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The Analysis of Black-Scholes Option Pricing

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Abstract –This paper makes a more detailed description of the assumptions of the model and the mathematics derivation process of the formula, and analyzes the sensitivity of the option value of each variable of the model. According to the empirical analysis, we find that the Black-Scholes Model has a strong practicality, but also has some limitimations, further research is needed.

Keywords –Option; Option prcing; Black-Scholes model; Brownian motion; Volatility

1. Introduction

Black and Scholes (1973) revolutionized the pricing theory of options by showing how to hedge continuously the exposure on the short position of an option. Even before F.Black and M. Scholes published their model, evidence was accumulating that asset returns are not normally distributed (or equivalently that asset prices do not follow Geometric Brownian Motion with constant volatility). It is now reasonably well established that, although Geometric Brownian Motion is a good approximation in most cases, there are various important deviations for different asset classes. This problem was addressed by J.C. Cox and S.A. Ross (1976) who introduced the constant elasticity of variance (CEV) model. R.C. Merton (1976) also addressed this problem. Robert Merton started continuous-time financial modeling with his explicit dynamic programming solution for optimal portfolio and consumption polices. This sets the stage for his general equilibrium model of security prices, another milestone (1973). His major contribution was his arbitrage-based proof of the option pricing formula introduced by Fisher Black and Myron Scholes (1973), and his continual development of that approach to derivative pricing. The original argument of F. Black and M. Scholes (1973) is based on a "local no arbitrage" assumption. The solution to the general model using the Feynman-Kac Formula is from D. Duffie (1988), where extensions and numerical approximations are noted. This solution extends in application (only in this Brownian Motion setting) beyond that of J.M. Harrison and D.M. Kreps (1979), in that it allows stochastic interest rates and applies under weaker conditions on the security process, which need not "span" by continuous trading. We have implicitly taken the "risk-neutral" pricing concept that the options being priced are European options, in that they may be exercised only at expiry. M. Parkinson (1977) and R. Geske and H. Jphnson (1984) have results on the pricing of American options. D. Duffie (1988) derived the Black-Scholes Formula in five ways! These are: (1) by a limit from discrete-time as the length of a time interval shrinks to zero, using the Lindeberg's Central Limit Theorem on a triangular array of random variables; (2) by a direct solution to the partial differential equation (PDE) derived from an absence of arbitrage which is a Cauchy problem of the parabola equation, using Fourier transform or four standard form transforms; (3) by an indirect solution of this partial differential equation using the Feynmen-Kac Formula; (4) by a limit of the underlying return process from discrete-time using Donsker's Theorem; and (5) by a change of probability measure using Girsanov's Theorem. These methods have extensions well beyond the pricing of an option, but their extensions have different ranges of application, and individually offer different insights and methods. An actuarial approach of option pricing has been proposed by Mogens Bladt and Hina Hviid Rydberg (1998). The basic idea is the following: discount risk-free asset future prices according to the risk-less interest rate and risk asset prices according to their expected rate of return. This paper studies the Black-Scholes Model of option pricing. It makes a more detailed description of the assumptions of the model and the mathematics derivation process of the formula. And then the paper analyzes the sensitivity of the option value of each variable of the model.

2. The Theoretical Analysis of the Model

2.1The Black–Scholes Model Formulation

We illustrate how to use the riskless hedging principle to derive the governing partial differential equation for the price of a European call option. In their seminal paper (1973), Black and Scholes made the following assumptions on the financial market.

(i) Trading takes place continuously in time.
(ii) The riskless interest rate is known and constant over time.
(iii) The asset pays no dividend.
(iv) There are no transaction costs in buying or selling the asset or the option, and no taxes.
(v) The assets are perfectly divisible.
(vi) There are no penalties to short selling and the full use of proceeds is permitted.
(vii) There are no riskless arbitrage opportunities

In 1973, Fischer Black and Myron Scholes gave the famous Black-Scholes model:

$dS_t = \mu S_t dt + \sigma S_t d\hat{W}_t$, μ, σ are constants, (1)

Where $\hat{W}_t$ is a standard Brownian motion on (Ω, F, P). $(S_t)_{t\in\Gamma}$ is defined on (Ω, F, P), and is adapted to a filtration $F = (F_t)_{t\in\Gamma}$, where Γ is a time index set and $\Gamma \subseteq [0,T]$ for some T>0.

Under the neutral-risk probability Q, the model is $dS_t=rS_tdt + \sigma S_t dW_t$, that is $S_t=S_0e^{\sigma W_t+(r-\sigma^2/2)t}$, where W_t is a standard Brownian motion with respect to Q and $W_t=\hat{W}_t+[(\mu-r)/\sigma]t$, r is the risk-free interest rate, and σ is the volatility of the underlying stock. Mathematically, the prce of the option is given by: $c=E[e^{rT}f(S_T)\,|\,F_0)]$, where f is the payoff function, $f\in L^2(S_T)$, $F_t=\sigma(W_s, 0\le s\le t)$, $t\in[0,\infty)$. E denotes the expectation corresponding to Q.

Take $V(t,y) = E_y(e^{-r(T-t)}f(S_{T-t}))$, then $c=V(0,y)$ and $V(t,y)$ solves the Cauchy problem:

$$\begin{cases}\frac{\partial}{\partial t}V(t,y)+\frac{1}{2}\sigma^2y^2\frac{\partial^2}{\partial y^2}V(t,y)+ry\frac{\partial}{\partial y}V(t,y)-rV(t,y)=0,(t,y)\in[0,T)\times(0,\infty);\\ V(T,y)=f(y),y\in(0,\infty).\end{cases}$$

Let $\tilde{V} = e^{-rt}V$, $x = e^{-rt}y$, then

$$\begin{cases}\frac{\partial}{\partial t}\tilde{V}(t,x)+\frac{1}{2}\sigma^2x^2\frac{\partial^2}{\partial x^2}\tilde{V}(t,x)=0,(t,x)\in[0,T)\times(0,\infty);\\ \tilde{V}(T,x)=e^{-rT}f(e^{rT}x), x\in(0,\infty)\end{cases}$$

Now apply Itô's formula to $\tilde{V}(t, x)$, we conclude

$$e^{-rT}f(S_T) = c+\int_0^T \varphi_t d\tilde{S}_t, \quad (2)$$

where $\tilde{S}_t = e^{-rt}S_t$ is the discounted price of the risky asset, and $d\tilde{S}_t= \sigma\tilde{S}_t dW_t$, and it is easily known $\tilde{S}_t$ is a martingale with respect to the filtration F and the probability measure Q; the delta hedging strategy φ_t is given by $\varphi_t = (\partial/\partial y)\ V(t,S_t)$, $0\le t\le T$, and $\varphi\in C^{1,2}([0,T)\times \mathbf{R}^+)$, where $C^{p,q}([a,b)\times B)=\{f:[a,b)\times B\to \mathbf{R}$, all possible partial derivatives of f, where one differentiates at most p-times with respect to the first variable and at most q-times with respect to the second one, exist and continuous on $[a,b)\times B\}$

Take $\Psi_t=\Psi(t, S_t)=(\varphi_t-\upsilon_a)\,\sigma\tilde{S}_t$, $0\le a\le t\le b$, where a is some close time, b is the next open time after a. $\upsilon_a\in F_a$, υ_a is the optimal variance hedging strategy, that is, considering we can't trade just in the time interval (a,b), υ_a is the hedging strategy $\tilde{\varphi}$ we can take at time a which minimize

$$E_{Sa}\left|\int_a^b(\varphi_t-\tilde{\varphi}_t)d\tilde{S}_t\right|^2. \quad (3)$$

It is known by Zhang[13]

$$\upsilon_a = \frac{E\left(\int_a^b \varphi_u\tilde{S}_u^2\,du\,\middle|\,F_a\right)}{E\left(\int_a^b \tilde{S}_u^2\,du\,\middle|\,F_a\right)}. \quad (4)$$

We consider European call option. Assume the price of the underlying stock satisfy (1.1) and $\sup E\,|\Psi(u,S_u)|^2<\infty$. For an European call option with maturity T and strike price X,

$$V_c(t,S_t)=S_tN(d_1)-X\,e^{-r(T-t)}N(d_2), \quad (5)$$

For an European put option with maturity T and strike price X,

$$V_p(t,S_t)= X\,e^{-r(T-t)}[1-N(d_2)] - S_t[1-N(d_1)], \quad (6)$$

Where

$$d_1=\frac{\ln(S_t/X)+[r+(\sigma^2/2)](T-t)}{\sigma\sqrt{T-t}}, \quad (7)$$

$$d_2=d_1-\sigma\sqrt{T-t}, \quad (8)$$

N (d_1) and N (d_2) represent normal probabilities based on the values of d_1 and d_2. We can easily calculate d_1 and d_2, and then look them up in a normal probability table to obtain N (d_1) and N (d_2), and then insert the values of N (d_1) and N (d_2) into the above formula.

Consider the following example.

Use the Black-Scholes-Merton Model to calculate the prices of European call and put options on an asset priced at 68.5. The exercise price is 65, the continuously compounded risk-free rate is 4 percent, the options expire in 110 days, and the volatility is 0.38. There are no cash flows on the underlying.

Solution: The time to expiration will be T= 110/365 = 0.3014. Then d_1 and d_2 are

$d_1=0.4135$
$d_2=0.2049$

Looking up in the normal probability table, we have

N (0.41) =0.6591
N (0.20) =0.5793

Plugging into the option price formula

C=7.95
P=3.67

2.2 Inputs to the Black-Scholes Model

The Black-Scholes Model has five inputs: the underlying price, the exercise price, the risk-free rate, the time to expiration, and the volatility. Let us now take a look at the various inputs required in the Black-Scholes Model. We need to know where to obtain the inputs and how the option price varies with these inputs.

2.2.1 The Underlying Price

The price should generally be obtained as a quote or trade price from a liquid, open market. Call option prices should be higher the higher the underlying price and the put option price should be lower.

The relationship between the option price and the underlying price has a special name: It is called the option delta. In fact, the delta can be obtained approximately from the Black-Scholes formula as the value of N (d_1) for calls and N (d_1)-1 for puts. More formally, the delta is defined as

$$\text{Delta}=\frac{\text{Change in option price}}{\text{Change in underlying price}}$$

The above definition for delta is exact; the use N (d_1) for calls and N (d_1)-1 for puts is approximate.

Delta is improtant as arisk measure. Traders, especially dealers in options, use delta to construct hedges to offset the risk they have assumed by buying and selling options.

Gamma is a numerical measure of how sensitive the delta is to a change in the underlying—in other words, how much the delta changes. When gamma is large, the delta changes rapidly and cannot provide a good approximation of how much the option moves for each unit of movement in the underlying. We shall not concern ourselves with measuring and using gamma, but we should know a few things about the gamma and, therefore, about the behavior of the delta.

Gamma is larger when there is more uncertainty about whether the option will expire in-or out-of-the-money. This means that gamma will tend to be large when the option is at-the-money and close to expiration. In turn, this statement means that delta will be a poor approximation for the option's price sensitivity when it is at-the-money and close to the expiration day. Thus, delta hedges will wok poorly. When the gamma is large, we may need to use a gamma-based hedge, which would require that we add a position in another option to the delta-hedge position of the underlying and the option. We shall not take up this advanced topic here.

2.2.2 The Exercise Price

The exercise price is easy to obtain. It is specified in the option contract and does not change. Therefore, it is not worthwhile to speak about what happens when the exercise price changes, but we can talk about how the option price would differ if we choose an option with a different exercise price. The call option price will be lower the higher the exercise price and the put option price will be higher.

2.2.3 The Risk-Free Rate

Call options increase in value as the risk-free rate increases. Put options decrease in values as the risk-free rate increases. Indeed, the price of a European call or put option does not change much if we use different inputs for the risk-free rate.

Suppose the discrete risk-free rate quoted in annual terms is r. Then continuous rate is $r^c=\ln(1+r)$.

The sensitivity of the option price to the risk-free rate is called the rho. We shall not concern ourselves with the calculation of rho.

2.2.4 Time to Expiration

Time to expiration is an easy input to determine. An option has a definite expiration date specified in the contract. We simply count the number of days until expiration and divide by 365, as we have done with forward and futures contracts.

As expiration approaches, the option price moves toward the payoff value of the option at expiration, a process known as time value decay. The rate at which the time value decays is called the option's theta. We shall not concern ourselves with calculating the specific value of theta, but be aware that if the option price decreases as time moves forward, the theta will be negative.

Note that both call and put values decrease as the time to expiration decreases. We previously noted that European put options do not necessarily do this. For some cases, European put options can increase in value as the time to expiration decreases, the case of a positive theta, but that is not so for our put. Most of the time, option prices are higher the longer the time to expiration. For European puts, however, some exceptions exist.

2.2.5 Volatility

Volatility is the standard deviation of the continuously compounded return on the stock. We have also noted that the volatility is an extremely important variable in the valuation of an option. In addition, as we illustrate here, option prices ae extremely sensitive to the volatility. Call option prices should be higher the higher the volatility. Put option does too.

Volatility is the only variable that cannot be obtained easily and directly from another source. We can calculate it base on the historical data of the company value.

The formula for the volatility is estimated as follows

$$s=\sqrt{\frac{1}{n-1}\sum_{i=1}^{n}u_i^2-\frac{1}{n(n-1)}(\sum_{i=1}^{n}u_i)^2}\ ,(i=1,2,...,n) \quad (9)$$

$$u_i=\ln(S_i/S_{i-1})\ ,\quad (i=1,2,...,n) \quad (10)$$

$$S_i=S_{i-1}e^{u_i}\ ,\ (i=1,2,...,n) \quad (11)$$

Where

n+1 = the number of observations

S_i = assessed value for company of the end of the i-th interval

u_i = continuous compounding return

As the standard deviation of u_i is $\sigma\sqrt{\tau}$ and s is estimates of $\sigma\sqrt{\tau}$,σ can be estimated as s^*.

$$s^*=\frac{s}{\sqrt{\tau}}\ , \quad (12)$$

Where:

τ = the length of the interval (annual)

The standard error of this estimates approximately $S^*/\sqrt{2n}$.

The relationship between option price and volatility is called the vega, which—albeit considered an option Greek—is not actually a Greek word. We shall not concern ourselves with the actual calculation of the Vega, but know that the Vega is positive for both calls and puts, meaning that if the volatility increases, both call and put prices increase. Also, the Vega is larger the closer the option is to being at-the-money.

Figure 1. Direction of Black-Scholes European Option Prices for a Change in the Five Model Inputs

Sensitivity Factor	Inputs	Calls	Puts
Delta	Asset price	Positively related Delta>0	Negatively related Delta<0

Vega	Volatility	Positively related Vega>0	Positively related Vega>0
Rho	Risk-free rate	Positively related Rho>0	Negatively related Rho<0
Theta	Time to expiration	Value→0 as call→maturity Theta<0	Value→0 as put→maturity Theta<0
	Exercise price	Negatively related	Positively related

3. Applications of the Black-Scholes Model

The convertible bonds give the holder a call option right, so that we can use the option pricing theory to establish its valuation model. The Black-Scholes option pricing model is more mature and widely recognized as the option pricing method. There are many scholars using the Black-Scholes Model to price convertible bonds.

We can select three convertible bonds in the market; the data is shown in Table 1.

Table 1.convertible bonds samples (January 5, 2009)

Company	Dahuang	Haima	Wuzhou
Code	110598	125572	110368
maturity	5	5	5
Coupon rate	1.5/1.8/2.1/2.4/2.7	1.5/1.8/2.2/2.5/2.7	1.3/1.5/1.7/1.9/2.1
Listing date	December 28,2007	January 30,2008	March 14,2008
Conversion date	June 19,2008	July 16,2008	August 29,2008
Initial conversion price	14.32	18.33	10.14
Latest conversion price	10.08	3.6	4.73

Data sources: Shanghai Stock Exchange

(i) Calculate the price of pure bond

Because the maturities of the convertible bonds that we choose are 5 years, the risk-free interest rate we use is the rate of trading treasury bonds with the same maturity. So the risk-free rate is 2.51%. We can calculate the theoretical price of pure bond which is shown in Table 2

Table 2. The price of pure bond

Company	Dahuang	Haima	Wuzhou
price	98.01	98.19	99.19

(ii) Calculate the price of the option component

We can calculate the theoretical price of the option component based on Black-Scholes Model. There are two parameters, risk-free rate and stock price volatility, to be calculated. Since the risk-free interest rate used in the model is a continuous compound rate, we can calculate the risk-free rate $r = \ln(1+2.51\%) = 0.0248$. In the empirical analysis, we choose 247 days. We can calculate the stock price volatilities, which are shown in Table 3.

Table 3. The stock price volatilities

Company	volatility
Dahuang	0.533
Haima	0.682
wuzhou	0.631

Based on the Black-Sholes formula (5), we calculate the theoretical price of the option shown in Table 4.

Table 4. The price of option component

Company	Stock price	Conversion price	Volatility	Option price
Dahuang	11.44	10.08	0.533	5.3879
Haima	3.31	3.60	0.682	3.6872
Wuzhou	4.49	4.73	0.631	3.2102

According to the analysis, we can plus the price of pure bond and the price of the option component, and then compare them with the market price. The results are shown in Table 5.

Table 5. Comparison of the theoretical price and market price

Company	Theoretical price	Market price
Dahuang	103.4	120.68
Haima	111.88	111.15
Wuzhou	112.4	111.78

From the above analysis, we can see that the operability of calculation for convertible bonds based on Black-Scholes Model is relatively strong. Because the Black-Scholes Model does not include variables which reflect investors' risk appetite, the model does not need to estimate the expected rate of return for investors. It does not need to forecast future cash flows of dividends and the expected growth of dividends, which, to some extent, overcome the defects of the traditional option pricing methods. However, because the Black-Scholes Model is under certain assumptions which cannot accurately describe the actual situation of the real market, the Black-Scholes Model also has its limitations that may make deviations to occur. Therefore, the application of the Black-Scholes Model is not a complete denial of the traditional option methods, nevertheless, it is an ideological update.

4. Conclusions

This paper intends to study option pricing problems based on the Black-Scholes Model. At the beginning of seventy years , Fiseher Black and Myron Scholes have made the unprecedented work in the domain of the option pricing theory, they proposed the first complete option pricing model, which named the Black-Scholes Model, widely accepted by the theory and the industrial world. The formula of Black-Scholes analyzes the pricing of option and risk management from the quantitative view. This is a powerful sustainment for the option's popularization. Afterwards, breakthrough is achieved in the research of financial phenomenon, which enters the stage of quantitative study from the qualitative investigation. However, in the reality financial market, their idealized condition has limitation. A mass of finance practice has indicated that there is a serious warp between the hypothesis of Black-Seholes model about the underlying asset Price and the realistic markets. Therefore ,many scholars put forward many new kinds of option pricing models by relaxing some assuming conditions of Black-Scholes model. The option pricing theory is becoming mature. Such as, according to the hypothesis of Geometric Brownian Motion that the price of object is supposed to comply with log-normal distribution, some experts pointed that the factual distribution with fat-tail, which was not in accordance with log-normal distribution, should be improved in the light of the results of marketed empirical study. J.C. Cox and S.A. Ross asserted that the movement of the price of object was not constantly changeable but displayed an array of bounces, which confirmed to Poisson distribution (J.C. Cox and S.A. Ross, 1976). Merton maintained that the change of the price of object was still continuous in an objective manner after being fully diffused.

As it is, a variety of researches involved the method of option valuation are increasingly discussed and advanced because the development of option is everlasting in theory. In reality, option is complicated and with widespread use. Consequently, studying the universality and individuality of the methods of option valuation is of great significance for the research of intricate option valuation [17-24].

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