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Optimal Faure sequence via mix Faure with the best scrambling schemes

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Abstract – The Faure sequence is one of the most well-known quasi-random sequences used in quasi-Monte Carlo applications. In its original and most basic form, the Faure sequence suffers from correlate different dimensions but when differences of sample size and dimension is high, the Faure sequence operates better than randomly scrambled version. In this paper we analyze various scrambling methods and propose a new method by mixing Faure sequence with the best available scrambled version. We demonstrate the usefulness of our method by integration problems.

Keywords – Faure sequence; Low-discrepancy sequences; (quasi)-Monte Carlo; Linear scrambling; Nonlinear scrambling

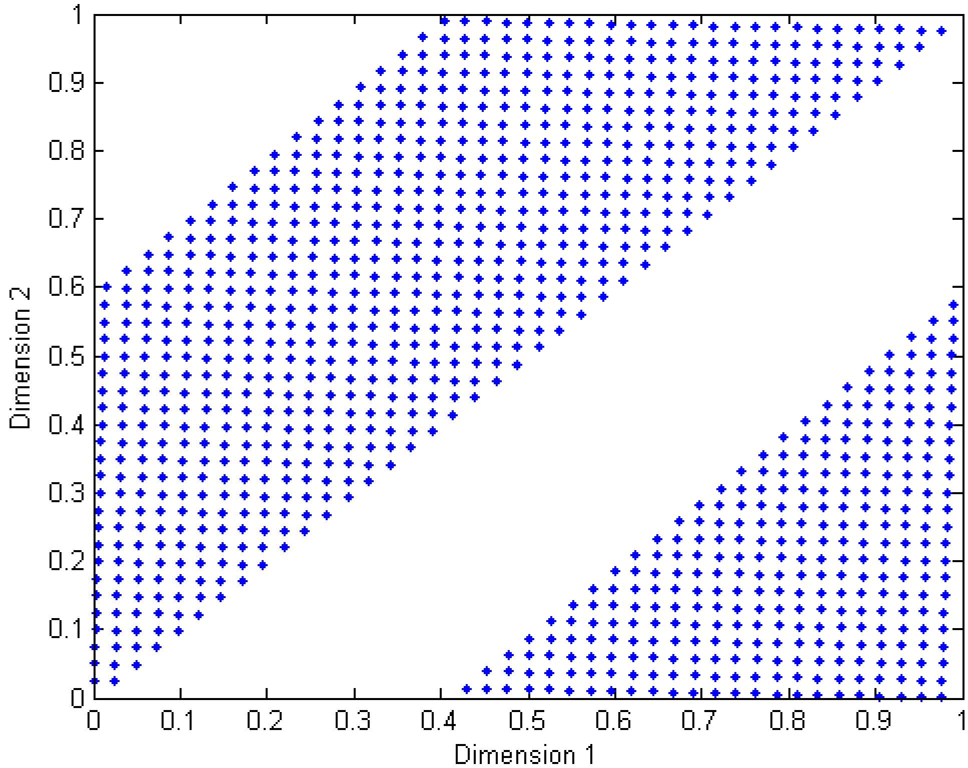
1. Introduction

The main problem of the Faure sequence is that when differences in sample size and dimension are small, integral error increases sharply. Other problem of this sequence is running time. The purpose of this article is providing new method that improves the results. We design their algorithms by use of nested functions to be performed time less. We compare these methods with other methods by error and time tables and a set of test-integrals. Faure sets $b_j = b$ for $j = 1, \ldots, s$ and uses powers of the upper triangular Pascal matrix modulo b for the generator matrices. The nth element of the Faure sequence is expressed as

$$x_n = (\varphi_b(P^0 n'), \varphi_b(P^1 n'), \ldots, \varphi_b(P^{s-1} n'))$$

Here $\varphi_b(n')$ is the radical inverse function in base b and $n' = (n_0, n_1, \ldots, n_m)^T$ is the digit vector of the b-adic representation of n. b is a prime number greater than or equal to the dimension s and P is the Pascal matrix modulo b whose (i, j)- element is equal to $\binom{j-1}{i-1}$ mod b. The matrix-vector products $P^j n'$ for $j = 0, \ldots, s-1$ are done in modulo b arithmetic. Fig.1 illustrates a disadvantage of the Faure sequence: the above construction leads to a sequence that has correlations between its individual coordinates. This leads among others to bad two- dimensional projections and also has its consequences when the sequence is used for numerical integration .

Fig.1.Projection of the first 1024 points from a 40-dimansional Faure sequence in base 41.



A solution to this problem consists of randomly scrambling the Faure sequence. The next section surveys known scrambling schemes.

2. Scrambling the Faure sequence

2.1. GFaure sequence

Tezuka [12, 14] proposed the generalized Faure sequence, GFaure, with the jth dimension generator matrix $C^{(j)} = A^{(j)}P^{j-1}$ and where the $A^{(j)}$ for $j = 1, \ldots, s$ are arbitrary non-singular lower triangular matrices over F_b. An implementation of GFaure in the C programming language is given in [14].

2.2. Faure sequence with random linear (digit) scrambling

Matousek [6] discusses several simplified versions of Owen's scrambling in which matrices and shift vectors are used. The points of a sequence constructed with Matousek's methods have the form

$x_n = (\varphi_b(A^{(1)}P^0 n' + g_1), \varphi_b(A^{(2)}P^1 n' + g_2), \ldots, \varphi_b(A^{(s)}P^{s-1} n' + g_s))$.

$$A^{(j)} = \begin{pmatrix} h_{1,1} & 0 & 0 & 0 \\ h_{2,1} & h_{2,2} & 0 & 0 \\ h_{3,1} & h_{3,2} & h_{3,3} & 0 \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad g_j = \begin{pmatrix} g_1 \\ g_2 \\ g_3 \\ \vdots \end{pmatrix}$$

where the g_j's and the $h_{i,j}$'s with $i > j$ are chosen randomly and independently from $\{0, 1, \ldots, b-1\}$ and the $h_{j,j}$'s are chosen randomly and independently from $\{1, 2, \ldots, b-1\}$. Another special ca of the above scrambling is the random linear digit scrambling where the matrices A(j) and the vectors g_j have the form

$$A^{(j)} = \begin{pmatrix} h_1 & 0 & 0 & 0 \\ 0 & h_2 & 0 & 0 \\ 0 & 0 & h_3 & 0 \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad g_j = \begin{pmatrix} g_1 \\ g_2 \\ g_3 \\ \vdots \end{pmatrix}$$

With the $h_j \in \{1, 2, \ldots, b-1\}$ and the $g_j \in \{0, 1, \ldots, b-1\}$ chosen uniformly and independently at random.

2.3. Faure sequence with I-binomial scrambling

Here, the $A^{(j)}$ and the g_j for $j = 1, \ldots, s$ are defined as

$$A^{(j)} = \begin{pmatrix} h_1 & 0 & 0 & 0 \\ h_2 & h_1 & 0 & 0 \\ h_3 & h_2 & h_1 & 0 \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad g_j = \begin{pmatrix} g_1 \\ g_2 \\ g_3 \\ \vdots \end{pmatrix}$$

Where h_1 is chosen randomly and independently from $\{1, 2, \ldots, b - 1\}$ and h_r $(r > 1)$ and g_r $(r \geq 1)$ are chosen randomly and independently from $\{0, 1, \ldots, b - 1\}$. [13]

2.4. Striped matrix scrambling

Owen [1] proposed a scrambling method with matrices $A^{(j)}$ and shift vectors g_j of the form

$$A^{(j)} = \begin{pmatrix} h_1 & 0 & 0 & 0 \\ h_1 & h_2 & 0 & 0 \\ h_1 & h_2 & h_3 & 0 \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad g_j = \begin{pmatrix} g_1 \\ g_2 \\ g_3 \\ \vdots \end{pmatrix}$$

Where the h_i are chosen randomly and independently from $\{1, 2, \ldots, b - 1\}$ and the g_i are chosen randomly and independently from $\{0, 1, \ldots, b - 1\}$.

2.5. Nonlinear scrambling

For which the jth coordinate of the nth point has the general form:

$$x_n^{(j)} = \varphi_b\left(\Phi^{-1}\left(A^{(j)} \Psi\left(P^{j-1} n' \right) + g_j \right) \right)$$

The matrices $A^{(j)}$ and the vectors g_j for $j = 1, \ldots, s$ may be chosen as in the previously discussed scrambling methods. Note that when random linear digit scrambling is combined with the bijections Φ and Ψ, this form of scrambling becomes a special case of the generalized linear random scrambling as given in Definition2.3 of [1].
If Ψ is applied and we take Φ to be the identity mapping, then denote this by "pre-inversive scrambling". If Ψ is the identity mapping and Φ is applied, then call this "post-inversive scrambling".

2.6. Optimal Faure sequence

Currently, various scrambling methods have been proposed, that to some extent solves problem of the Faure sequence dependency in high-dimensional. Almost these scrambling methods operate the same. Provide a new method that will operate similar to the previous scrambling method is not difficult. We must seek ways to improve the situation. With review two-dimensional projections of Faure sequences in different dimensions and comparison with the scrambling methods also comparison numerical integration's error, we realize that when difference of sample size and dimension is high, The Faure sequence works better than randomly scrambled version. On this basis, we present a new method. In all algorithms, we use nested functions and runtime have improved somewhat.

3. MFaure sequence

In this section, we use the Faure sequence when difference of sample size and dimension is high and best existing scramble when difference of sample size and dimension is low. But the question is: What is the difference. We consider the difference equal to (b^4-1) and for the second time consider equal to (b^2) and compare the results. Here b is the smallest prime number greater than or equal dimension s. The nth element of the MFaure sequence has the general form:

$$X_n = \begin{cases} \text{Faure}(n, s, b), & n > b^4 - 1 \\ \text{best scrambling}, & \text{otherwise} \end{cases} \quad (1)$$

We will call this ''MFaure sequence''. Another method is the following

$$X_n = \begin{cases} \text{Faure}(n, s, b), & n > b^2 \\ \text{best scrambling}, & \text{otherwise} \end{cases} \quad (2)$$

We will call this ''M2Faure sequence''. Figs. 2 and 3 show two-dimensional projections of 1024 points of a 40-dimensional MFaure sequence respectively M2Faure sequence.

Fig.2.Projection of the first 1024 points from a 40-dimansional MFaure sequence in base 41.

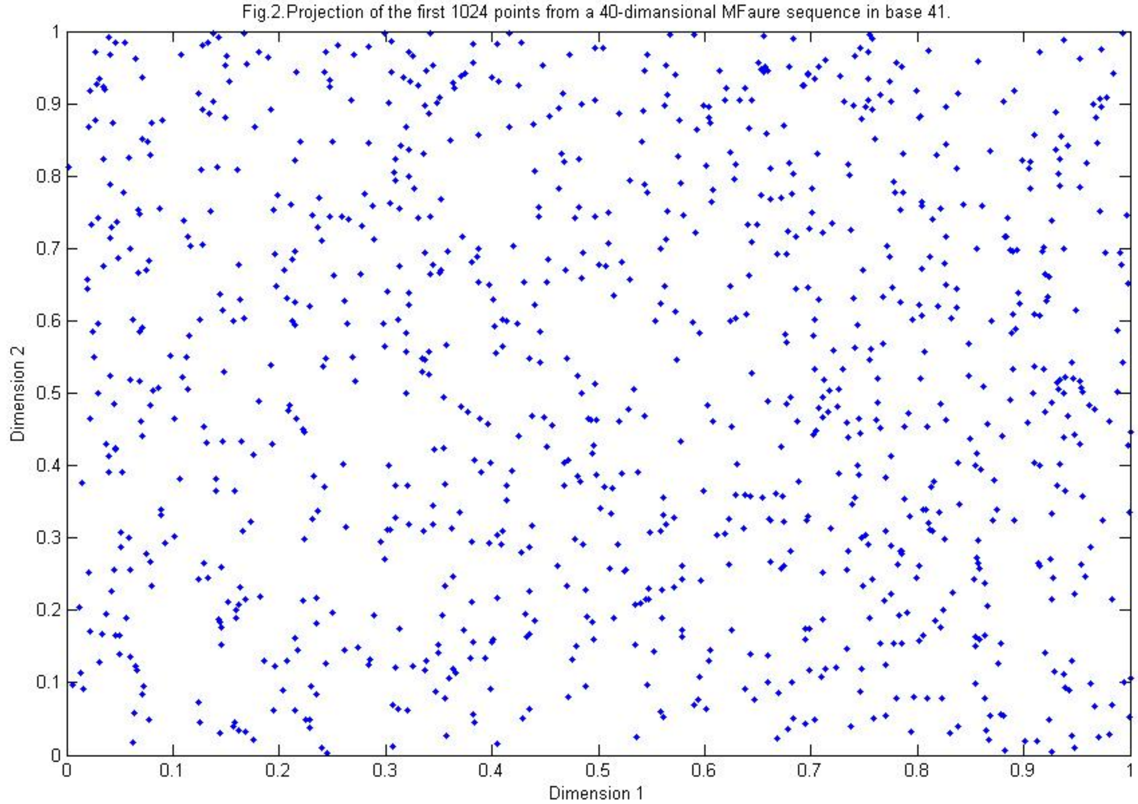
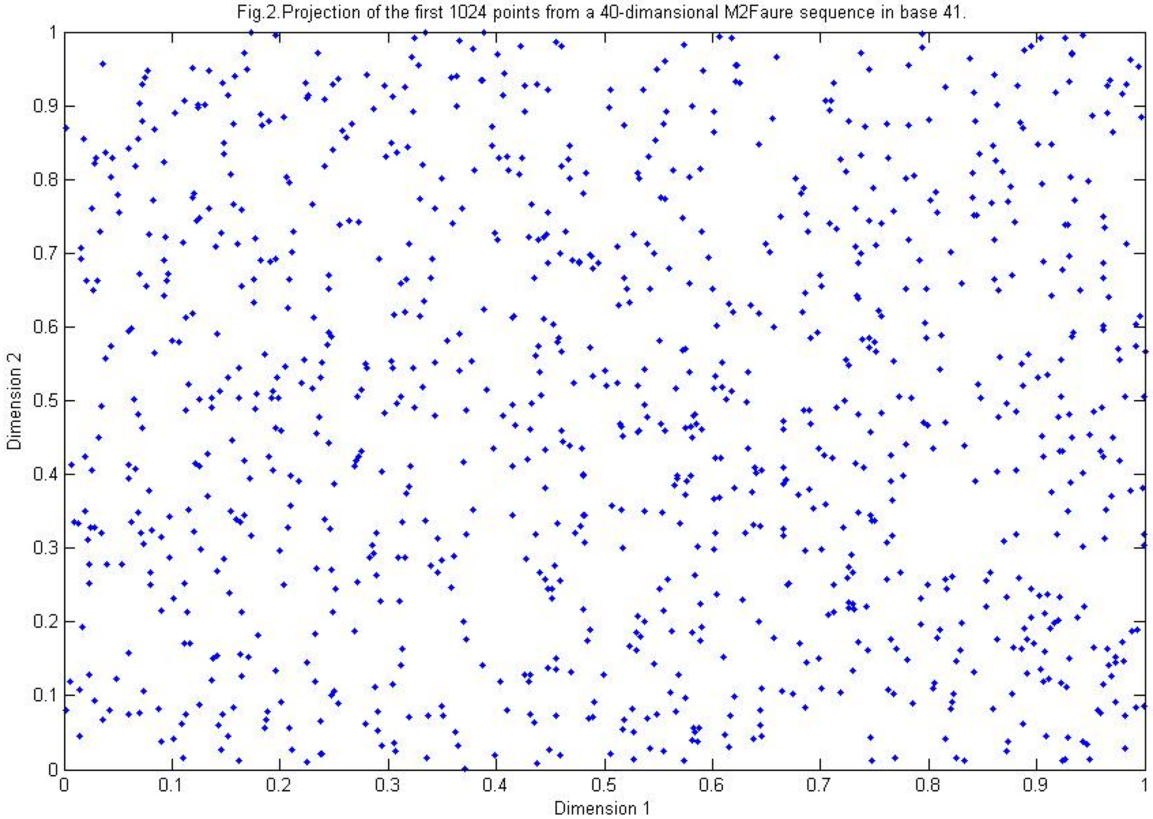


Fig.2.Projection of the first 1024 points from a 40-dimansional M2Faure sequence in base 41.



4. Numerical integration

A method to compare the quality of the sequences is to apply the sequences in high-dimensional integration problems [10].

Consider the following test-integrals:

$$I_1(f) = \int_0^1 \cdots \int_0^1 \prod_{i=1}^{s} \frac{\pi}{2} \frac{|4x_i - 2| + 1}{2} \, dx_1 \cdots dx_s = 1 \qquad (3)$$

$$I_2(f) = \int_0^1 \cdots \int_0^1 \prod_{i=1}^{s} \frac{\pi}{2} \frac{|4x_i - 2| + i^2}{1 + i^2} \, dx_1 \cdots dx_s = 1 \qquad (4)$$

$$I_3(f) = \int_0^1 \cdots \int_0^1 \prod_{i=1}^{s} \frac{\pi}{2} (\sin \pi x_i) \, dx_1 \cdots dx_s = 1 \qquad (5)$$

For (3), all variables are equally important and the truncation dimension is approximately the same as the nominal dimension. This is the most difficult case for numerical integration. For (4), the importance of the successive variables is decreasing. We refer to [15] for more details. The relative integration errors for the test functions (3) and (4) and (5) are given in Figs. 4 and 5 and 6 respectively. As can be seen in Figs, The MFaure and M2Faure sequences operate better than others. And the M2Faure sequence operates better than the MFaure sequence. There is the fact that no scrambling method outperforms the others, all scrambling methods have similar behavior.

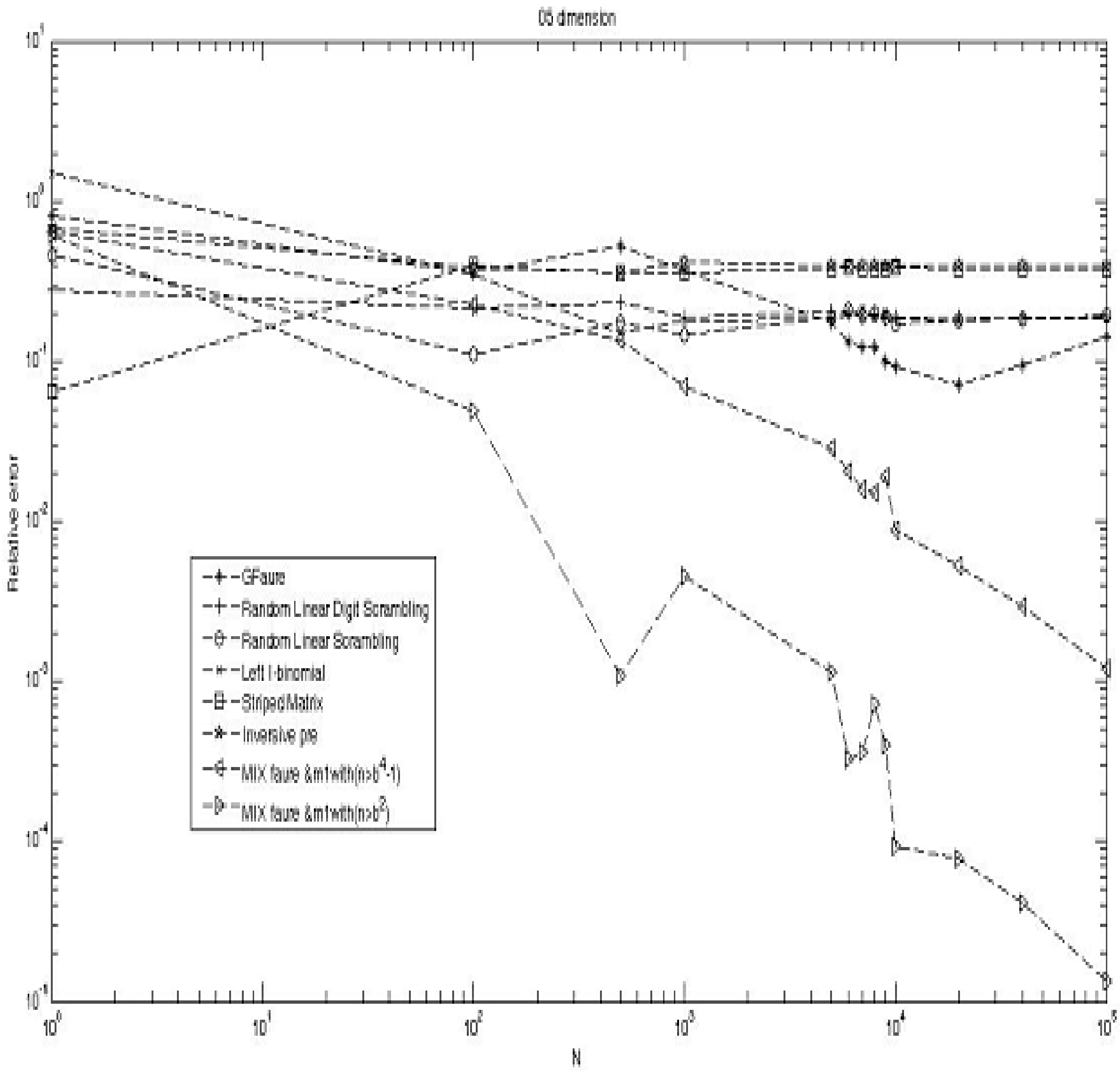


Fig4. Relative error against number of sample points for test-integral (3) for i = 1, . . . , s in 5 dimension.

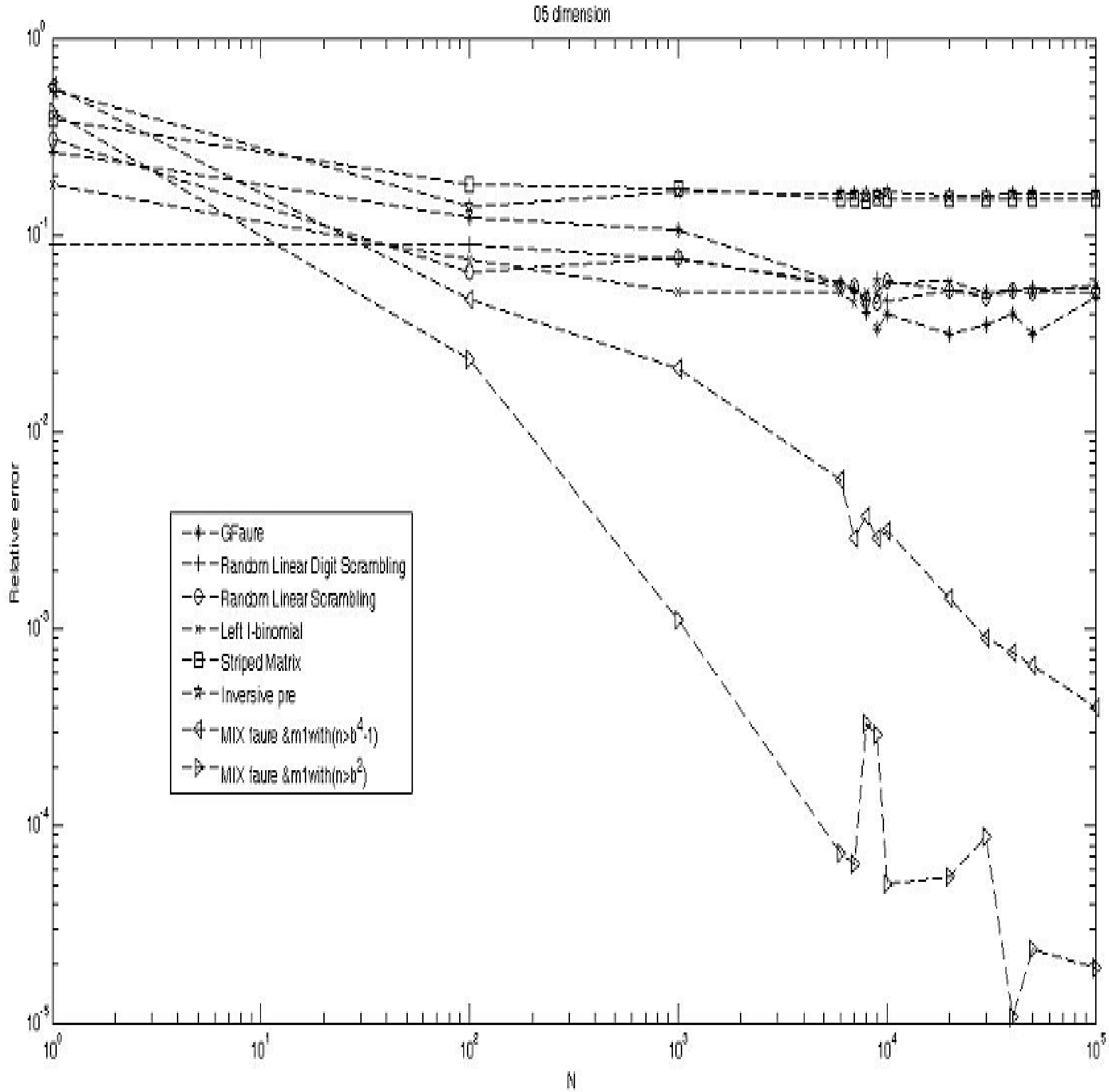


Fig5. Relative error against number of sample points for test-integral (4) for i = 1, . . . , s in 5 dimension.

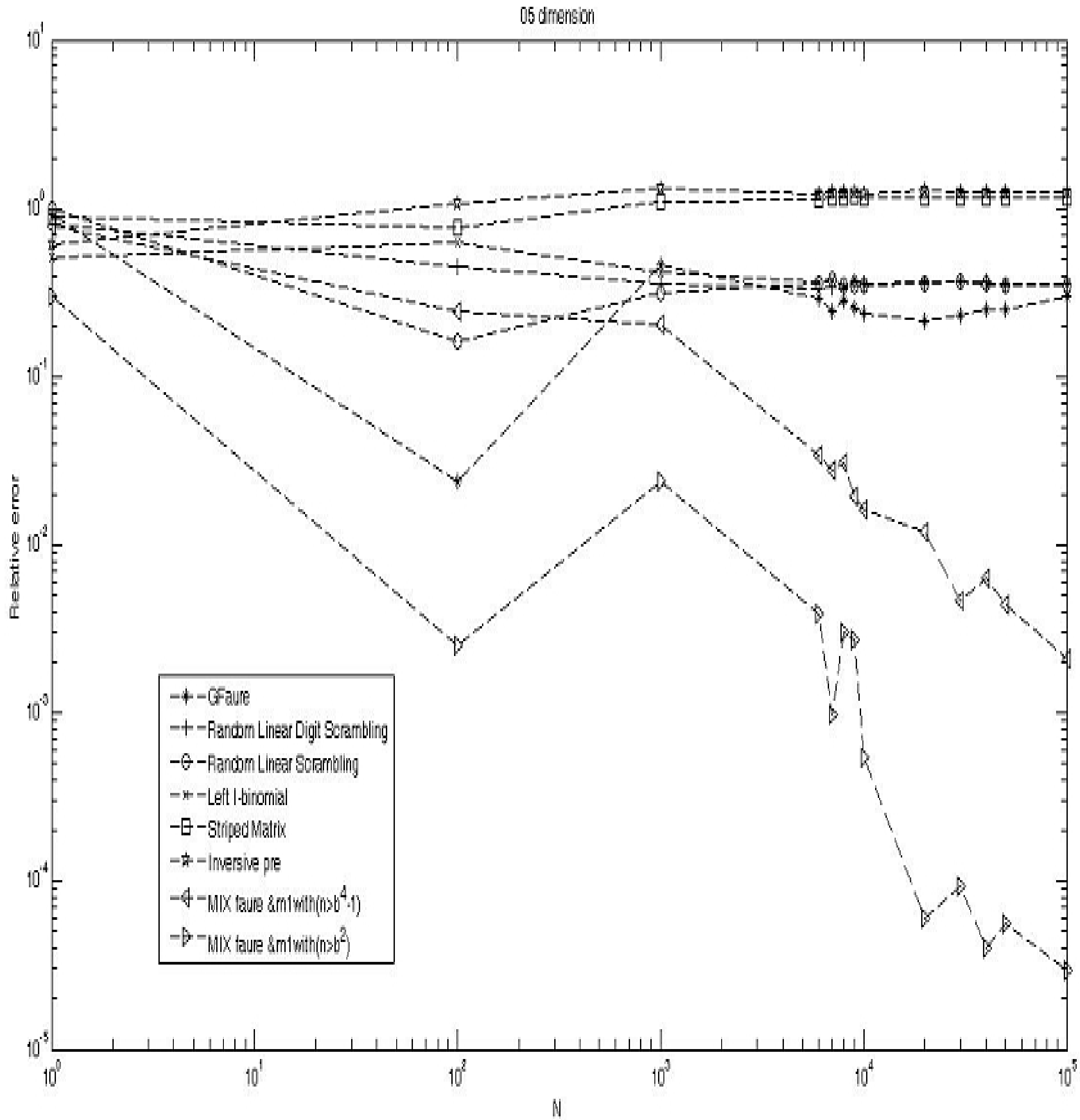


Fig6. Relative error against number of sample points for test-integral (5) for i = 1, . . . , s in 5 dimension.

6. Conclusions

A first conclusion to be drawn from our experiments is that if differences of sample size and dimension are high, clearly our methods operate better than others. Secondly, if this difference is small, operate like the best available scrambling method. Using the evaluation of numerical integrals, we did not find any significant differences between the different scrambling methods.

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