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On An Interesting Integral Involving Gauss's Hypergeometric Function

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ABSTRACT: Around five decades ago, MacRobert, in his very popular, useful and interesting research paper, obtained certain new class of finite integrals. He, then used the integrals to evaluate integrals involving E-functions. The aim of this short research paper is to find an interesting integral involving hypergeometric function by employing one of the integrals obtained by MacRobert. The beauty of our result is that it comes out in terms of products of the ratios of gamma function. Also, we know that whenever the value of an integral comes in terms of gamma functions, the results are very useful from the applications point of view. On specializing the parameters, we can easily obtain the known integrals due to Rathie and integral given in the book of Mathai and Saxena. It is not out of place to mention here that, for other integrals, transformation and summation formulas involving hypergeoemtric and generalized hypergeometric series, we refer the work by Ali. Applications of this integral in obtaining integrals involving Inayat Hussain's $\overline{\mathrm{H}}$-function, Rathie's $\overline{\mathrm{I}}$-function etc. are under investigations and will be published soon.

Key Words: Hypergeometric functions; Integrals.

AMS subject classification: 33C60.

1. INTRODUCTION

We start with the definition Gauss's hypergeometric function [7].

$$1+\frac{a.b}{c}.\frac{z}{1!}+\frac{a(a+1)b(b+1)}{c(c+1)}.\frac{z^2}{2!}+\ldots \tag{1.1}$$

Here a, b and c may be real on complex numbers with an exception that c is neither zero nor a negative integer. z is called the parameter of the series.

The above series (1.1) can be written as

$$\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}.\frac{z^n}{n!}$$

where

$$(a)_n = a\,(a + 1) \ldots (a + n - 1) = \frac{\Gamma(a+n)}{\Gamma(a)}$$

$(a)_0 = 1$

is called the Pochhammer's symbol (or the shifted factorial). Since (1)n = n! and Γ is the well known Gamma function.

The series (1.1) is convergent for all values of z provided $|z| < 1$ and divergent if $|z| > 1$. When z = 1, the series is convergent if $\Re e\,(c - a - b) > 0$ and divergent if $\Re e\,(c - a - b) \le 0$. When z = - 1, the series is absolutely convergent when $\Re e\,(c - a - b) > 0$ and is convergent but not absolutely when $-1 < \Re e\,(c - a - b) \le 0$ and divergent when $\Re e\,(c - a - b) < -1$.

The series (1.1) is represented by the symbol ${}_2F_1\left(\begin{matrix}a,b\\c\end{matrix};z\right)$ and is called the Gauss's hypergeometric function. It is not out of place to mention here that

(i) For a = 1 and b = c or b = 1 and a = c, the series (1.1) reduces to the well known Geometric series and thus from the fact this series is called hypergeometric series.

(ii) When a or b or both is zero, the series becomes unity.

(iii) When a or b or both is a negative integer the series becomes a polynomial (i.e. containing a finite number of terms and the question of convergence does not arise).

The generalized hypergeometric functions with p numerator and q denominator parameters is defined by [7]

$${}_pF_q\left[\begin{matrix}a_1,\ldots,a_p\\b_1,\ldots,b_q\end{matrix};x\right] = \sum_{n=0}^{\infty}\frac{(a_1)_n\ldots(a_p)_n}{(b_1)_n\ldots(b_q)_n}.\frac{x^n}{n!} \tag{1.2}$$

For p = q and q = 1 it reduces to (1.1).

The aim of this short research note is to provide an integral involving hypergeometric function. A few known as well as integrals have also been obtained as limiting cases of our main findings.

2. Results Required

The following interesting and useful result due to MacRobert [5] will be required in our present investigation.

$$\int_0^1 x^{\alpha-1} (1-x)^{\beta-1} [ax+b(1-x)]^{-\alpha-\beta} dx = \frac{1}{a^\alpha b^\beta} \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \quad (2.1)$$

provided $\Re e\ (\alpha) > 0$ and $\Re e\ (\beta) > 0$ and a and b are non-zero constants and the expression [ax + b (1-x)], where $0 \le x \le 1$, is not zero.

3. Main Result

The following interesting integral involving hypergeometric function will be evaluated in this section.

$$\int_0^1 x^{\alpha-1} (1-x)^{\beta-1} [ax+b(1-x)]^{-\alpha-\beta} \, {}_2F_1\left[\begin{matrix} a,b \\ c \end{matrix} \;\middle|\; \frac{4abx(1-x)}{[ax+b(1-x)]^2}\right] dx$$
$$= \frac{1}{a^\alpha b^\beta} \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \, {}_4F_3\left[\begin{matrix} a,b,\alpha,\beta \\ c, \frac{1}{2}(\alpha+\beta), \frac{1}{2}(\alpha+\beta+1) \end{matrix} \;\middle|\; 1\right] \quad (3.1)$$

provided $\Re e\ (\alpha) > 0$, $\Re e\ (\beta) > 0$ and $\Re e\ (2c-2a-2b) > -1$. Also a and b are non-zero constants and the expression [ax + b (1 – x)], where $0 \le x \le 1$ is not zero.

4. Derivation

In order to derive our main integral (3.1), we proceed as follows.

Denoting the left-hand side of (3.1) by I, expressing ${}_2F_1$ as a series with the help of the definition (1.1), we have

$$I = \int_0^1 x^{\alpha-1} (1-x)^{\beta-1} [ax+b(1-x)]^{-\alpha-\beta} \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n \; n!} \frac{2^{2n} a^n b^n x^n (1-x)^n}{[ax+b(1-x)]^{2n}} dx \quad (4.1)$$

Changing the order of integration and summation, which is easily seen to be justified due to the uniform convergence of the series in the interval (0, 1), we have

$$I = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n \; n!} 2^{2n} a^n b^n \int_0^1 x^{\alpha+n-1} (1-x)^{\beta+n-1} [ax+b(1-x)]^{-\alpha-(\beta+2n)} dx \quad (4.2)$$

Evaluating the integral with the help of MacRobert's integral (2.1) and after some simplification, we have

$$I = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n \; n!} 2^{2n} \frac{(\alpha)_n (\beta)_n}{(\alpha+\beta)_{2n}} \quad (4.3)$$

Finally using the result

$$(a)_{2n} = 2^{2n} \left(\frac{1}{2}a\right)_n \left(\frac{1}{2}a + \frac{1}{2}\right)_n \quad (4.4)$$

We have

$$I = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \sum_{n=0}^{\infty} \frac{(a)_n (b)_n (\alpha)_n (\beta)_n}{(c)_n \left(\frac{1}{2}(\alpha+\beta)\right)_n \left(\frac{1}{2}(\alpha+\beta+1)\right)_n} \quad (4.5)$$

Summing up the series, we easily arrive at the right-hand side of (3.1).

This completes the proof of our main result (3.1).

5. Special Cases

In this section, we shall mention a few known as well as unknown results as special cases of our main result.

(1) If in (3.1), we take $a = \delta$, $b = \delta + \frac{1}{2}$ and $c = \gamma$, we get

$$\int_0^1 x^{\alpha-1} (1-x)^{\beta-1} [ax+b(1-x)]^{-\alpha-\beta} \, {}_2F_1\left[\begin{matrix} \delta, \delta+\frac{1}{2} \\ \gamma \end{matrix} \;\middle|\; \frac{4abx(1-x)}{[ax+b(1-x)]^2}\right] dx$$
$$= \frac{1}{a^\alpha b^\beta} \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \, {}_4F_3\left[\begin{matrix} \delta, \delta+\frac{1}{2}, \alpha, \beta \\ \gamma, \frac{1}{2}(\alpha+\beta), \frac{1}{2}(\alpha+\beta+1) \end{matrix} \;\middle|\; 1\right] \quad (5.1)$$

(2) In (5.1), if we take $\delta = \frac{1}{2}(\alpha+\beta)$, we get

$$\int_0^1 x^{\alpha-1} (1-x)^{\beta-1} [ax+b(1-x)]^{-\alpha-\beta} \, {}_2F_1\left[\begin{matrix} \frac{1}{2}(\alpha+\beta), \frac{1}{2}(\alpha+\beta+1) \\ \gamma \end{matrix} \;\middle|\; \frac{4abx(1-x)}{[ax+b(1-x)]^2}\right] dx$$
$$= \frac{1}{a^\alpha b^\beta} \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \, {}_2F_1\left[\begin{matrix} \alpha, \beta \\ \gamma \end{matrix} \;\middle|\; 1\right] \quad (5.2)$$

Using classical Gauss's theorem, we have

$$\int_0^1 x^{\alpha-1} (1-x)^{\beta-1} [ax+b(1-x)]^{-\alpha-\beta}$$

$$ {}_2F_1\left[\begin{matrix}\frac{1}{2}(\alpha+\beta), \frac{1}{2}(\alpha+\beta+1) \\ \gamma\end{matrix} \;\middle|\; \frac{4abx(1-x)}{[ax+b(1-x)]^2}\right] dx $$

$$ = \frac{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{a^{\alpha}b^{\beta}\Gamma(\alpha+\beta)\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)} \tag{5.3} $$

Note: The results (5.1) to (5.3) are given by Rathie [8]. Similarly other results can be obtained.

6. Concluding Remark

It is not out of place to mention here that integrals and transformation formulas evaluated with help of summation theorems such as those of Gauss, Gauss second, Bailey and Kummer for the series ${}_2F_1$; Watson, Dixon, Whipple and Saalschutz for the series ${}_3F_2$ have wide applications in the theory of hypergeometric and generalized hypergeometric series of one and more variables. For applications of these classical summation theorem and their generalizations by Lavoie et al., Rathie and Rakha we refer original research papers by Ali, S. [1,2,3,4], Modi, M. [6] and Sharma, M. [9].

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