

Local Fractional Integral Equations and Their Applications

Xiao-Jun Yang

Department of Mathematics and Mechanics, China University of Mining and Technology, Xuzhou Campus, Xuzhou, Jiangsu, 221008, P. R. China

Email: dyangxiaojun@163.com

Abstract – This letter outlines the local fractional integral equations carried out by the local fractional calculus (LFC). We first introduce the local fractional calculus and its fractal geometrical explanation. We then investigate the local fractional Volterra/ Fredholm integral equations, local fractional nonlinear integral equations, local fractional singular integral equations and local fractional integro-differential equations. Finally, their applications of some integral equations to handle some differential equations with local fractional derivative and local fractional integral transforms in fractal space are discussed in detail.

Keywords – Local fractional calculus; Volterra/ Fredholm integral equations; Nonlinear integral equations; Singular integral equations; Integro-differential equations

1. Introduction

The theory of integral equations is one of the most useful mathematical tools in both pure and applied mathematics. It has enormous applications in many physical problems. Many initial and boundary value problems associated with ordinary differential equation (ODE) and partial differential equation (PDE) can be transformed into problems of solving some approximate integral equations. However, some initial and boundary value domains are fractal curves, which are everywhere continuous but nowhere differentiable. As a result, we cannot employ the classical calculus, which requires that the defined functions should be differentiable, to process ordinary local fractional differential equation (OLFDE) and local fractional partial differential equation (LFPDE) with fractal conditions. The theory of local fractional integrals and derivatives (fractal calculus)[1-17], as one of useful tools to handle the fractal and continuously non-differentiable functions, was successfully applied in local fractional Fokker-Planck equation[1, 2], anomalous diffusion and relaxation equation in fractal space[3, 4], the fractal heat conduction equation [5, 6], fractal-time dynamical systems[7, 8], fractal elasticity [9-10], local fractional diffusion equation [11], local fractional Laplace equation [20], local fractional ordinary differential equations[17, 18], local fractional partial differential equation[17, 18, 20, 26], local fractional integral equations[25, 30], fractal signals [17, 18, 19, 23], fractional Brownian motion in local fractional derivatives sense[16], fractal wave equation [20, 26].

This letter is to suggest some models for integral equations based on the local fractional calculus, and discuss their applications. The structure of this paper is as follows. In section 2, the preliminary results on the local fractional calculus and its fractal geometrical explanation. Local fractional Volterra integral equations and their applications are investigated in section 3. Local fractional Fredholm integral equations and their applications are investigated in section 4. Local fractional nonlinear

integral equations and their applications are in section 5. Local fractional singular integral equations and their applications are investigated in section 6. Local fractional integro-differential equations and their applications are in section 7. Conclusions are in section 8.

2. Preliminary results

To begin with we will provide a brief introduction to local fractional calculus.

2.1. Local fractional continuity of functions

Definition 1 If there exists the relation [17-24, 30]

$$|f(x) - f(x_0)| < \varepsilon^\alpha \quad (2.1)$$

with $|x - x_0| < \delta$, for $\varepsilon, \delta > 0$ and $\varepsilon, \delta \in \mathbb{R}$. Now $f(x)$ is called local fractional continuous at $x = x_0$, denote by $\lim_{x \rightarrow x_0} f(x) = f(x_0)$. Then $f(x)$ is called local fractional continuous on the interval (a, b) , denoted by

$$f(x) \in C_\alpha(a, b). \quad (2.2)$$

Definition 2 A function $f(x)$ is called a non-differentiable function of exponent α , $0 < \alpha \leq 1$, which satisfy Hölder function of exponent α , then for $x, y \in X$ such that [17-24, 30]

$$|f(x) - f(y)| \leq C|x - y|^\alpha. \quad (2.3)$$

Definition 3 A function $f(x)$ is called to be continuous of order α , $0 < \alpha \leq 1$, or shortly α continuous, when we have the following relation [17-24, 30]

$$f(x) - f(x_0) = o\left((x - x_0)^\alpha\right). \quad (2.4)$$

Remark 1. Compared with (2.4), (2.1) is standard definition of local fractional continuity. Here (2.3) is unified local fractional continuity.

Lemma1(See [31])

If (Ω, d) and (Ω', d') are metric spaces, $E \subset \mathbb{R}$ and $f : E \rightarrow \Omega'$ satisfies

$$\rho d(x, y) \leq d'(f(x), f(y)) \leq \tau d(x, y) \quad (2.5)$$

where ρ and τ are positives and finite constants, then

$$\rho^s H^s(E) \leq H^s(f(E)) \leq \tau^s H^s(E) \quad (2.6)$$

where each $s \geq 0$ and H^s is the s -dimensional Hausdorff measures.

Suppose (Ω, d) and (Ω', d') are metric spaces. A bijection $f : (\Omega, d) \rightarrow (\Omega', d')$ is said to be a bi-Lipschitz mapping, if there are constants $\rho, \tau > 0$ such that for all $x_1, x_2 \in \Omega$,

$$\rho d(x_1, x_2) \leq d'(f(x_1), f(x_2)) \leq \tau d(x_1, x_2). \quad (2.7)$$

The following lemma is also a standard result in fractal geometry (see for example [32- 35]).

Lemma 2 (See [32])

If $f : (\Omega, d) \rightarrow (\Omega', d')$ is a bi-Lipschitz mapping, then

$$\dim_H(A) = \dim_H(f(A)) \quad (2.8)$$

for all $A \in \Omega$.

Lemma 3

Let F be a subset of the real line and be a fractal. If $f : (F, d) \rightarrow (\Omega', d')$ is a bi-Lipschitz mapping, then there is for constants $\rho, \tau > 0$ and $F \subset \mathbb{R}$,

$$\rho^s H^s(F) \leq H^s(f(F)) \leq \tau^s H^s(F)$$

such that for all $x_1, x_2 \in F$,

$$\rho^\alpha |x_1 - x_2|^\alpha \leq |f(x_1) - f(x_2)| \leq \tau^\alpha |x_1 - x_2|^\alpha. \quad (2.9)$$

This result is directly deduced from fractal geometry. From **Lemma 1** and **Lemma 2** it is observed that that $\dim_H(F) = \dim_H(f(F)) = s$.

Theorem 4

Let F be a subset of the real line and be a fractal. If $f : ((\zeta, \xi), d) \rightarrow ((\eta, \nu), d')$ is a bi-Lipschitz mapping, then there is for constants $\rho, \tau > 0$,

$$\rho^\alpha |x_1 - x_2|^\alpha \leq |f(x_1) - f(x_2)| \leq \tau^\alpha |\zeta - \xi|^\alpha. \quad (2.10)$$

where $E = (\eta, \nu)$.

Proof. Let

$$H^s(F \cap (\zeta, \xi)) = (\xi - \zeta)^s = |\zeta - \xi|^s \quad [36],$$

by **Theorem 3** we get the result.

Theorem 5

Let F be a subset of the real line and be a fractal. If $f(\Omega)$ is a bi-Lipschitz mapping, then there are any $x_1, x_2 \in \Omega \subset \mathbb{R}$ and positive constant ν such that

$$|f(x_1) - f(x_2)| \leq \nu |x_1 - x_2|^\alpha. \quad (2.11)$$

Proof. By using **Theorem 4**, considering $\gamma = \max(\rho, \tau)$ and $\nu = \gamma^\alpha$ we obtain the result.

Remark 2. if $f(x) \in C_\alpha(a, b)$,

then $\dim_H(F \cap (a, b)) = \dim_H(C_\alpha(a, b)) = \alpha$ and

$C_\alpha(a, b) = \{f : f(x) \text{ is local fractional continuous, } x \in F \cap (a, b)\}$.

2.2. Local fractional derivatives

Definition 4 Setting $f(x) \in C_\alpha(a, b)$, local fractional derivative of $f(x)$ of order α at $x = x_0$ is defined [17-24, 30, 36, 37]

$$f^{(\alpha)}(x_0) = \frac{d^\alpha f(x)}{dx^\alpha} \Big|_{x=x_0} = \lim_{x \rightarrow x_0} \frac{\Delta^\alpha (f(x) - f(x_0))}{(x - x_0)^\alpha}, \quad (2.12)$$

where $\Delta^\alpha (f(x) - f(x_0)) \cong \Gamma(1 + \alpha) \Delta (f(x) - f(x_0))$.

For any $x \in (a, b)$, there exists

$$f^{(\alpha)}(x) = D_x^{(\alpha)} f(x),$$

denoted by

$$f(x) \in D_x^{(\alpha)}(a, b).$$

Local fractional derivative of high order is written in the form

$$f^{(k\alpha)}(x) = \overbrace{D_x^{(\alpha)} \dots D_x^{(\alpha)}}^{k \text{ times}} f(x), \quad (2.6)$$

and local fractional partial derivative of high order

$$\frac{\partial^{k\alpha} f(x)}{\partial x^{k\alpha}} = \overbrace{\frac{\partial^\alpha}{\partial x^\alpha} \dots \frac{\partial^\alpha}{\partial x^\alpha}}^{k \text{ times}} f(x). \quad (2.7)$$

2.3. Local fractional integrals

Definition 5 Setting $f(x) \in C_\alpha(a, b)$, local fractional integral of $f(x)$ of order α in the interval $[a, b]$ is defined [17-30, 36, 37]

$$\begin{aligned} & {}_a I_b^{(\alpha)} f(x) \\ &= \frac{1}{\Gamma(1 + \alpha)} \int_a^b f(t) (dt)^\alpha, \quad (2.8) \\ &= \frac{1}{\Gamma(1 + \alpha)} \lim_{\Delta t \rightarrow 0} \sum_{j=0}^{N-1} f(t_j) (\Delta t_j)^\alpha \end{aligned}$$

where $\Delta t_j = t_{j+1} - t_j$, $\Delta t = \max\{\Delta t_1, \Delta t_2, \Delta t_j, \dots\}$ and $[t_j, t_{j+1}]$, $j = 0, \dots, N - 1$, $t_0 = a, t_N = b$, is a partition of the interval $[a, b]$. For any $x \in (a, b)$, there exists

$${}_a I_x^{(\alpha)} f(x),$$

denoted by

$$f(x) \in I_x^{(\alpha)}(a, b).$$

Remark 3. If $f(x) \in D_x^{(\alpha)}(a, b)$, or $I_x^{(\alpha)}(a, b)$, we have

$$f(x) \in C_\alpha(a, b).$$

Here, it follows that

$${}_a I_a^{(\alpha)} f(x) = 0 \text{ if } a = b; \quad (2.2)$$

$${}_a I_b^{(\alpha)} f(x) = -{}_b I_a^{(\alpha)} f(x) \text{ if } a < b; \quad (2.3)$$

$$\text{and } {}_a I_a^{(0)} f(x) = f(x) \quad (2.4)$$

We only need here the following:

For any $f(x) \in C_\alpha(a, b)$, $0 < \alpha \leq 1$, we have local fractional multiple integrals, which is written as[27]

$${}_{x_0} I_x^{(k\alpha)} f(x) = \overbrace{{}_{x_0} I_x^{(\alpha)} \dots {}_{x_0} I_x^{(\alpha)}}^{k \text{ times}} f(x) \quad (2.5)$$

For $0 < \alpha \leq 1$, $f^{(k\alpha)}(x) \in C_\alpha^k(a, b)$, then we have[27]

$$\left({}_{x_0} I_x^{(k\alpha)} f(x) \right)^{(k\alpha)} = f(x), \quad (2.7)$$

where ${}_{x_0} I_x^{(k\alpha)} f(x) = \overbrace{{}_{x_0} I_x^{(\alpha)} \dots {}_{x_0} I_x^{(\alpha)}}^{k \text{ times}} f(x)$ and

$$f^{(k\alpha)}(x) = \overbrace{D_x^{(\alpha)} \dots D_x^{(\alpha)}}^{k \text{ times}} f(x).$$

The results are valid [37]:

(I) If $\psi(x, y) \in C_\alpha(a, b) \times C_\alpha(c, d)$, then

$${}_a I_b^{(\alpha)} {}_c I_d^{(\alpha)} \psi(x, y) = {}_c I_d^{(\alpha)} {}_a I_b^{(\alpha)} \psi(x, y).$$

(II) If $\psi(x, y, z) \in C_\alpha(a, b) \times C_\alpha(c, d) \times C_\alpha(e, f)$, then

$$\begin{aligned} &{}_a I_b^{(\alpha)} {}_c I_d^{(\alpha)} {}_e I_f^{(\alpha)} \psi(x, y, z) \\ &= {}_e I_f^{(\alpha)} {}_a I_b^{(\alpha)} {}_c I_d^{(\alpha)} \psi(x, y, z). \\ &= {}_c I_d^{(\alpha)} {}_e I_f^{(\alpha)} {}_a I_b^{(\alpha)} \psi(x, y, z) \end{aligned}$$

2.4. Its fractal geometrical explanation

Definition 6 Let a be an arbitrary but fixed real number. The integral staircase function $S_F^\alpha(x)$ of order α for a set F is given by [7, 8, 36]

$$S_F^\alpha(x) = \begin{cases} \gamma^\alpha [F, a, x], & \text{if } x \geq a; \\ -\gamma^\alpha [F, x, a], & \text{if } x < a. \end{cases}$$

Then we have the following results:

(a) The fractal mass function $\gamma^\alpha [F, a, b]$ can written as [36]

$$\begin{aligned} &\gamma^\alpha [F, a, b] \\ &= \lim_{\max_{0 < \delta_i < \alpha-1} (x_{i+1} - x_i) \rightarrow 0} \sum_{i=0}^{n-1} \frac{(x_{i+1} - x_i)^\alpha}{\Gamma(1+\alpha)} \\ &= \frac{1}{\Gamma(1+\alpha)} H^\alpha (F \cap (a, b)) \\ &= \frac{(b-a)^\alpha}{\Gamma(1+\alpha)} \end{aligned}$$

(b) we have [36]

$$\begin{aligned} &S_F^\alpha(y) - S_F^\alpha(x) \\ &= \gamma^\alpha [F, x, y] \\ &= \lim_{\max_{0 < \delta_i < \alpha-1} (x_{i+1} - x_i) \rightarrow 0} \sum_{i=0}^{n-1} \frac{(x_{i+1} - x_i)^\alpha}{\Gamma(1+\alpha)} \\ &= \frac{(y-x)^\alpha}{\Gamma(1+\alpha)} \end{aligned}$$

(c) if $a < b < c$, we have

$$\gamma^\alpha [F, a, b] + \gamma^\alpha [F, b, c] = \gamma^\alpha [F, a, c] \quad [7, 8].$$

Remark 4. From formula (a) we obtain that

$$\begin{aligned} &\gamma^\alpha [F, a, b] \\ &= \frac{1}{\Gamma(1+\alpha)} \int_a^b (dt)^\alpha \\ &= \frac{1}{\Gamma(1+\alpha)} \lim_{\Delta t \rightarrow 0} \sum_{j=0}^{N-1} (\Delta t_j)^\alpha \\ &= \frac{(b-a)^\alpha}{\Gamma(1+\alpha)} \end{aligned}$$

Remark 5. From formula (c) we deduce to $(b-a)^\alpha + (c-b)^\alpha = (c-a)^\alpha$, which is called the theory of fractional set [18, 38]. Hence, we can understand it by fractal geometry:

$$H^\alpha (F \cap (b-a)) + H^\alpha (F \cap (c-b)) = H^\alpha (F \cap (c-a))$$

ie. $1^\alpha + 2^\alpha = 3^\alpha$. That is, the fractal geometric representation is that cantor set $[0, 3]$ is equivalent to the sum of cantor set $[0, 1]$ and cantor set $[1, 3]$. The dimension of cantor set is α , for $0 < \alpha \leq 1$ and, 1^α , 2^α and 3^α are real line numbers on a fractional set [18, 38, 39].

3. Local fractional Volterra integral equations

3.1. Local fractional Volterra integral equations

The most standard form of Volterra linear local fractional integral equations is of the form [30]

$$u(x) = f(x) + \frac{\lambda^\alpha}{\Gamma(1+\alpha)} \int_a^x K(x, t) u(t) (dt)^\alpha \quad (3.1)$$

$$\text{or } f(x) = \frac{\lambda^\alpha}{\Gamma(1+\alpha)} \int_a^x K(x, t) u(t) (dt)^\alpha \quad (3.2)$$

where $K(x, t)$ is the kernel of the local fractional integral equation, $f(x)$ a local fractional continuous function of x , and λ^α a parameter. The limits of integration are function of x and the unknown function $u(x)$ appears linearly under the integral sign. The equation (3.1) is called Volterra local fractional integral equation of second kind; the equation (3.2) is called Volterra local fractional integral equation of first kind.

3.2. Applications of local fractional Volterra integral equations

We directly observe that the local fractional differential equation of α order

$$\frac{d^\alpha \phi}{dx^\alpha} = f(x, \phi), (a \leq x \leq b) \quad (3.3)$$

can be written immediately as the local fractional Valtterra integral equation of second kind

$$\phi(x) = \phi(a) + \frac{1}{\Gamma(1+\alpha)} \int_a^x f[t, \phi(t)] (dt)^\alpha. \quad (3.4)$$

We observe that the local fractional differential equation of 2α order

$$\frac{d^{2\alpha}\phi}{dx^{2\alpha}} = f(x, \phi), (a \leq x \leq b) \quad (3.5)$$

carrying out an integration by parts, can be expressed immediately as

$$\begin{aligned} \phi(x) = & \phi(a) + \frac{(x-a)^\alpha}{\Gamma(1+\alpha)} \phi^{(\alpha)}(a) \\ & + \frac{1}{\Gamma(1+\alpha)} \int_a^x \frac{(x-s)^\alpha}{\Gamma(1+\alpha)} f[t, \phi(t)] (dt)^\alpha \end{aligned} \quad (3.6)$$

4. Local fractional Fredholm integral equations

4.1. Local fractional Fredholm integral equations

The most standard form of Fredholm linear local fractional integral equations is given by the form

$$u(x) = f(x) + \frac{\lambda^\alpha}{\Gamma(1+\alpha)} \int_a^b K(x,t) u(t) (dt)^\alpha \quad (4.1)$$

or

$$f(x) = \frac{\lambda^\alpha}{\Gamma(1+\alpha)} \int_a^b K(x,t) u(t) (dt)^\alpha \quad (4.2)$$

where $K(x,t)$ is the kernel of the local fractional integral equation, $f(x)$ a local fractional continuous function of x , and λ^α a parameter. The limits of integration a and b are constants and the unknown function $u(x)$ appears linearly under the integral sign. The equation (4.1) is called Fredholm local fractional integral equation of second kind; this equation (4.2) is called Fredholm local fractional integral equation of first kind.

4.2. Applications of local fractional Fredholm integral equations

We consider the following boundary value problem:

$$\begin{cases} \frac{d^\alpha \phi}{dx^\alpha} = f(x, \phi), a \leq x \leq b \\ \phi(a) = \phi_a, \phi(b) = \phi_b \end{cases} \quad (4.3)$$

carrying out an integration by parts, can be expressed immediately as

$$\begin{aligned} \phi(x) = & \phi(a) + \frac{(x-a)^\alpha}{\Gamma(1+\alpha)} [\phi(b) - \phi(a)] \\ & + \frac{1}{\Gamma(1+\alpha)} \int_a^b K(x,t) f[t, \phi(t)] (dt)^\alpha \end{aligned} \quad (4.4)$$

where

$$K(x,t) = \begin{cases} \frac{(b-x)^\alpha (t-a)^\alpha}{(b-a)^\alpha \Gamma(1+\alpha)}, 0 \leq t \leq x \leq b \\ \frac{(x-a)^\alpha (b-t)^\alpha}{(b-a)^\alpha \Gamma(1+\alpha)}, 0 \leq x \leq t \leq b \end{cases}$$

5. Local fractional nonlinear integral equations

5.1. Local fractional nonlinear integral equations

If the unknown function $u(t)$ appearing under the integral sign is given in the functional form $F(u(t))$ such as the power of $u(t)$ is no longer unity. Then the Volterra and Fredholm local fractional integral equations are classified as nonlinear local fractional integral equations. In general, a nonlinear local fractional integral equation is defined as given in the following equations:

$$u(x) = f(x) + \frac{\lambda^\alpha}{\Gamma(1+\alpha)} \int_a^x K(x,t) F(u(t)) (dt)^\alpha \quad (5.1)$$

or

$$u(x) = f(x) + \frac{\lambda^\alpha}{\Gamma(1+\alpha)} \int_a^b K(x,t) F(u(t)) (dt)^\alpha \quad (5.2)$$

Equations (5.1) and (5.2) are called nonlinear Volterra local fractional integral equations and nonlinear Fredholm local fractional integral equations, respectively. If we set $f(x) = 0$, in Volterra or Fredholm local fractional integral equations, then the resulting

$$u(x) = \frac{\lambda^\alpha}{\Gamma(1+\alpha)} \int_a^x K(x,t) F(u(t)) (dt)^\alpha \quad (5.3)$$

and

$$u(x) = \frac{\lambda^\alpha}{\Gamma(1+\alpha)} \int_a^b K(x,t) F(u(t)) (dt)^\alpha \quad (5.4)$$

The equations (5.3) and (5.4) are called homogeneous local fractional integral equation; otherwise it is called nonhomogeneous local fractional integral equation.

5.2. Applications of local fractional nonlinear integral equations

The Volterra or Fredholm nonlinear local fractional integral equation can be written in the form

$$u(x) = f(x) + \frac{\lambda^\alpha}{\Gamma(1+\alpha)} \int_0^x K(x,t) u^{2n}(t) (dt)^\alpha \quad (5.5)$$

or

$$u(x) = f(x) + \frac{\lambda^\alpha}{\Gamma(1+\alpha)} \int_a^b K(x,t) u^{2n}(t) (dt)^\alpha, n > 1. \quad (5.6)$$

Equations (4.1) and (4.2) are called nonlinear Volterra local fractional integral equations and nonlinear Fredholm local fractional integral equations, respectively.

6. Local fractional singular integral equations

6.1. Local fractional singular integral equations

A singular local fractional integral equation is defined as an integral with the infinite limits or when the kernel

of the integral becomes unbounded at a certain point in the interval.

Local fractional singular integral equation of first kind

$$f(x) = \frac{\lambda^\alpha}{\Gamma(1+\alpha)} \int_{b(x)}^{a(x)} K(x,t)u(t)(dt)^\alpha \quad (6.1)$$

or the local fractional singular integral equation of second kind

$$u(x) = f(x) + \frac{\lambda^\alpha}{\Gamma(1+\alpha)} \int_{b(x)}^{a(x)} K(x,t)u(t)(dt)^\alpha \quad (6.2)$$

is called singular if $a(x)$, or $b(x)$, or both limits of integration are infinite. The above are classified as the singular local fractional integral equations.

6.2. Applications of singular local fractional integral equations

$$u(x) = E_\alpha(x^\alpha) + \frac{1}{T(1+\alpha)} \int_0^\infty E_\alpha(x^\alpha)u(t)(dt)^\alpha; \quad (6.3)$$

$$F_\alpha\{f(x)\} = \frac{1}{\Gamma(1+\alpha)} \int_{-\infty}^\infty E_\alpha(-i^\alpha \omega^\alpha x^\alpha) f(x)(dx)^\alpha \quad [17-20, 22-25, 30, 36]; \quad (6.4)$$

$$L_\alpha\{f(x)\} = \frac{1}{\Gamma(1+\alpha)} \int_0^\infty f(x)E_\alpha(-s^\alpha x^\alpha)(dx)^\alpha \quad [18, 21, 24, 30, 36]; \quad (6.5)$$

$$F_\alpha\{f(x)\} = \frac{1}{\Gamma(1+\alpha)} \int_{-\infty}^\infty f(x)E_\alpha(-i^\alpha h_0^\alpha x^\alpha \omega^\alpha)(dx)^\alpha \quad [18] \quad (6.6)$$

if we have $h_0^\alpha = \frac{(2\pi)^\alpha}{\Gamma(1+\alpha)}$.

7. Local fractional integro-differential equations

7.1. Local fractional integro-differential equations

In this type of equations, the unknown function $u(x)$ appears as the combination of the ordinary local fractional derivative and under the local fractional integral sign.

Local fractional Volterra type integro-differential equation is written in the form

$$u^{(2\alpha)}(x) + u^{(\alpha)}(x) = f(x) + \frac{\lambda^\alpha}{\Gamma(1+\alpha)} \int_a^x g(x)u(x)(dt)^\alpha, \quad (7.1)$$

or local fractional Fredholm type integro-differential equation is written in the form

$$u^{(2\alpha)}(x) + u^{(\alpha)}(x) = f(x) + \frac{\lambda^\alpha}{\Gamma(1+\alpha)} \int_a^b g(x)u(x)(dt)^\alpha. \quad (7.2)$$

Equation (7.1) is of the Volterra local fractional integro-differential equations, whereas equation (7.2) is of the Fredholm local fractional integro-differential equations.

7.2. Applications of local fractional integro-differential equations

The Volterra local fractional integro-differential equation can be written in the for

$$u^{(2\alpha)}(x) + u^{(\alpha)}(x) = E_\alpha(x^\alpha) + \frac{x^\alpha}{\Gamma(1+\alpha)} \int_0^x u(x)(dt)^\alpha, \quad x \in [0,1] \quad (7.3)$$

with the initial conditions

$$u(0) = 1 \text{ and } u^{(\alpha)}(0) = 1.$$

The Fredholm linear local fractional integro-differential equation can be written as

$$u^{(2\alpha)} + u^{(\alpha)} = \frac{x^\alpha}{\Gamma(1+\alpha)} \int_0^1 E_\alpha(t^\alpha)u(t)(dt)^\alpha, \quad x \in [0,1] \quad (7.4)$$

with the initial conditions

$$u(0) = 1 \text{ and } u^{(\alpha)}(0) = 0.$$

The Volterra nonlinear local fractional integro-differential equation can be written in the form

$$u^{(2\alpha)} + u(x)u^{(\alpha)} = E_\alpha(x^\alpha) + \frac{x^\alpha}{\Gamma(1+\alpha)} \int_0^x u^2(x)(dt)^\alpha, \quad x \in [0,1] \quad (7.5)$$

with the initial conditions

$$u(0) = 1 \text{ and } u^{(\alpha)}(0) = 1.$$

The Fredholm nonlinear local fractional integro-differential equation can be written as

$$u^{(2\alpha)} + u(x)u^{(\alpha)} = \frac{x^\alpha}{\Gamma(1+\alpha)} \int_0^1 E_\alpha(t^\alpha)u^2(t)(dt)^\alpha, \quad x \in [0,1] \quad (7.6)$$

with the initial conditions

$$u(0) = 1 \text{ and } u^{(\alpha)}(0) = 0.$$

8. Conclusions

In this letter, we study the local fractional calculus and its fractal geometrical explanation. The theory of integral equations in the local fractional calculus is one of the most useful mathematical tools in both pure and applied mathematics in the fractal fields. Here, we investigate the local fractional integral equations, such as local fractional Volterra/ Fredholm integral equations, local fractional nonlinear integral equations, local fractional singular integral equations and local fractional integro-differential equations. It is useful for engineers and scientists to handle some differential equations with local fractional derivative and local fractional integral transforms in fractal domains, which is applied to deal with the fractal problems, i.e., the fractal differential equations, fractal signals and the governing equations in fractal media. Hence, the theory of local fractional equations is of great significance for engineers and scientists to handle the problems and nonlinear behaviors of the fractal mathematics and engineering [40-44].

References

- [1] K.M. Kolwankar, A.D. Gangal, Hölder exponents of irregular signals and local fractional derivatives, *Pramana J. Phys.*, 48 (1997) 49-68.
- [2] K.M. Kolwankar, A.D. Gangal, Local fractional Fokker-Planck equation, *Phys. Rev. Lett.*, 80 (1998) 214-217.
- [3] W. Chen, Time-space fabric underlying anomalous diffusion, *Chaos, Solitons, Fractals*, 28 (2006) 923-929.
- [4] W. Chen, X.D. Zhang, D. Korosak, Investigation on fractional and fractal derivative relaxation- oscillation models. *Int. J. Nonlin. Sci. Num.*, 11 (2010) 3-9.
- [5] J.H. He, A new fractal derivation, *Thermal Science*, 15(1) (2011) 2011-147.
- [6] J.H. He, S.K. Elagan, Z.B. Li, Geometrical explanation of the fractional complex transform and derivative chain rule for fractional calculus, *Phy. Lett. A*, 376(4) (2012) 257-259.
- [7] A. Parvate, A. D.Gangal, Fractal differential equations and fractal-time dynamical systems, *Pramana J. Phys.*, 64 (3) (2005) 389-409.
- [8] A. Parvate, A. D.Gangal, Calculus on fractal subsets of real line - I: formulation, *Fractals*, 17 (1) (2009) 53-81.
- [9] A. Carpinteri, B.Cornetti, K. M. Kolwankar, Calculation of the tensile and flexural strength of disordered materials using fractional calculus, *Chaos, Solitons, Fractals*, 21 (2004) 623-632.
- [10] A.V. Dyskin, Effective characteristics and stress concentration materials with self-similar microstructure, *Int. J. Sol. Struct.*, 42 (2005) 477-502.
- [11] X.J. Yang, Applications of local fractional calculus to engineering in fractal time-space: Local fractional differential equations with local fractional derivative, *ArXiv:1106.3010v1 [math-ph]*, 2011.
- [12] F.B. Adda, J. Cresson, About non-differentiable functions, *J. Math. Anal. Appl.*, 263 (2001) 721-737.
- [13] A. Babakhani, V.D. Gejji, On calculus of local fractional derivatives, *J. Math. Anal. Appl.*, 270 (2002) 66-79.
- [14] X.R. Li, Fractional Calculus, Fractal Geometry, and Stochastic Processes, Ph.D. Thesis, University of Western Ontario, 2003.
- [15] Y. Chen, Y. Yan, K. Zhang, On the local fractional derivative, *J. Math. Anal. Appl.*, 362 (2010) 17-33.
- [16] T. Christoph, Further remarks on mixed fractional Brownian motion, *Appl. Math. Sci.*, 38 (3) (2009) 1885 - 1901.
- [17] X.J. Yang, Local Fractional Integral Transforms, *Progr. in Nonlinear Sci.*, 4 (2011) 1-225.
- [18] X.J. Yang, Local Fractional Functional Analysis and Its Applications, Asian Academic publisher Limited, Hong Kong, 2011.
- [19] X.J. Yang, M.K. Liao, J.W. Chen, A novel approach to processing fractal signals using the Yang-Fourier transforms, *Procedia Eng.*, 29 (2012) 2950-2954.
- [20] X.J. Yang, Local fractional partial differential equations with fractal boundary problems, *Advances in Computational Mathematics and its Applications*, 1(1) (2012) 60-63.
- [21] X.J. Yang, A short introduction to Yang-Laplace Transforms in fractal space, *Advances in Information Technology and Management*, 1(2) (2012) 38-43.
- [22] X.J. Yang, Local fractional Fourier analysis, *Advances in Mechanical Engineering and its Applications*, 1(1) (2012) 12-16.
- [23] X.J. Yang, Generalized Sampling Theorem for Fractal Signals, *Advances in Digital Multimedia*, 1(2) (2012)88-92.
- [24] Y. Guo, Local fractional Z transform in fractal space, *Advances in Digital Multimedia*, 1(2) (2012) 96-102.
- [25] W.P. Zhong, F. Gao, X.M. Shen, Applications of Yang-Fourier transform to local Fractional equations with local fractional derivative and local fractional integral, *Advanced Materials Research*, 416 (2012) 306-310.
- [26] W. P. Zhong, F. Gao, Application of the Yang-Laplace transforms to solution to nonlinear fractional wave equation with local fractional derivative. In: *Proc. of the 2011 3rd International Conference on Computer Technology and Development, ASME*, 2011, pp.209-213.
- [27] X.J. Yang, Generalized local Taylor's formula with local fractional derivative, *ArXiv:1106.2459v2 [math-ph]*, 2011.
- [28] G. S. Chen, Mean value theorems for local fractional integrals on fractal space, *Advances in Mechanical Engineering and its Applications*, 1(1) (2012) 5-8.
- [29] G.S. Chen, The local fractional Stieltjes transform in fractal space, *Advances in Intelligent Transportation Systems*, 1(1) (2012) 29-31.
- [30] X. J. Yang, Local Fractional Kernel Transform in Fractal Space and Its Applications, *Advances in Computational Mathematics and its Applications*, 1(2) (2012) 86-93.
- [31] N. Castro, M. Reyes, Hausdorff measures and dimension on \mathbb{R}^∞ , *Proceedings of the American Mathematical Society*, 125(22) (1997) 3267-3273.
- [32] Q.L. Guo, H.Y. Jiang, L.F. Xi, Hausdorff Dimension of Generalized Sierpinski Carpet, *International Journal of Nonlinear Science*, 2 (3) (2006) 153-158.
- [33] J. E. Hutchinson: Fractals and self similarity, *Indiana University Mathematics Journal*, 30 (1981) 713-747.
- [34] K. J. Falconer, *Fractal Geometry—Mathematical Foundations and Application*, John Wiley, New York, 1997.
- [35] Z. Y. Wen, *Mathematical Foundations of Fractal Geometry*, Shanghai Scientific & Technological Education Publishing House, Shanghai, 2000.
- [36] X. J. Yang, Local fractional calculus and its applications, *Proceedings of FDA'12, The 5th IFAC Workshop Fractional Differentiation and its Applications*, 1-8, 2012.
- [37] X.J. Yang, Research on Fractal Mathematics and Some Applications in Mechanics, M.S. thesis, China University of Mining and Technology, 2009.
- [38] G.S. Chen, A generalized Young inequality and some new results on fractal space, *advances in Computational Mathematics and its Applications*, 1(1) (2012) 56-59.
- [39] X.J. Yang, Expression of generalized Newton iteration method via generalized local fractional Taylor series, *Advances in Computer Science and its Applications*, 1(2) (2012) 89-92.
- [40] Y. Zhang, S. Wang, L. Wu, G. Wei, Multi-channel Diffusion Tensor Image Registration via adaptive chaotic PSO, *Journal of Computers*, 6(4) (2011) 825-829
- [41] Y. Zhang, Y. Huo, Y. Q. Zhu, et al., Polymorphic BCO for Protein Folding Model. *Journal of Computational Information Systems*, 6(6) (2010) 1787-1794.
- [42] Y. Zhang, S. Wang, L. Wu, Y. Huo, PSNN used for Remote-Sensing Image Classification. *Journal of Computational Information Systems*, 6(13) (2010), 4417- 4425.
- [43] Y. Zhang, L. Wu, B.S. Peterson, Z. Dong, A Two-Level Iterative Reconstruction Method for Compressed Sensing MRI, *Journal of Electromagnetic Waves and Applications*, 25 (2011) 1081-1091.
- [44] Y. Zhang, L. Wu, A novel algorithm for APSP problem via a simplified delay Pulse Coupled Neural Network, *Journal of Computational Information Systems*, 7 (2011) 737-744.

Vitae



Mr. Yang Xiao-Jun was born in 1981. He worked as a scientist and engineer in CUMT. His research interest includes Fractal mathematics (Geometry, applied mathematics and functional analysis), fractal Mechanics (fractal elasticity and fractal fracture mechanics, fractal rock mechanics and fractional continuous mechanics in fractal media), fractional calculus and its applications, fractional differential equation, local fractional integral equation, local fractional differential equation, local fractional integral transforms, local fractional short-time analysis and wavelet analysis, local fractional calculus and its applications and local fractional functional analysis and its applications.