n(tp)]'| > 1.$$

Therefore tp is one n -period repelling point of $g(z)$, and thereby belongs to the Julia set of $g(z) = tf(\frac{1}{t}z)$.

Example 1 . Consider $z_{n+1} = \frac{1}{t}z_n^2 + tc, t \in R, c$ is

commutative hypercomplex number, then its Julia set is correctly the wholly magnifying or minifying t times of the Julia set of $z_{n+1} = z_n^2 + c$. We utilize the escape time algorithm and race tracing to get the Julia sets with different t values. The 3D Julia set of $z_{n+1} = z_n^2 + c$ is shown in Figure 1(a) while Figure 1(b) depicts the 3D Julia set at $t = 1.3$.



(a)



(b)

Figure 1. $z_{n+1} = \frac{1}{t}z_n^2 + tc$, $c = -1.0$, (a) $t = 1$, (b) $t = 1.3$.

Example 2 . Consider $z_{n+1} = \frac{1}{t^2}z_n^3 + tc, t \in R, c$ is commutative hypercomplex number, then its Julia set is correctly the wholly magnifying or minifying t times of the Julia set of $z_{n+1} = z_n^3 + c$. The 3D Julia set of $z_{n+1} = z_n^3 + c$ is shown in Figure 2(a) while Figure 2(b) demonstrates the 3D Julia set with $t = 1.5$.

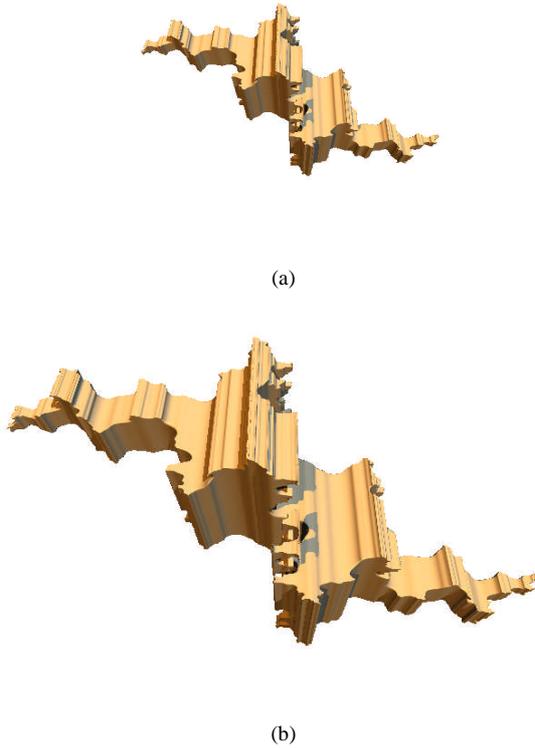


Figure 2. $z_{n+1} = \frac{1}{t^2} z_n^3 + tc$, $c = -0.2 + 0.8i + 0.02j$, (a) $t = 1$,
(b) $t = 1.5$.

3.2 Single-axis control of Julia sets

We now discuss the control of the single-axis control of Julia sets. It is convenient for us to prove the theorems by introducing matrix representations. We write the commutative hypercomplex number as $z = (q, x, y, w)^T$. Set

$$\begin{aligned} z_n &= (q_n, x_n, y_n, w_n)^T, \\ z_{n+1} &= (q_{n+1}, x_{n+1}, y_{n+1}, w_{n+1})^T, \\ c &= (a, b, d, e)^T. \end{aligned}$$

Then $z_{n+1} = z_n^2 + c$ can be rewritten as the form

$$\begin{aligned} q_{n+1} &= q_n^2 - x_n^2 - y_n^2 + w_n^2 + a, \\ x_{n+1} &= 2q_n x_n - 2y_n w_n + b, \\ y_{n+1} &= 2q_n y_n - 2x_n w_n + d, \\ w_{n+1} &= 2q_n w_n + 2x_n y_n + e. \end{aligned} \tag{1}$$

Transform (1) by

$$\begin{aligned} z_n' &= Az_n, \\ \begin{pmatrix} q_n' \\ x_n' \\ y_n' \\ w_n' \end{pmatrix} &= \begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 \\ 0 & 0 & r & 0 \\ 0 & 0 & 0 & s \end{pmatrix} \begin{pmatrix} q_n \\ x_n \\ y_n \\ w_n \end{pmatrix}, \quad (\alpha, \beta, r, s \in \mathbb{R}). \end{aligned} \tag{2}$$

Substitute (2) into (1), yields

$$\begin{aligned} q_{n+1}' &= \frac{1}{\alpha} q_n'^2 - \frac{\alpha}{\beta^2} x_n'^2 - \frac{\alpha}{r^2} y_n'^2 + \frac{\alpha}{s^2} w_n'^2 + \alpha a, \\ x_{n+1}' &= \frac{2}{\alpha} q_n' x_n' - \frac{2\beta}{rs} y_n' w_n' + \beta b, \\ y_{n+1}' &= \frac{2}{\alpha} q_n' y_n' - \frac{2r}{\beta s} x_n' w_n' + rd, \\ w_{n+1}' &= \frac{2}{\alpha} q_n' w_n' + \frac{2s}{\beta s} x_n' y_n' + se. \end{aligned} \tag{3}$$

The Julia set of dynamical system (3) is then the Julia set of system (1) with stretching or shrinking α, β, r, s times along the corresponding axes.

Theorem 2. The commutative hypercomplex dynamical system $z_{n+1} = f(z_n)$ can be represented by

$$\begin{aligned} z_{n+1} &= (q_{n+1}, x_{n+1}, y_{n+1}, w_{n+1})^T = f(z_n), \\ q_{n+1} &= f_q(z_n), \quad x_{n+1} = f_x(z_n), \\ y_{n+1} &= f_y(z_n), \quad w_{n+1} = f_z(z_n). \end{aligned} \tag{4}$$

If $z_0 = (q_0, x_0, y_0, w_0)^T$ belongs to the Julia set of (4), then $z_0' = Az_0 = (\alpha q_0, \beta x_0, r y_0, s w_0)^T$ belongs to the Julia set of $z_{n+1}' = g(z_n') = Af(A^{-1}z_n')$, where

$$\begin{aligned} z_{n+1}' &= (q_{n+1}', x_{n+1}', y_{n+1}', w_{n+1}')^T = Af(A^{-1}z_n'), \\ q_{n+1}' &= g_q(z_n') = \alpha f_q(A^{-1}z_n'), \\ x_{n+1}' &= g_x(z_n') = \beta f_x(A^{-1}z_n'), \\ y_{n+1}' &= g_y(z_n') = r f_y(A^{-1}z_n'), \\ w_{n+1}' &= g_z(z_n') = s f_z(A^{-1}z_n'). \end{aligned} \tag{5}$$

Proof. If $z_0 = (q_0, x_0, y_0, w_0)^T$ is one n -period point of (4), then $f^n(z_0) = z_0$. It follows from $g(z) = Af(A^{-1}z)$ that

$$\begin{aligned} g^n(z) &= g(g^{n-1}(z)) = Af(A^{-1}g^{n-1}(z)) \\ &= Af(A^{-1}g(g^{n-2}(z))) \\ &= Af(A^{-1}Af(A^{-1}g^{n-2}(z))) \\ &= Af^2(A^{-1}g^{n-2}(z)). \end{aligned}$$

The deduction ahead can yield

$$g^n(z) = Af^n(A^{-1}z). \tag{6}$$

Substitute $z_0' = Az_0$ into (6),

$$\begin{aligned} g^n(z_0') &= Af^n(A^{-1}z_0') = Af^n(A^{-1}Az_0) \\ &= Af^n(z_0) = Az_0 = z_0'. \end{aligned}$$

It implies that $z_0' = Az_0$ is one n -period point of (5).

We next show $z_0' = Az_0$ is one n -period repelling point of (5). Suppose z_0 is one n -period repelling point of $z_{n+1} = f(z_n)$, then $z_0 = f^n(z_0)$, $|(f^n(z_0))'| > 1$. It implies from (6) that

$$\begin{aligned} \left| \frac{d}{dz} (g^n(z)) \right|_{z=Az_0} &= \left| \frac{d}{dz} (Af^n(A^{-1}z)) \right|_{w=Aw_0} \\ &= |A(f^n(z_0))' A^{-1}| = |(f^n(z_0))'| > 1. \end{aligned}$$

Thus $z_0' = Az_0$ is one n -period repelling point.

Example 3 . Consider $z_{n+1} = z_n^2 + c$ and do the transformation (2) to the state variables, we can generate the Julia sets as shown in Figure 3. Figure 3(a) is of $\alpha = 1, \beta = 1.5, r = 1, s = 1$, and Figure 3(b) is of $\alpha = 1, \beta = 1, r = 1.3, s = 1$.



(a)



(b)

Figure 3. $z_{n+1} = z_n^2 + c, c = -1.0$.

3.3 General control of Julia sets

As a matter of fact, the commutative hypercomplex dynamical system can be transformed by a general form to achieve the general control of the Julia sets, such as shearing, rotation and compound affine transformations.

Theorem 3. The commutative hypercomplex dynamical system $z_{n+1} = f(z_n)$ can be written as

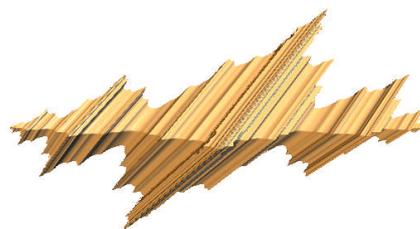
$$\begin{aligned} z_{n+1} &= (q_{n+1}, x_{n+1}, y_{n+1}, w_{n+1})^T = f(z_n), \\ q_{n+1} &= f_q(z_n), x_{n+1} = f_x(z_n), \\ y_{n+1} &= f_y(z_n), z_{n+1} = f_z(z_n). \end{aligned} \tag{7}$$

If $z_0 = (q_0, x_0, y_0, w_0)^T$ belongs to the Julia set of (7), then $z'_0 = Az_0$ belongs to the Julia set of $z'_{n+1} = g(z'_n) = Af(A^{-1}z'_n)$, where A is an invertible transformation matrix, including shearing, rotation and general compound affine transformation matrices. In this sense, we can realize the control of Julia sets by affine transformations according to Theorem 3. The proof is similar to that of Theorem 2.

Example 4 . Consider $z_{n+1} = z_n^2 + c$ and perform the shearing transformation $z'_n = Az_n$, one can get the Julia

sets shown in Figure 4, where A are $\begin{pmatrix} 1 & 0.8 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

and $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1.1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.



(a)



(b)

Figure 4. The Julia sets derived by $z_{n+1} = z_n^2 + c$ after shearing transformation.

Example 5 . Consider $z_{n+1} = z_n^2 + c$ and act the state variables by rotation transformation $z'_n = Az_n$. The generated Julia sets are shown in Figure 5, where A are

$$\begin{pmatrix} \cos \frac{\pi}{3} & -\sin \frac{\pi}{3} & 0 & 0 \\ \sin \frac{\pi}{3} & \cos \frac{\pi}{3} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \frac{\pi}{6} & -\sin \frac{\pi}{6} & 0 \\ 0 & \sin \frac{\pi}{6} & \cos \frac{\pi}{6} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$





Figure 5. The generated Julia sets of $z_{n+1} = z_n^2 + c$ by rotation transformation control.

Example 6 . Consider $z_{n+1} = z_n^2 + c$ and execute general affine transformation $z'_n = Az_n$, we get the Julia sets in Figure 6, where the transformation matrix A is

$$\begin{pmatrix} 1.1 & 0.5 & 0.6 & 0 \\ 0.6 & 1 & -0.5 & 0 \\ 0.1 & 1.1 & 1.2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

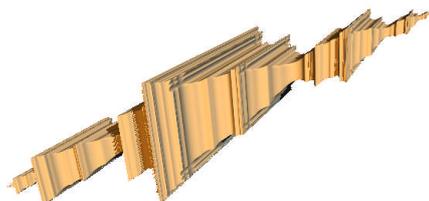


Figure 6. The Julia set of $z_{n+1} = z_n^2 + c$ by general transformation control.

4. Conclusions

In this paper, we use a proper algebraic transformation on the commutative hypercomplex dynamical systems to realize the whole magnification and minification, stretching or shrinking along the single-axis, shearing, rotation and general affine transformation of Julia sets, without changing the characters of the Julia set. We not only visualize the topological equivalent Julia sets, but also prove the topological equivalent properties of the Julia sets. The theoretical proof is simple and general which is suitable for more general topological transformations other than affine transformations.

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