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Study on Behavior of Fuzzy Markov Chains

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Abstract – We consider finite Markov chains where there are uncertainties in some of the transition probabilities. These uncertainties are modeled by fuzzy algebra and max-min algebra. We simulate fuzzy Markov chain using different sizes and Rand function of MATLAB software. It is observed that the most of fuzzy Markov chains not only do have an ergodic behavior, but also they are periodic.

Keywords – Fuzzy Markov Chains, Periodicity, Simulation, MATLAB Rand function

1. Introduction

The main motivation of this paper is to begin the expansion of fuzzy Markov chains based on possibilities. A reason is that one might prefer subjective possibilities over subjective probabilities to model the uncertainties.

This paper is divided into five sections. Section 1 is an introductory section. In section 2 some concepts of fuzzy Markov chains and their properties are defined [1] and [2]. In section 3, a linear transformation of the fuzzy Markov chains into a classic Markov chains is given [4]. In section 4 of this paper, we employ MATLAB Rand function to generate membership functions of fuzzy Markov chains and finally presents the concluding remarks.

2. Basic definitions of fuzzy Markov chains and their properties

The definition of a fuzzy Markov chain is based on a squared relational matrix represents the possibility that a discrete state at instant becomes into any state at next instant $t+1$as follows:

$$\bar{P}\left(X^{(t)}=s \middle| X^{(t-1)}\right)=x^{(t-1)}. \qquad (1)$$

Here, $\bar{P}\left(X^{(t)}\right)$ is a fuzzy distribution of the process characterized by a membership function. In this paper, we use the basic definitions about fuzzy set given by **J**. Buckley in [3] and fuzzy Markov chains given by Avrachenkov and Sanchez in [2].

Definition 2.1. Let $S=\{1,2,\dots,n\}$. A finite fuzzy set for a fuzzy distribution on S is defined by a mapping x from S to $[0,1]$ represented by a vector $x=\{x_1,x_2,\dots,x_n\}$, with $0\le x_i \le 1, i\subset S$.
In this definition, xi is the membership degree that a state i has regarding a fuzzy set S, $i\in S$ with cardinality m, $C(S)=m$. All relations and compositions are defined by fuzzy sets theory since are useful tools to find a fuzzy stationary distribution.

Now, a fuzzy relational matrix $\bar{P}$ is defined in a metric space S × S by a matrix $\{pij\}_{i,j=1}^{m}$ with $0\le pij \le 1$, $i,j\in S$. The complete set of all fuzzy sets is denoted by $F(S)$ where $C(S)=m$. We note that it dose not need the addition of elements of each row of the matrix $\bar{P}$be equal to one.[3]
This fuzzy matrix $\bar{P}$ allows to define all relations among the m states of the fuzzy Markov chain at each time instant t, $t=1,2,\dots,n$ as follows.

$$x_j^{(t+1)}=max_{i\in S}\{x_j^t\wedge \bar{p}_{ij}\},\quad j\in S \qquad (2)$$

$$x^{(t+1)}=x^{(t)}o\bar{P}\,. \qquad (3)$$

where i and j, $i,j=1,2,\dots,m$ are the initial and final states of the transitionand $x^{(0)}$ is the initial distribution.

Also,

$$\bar{P}^{t}_{ij} = max_{k\in S}\{\bar{P}_{ik} \wedge \bar{P}^{t-1}_{kj}\}, \quad i,j \in S \quad (4)$$

$$\bar{P}^{t} = \bar{P} \circ \bar{P}^{t-1}. \quad (5)$$

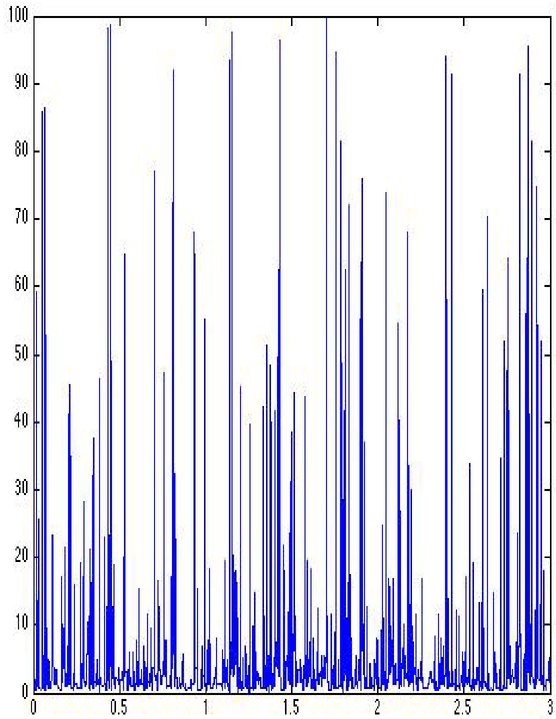
Thomason in [9] showed that the powers of a fuzzy matrix are stable if it is used the max-min operator. more information about powers of a fuzzy matrix are, see in [2, 5]. Now, a stationary distribution of a fuzzy matrix defined as follow [6].

Definition 2.3 (Stationary distribution). Let the powers of the fuzzy transition matrix $\bar{P}$ converge in τ steps to a non-periodic solution, then the associatedfuzzy Markov chain is called aperiodic fuzzyMarkov chain and $\bar{P}^{*} = \bar{P}^{\tau}$ is its stationary fuzzy transition matrix.

Definition 2.4 (Strong Ergodicity and Week Ergodicity). A fuzzy Markov chain is called strong Ergodic if it is aperiodic and its stationary transition matrix has identical rows.

A fuzzy Markov chain is called weakly Ergodic if it is aperiodic and its stationary transition matrix is stable with no identical rows.

Now, if the stationary distribution of $\bar{P}$ is given by $\bar{P}^{*} = \bar{P}^{\tau}$ where $\lim_{n\to\tau} \bar{P}^{n} = \bar{P}^{*}$, then $\bar{P}$ becomes an idempotent matrix. Sanchez defined its stationary distribution by its **Eigen fuzzy set**, see in [8].



3. Converting a fuzzy Markov chain to a classic Markov chain

The scalar cardinality and the cumulative membership function of a fuzzy set is used to define a conversion of fuzzy Markov chain into a classic Markov chain [6].

Definition 3.1 (Scalar cardinality of a fuzzy set). The well known scalar cardinality of a fuzzy set namely $|S|$ is a measure of amount and represents the total size of the membership function of $S\ (\mu_S)$ as follows:

$$\Lambda_i = |S_i| = \sum_{j=1}^{m} \mu_{S_{ij}}, \qquad i \in m. \qquad (6)$$

where Λ is a diagonal squared matrix whose components are the scalar cardinality of each set $|S_i|$ by row, denoted by Λ_i, $i \in m$.

Definition 3.2 (Cumulative membership function). The cumulative membership function is defined as:

$$\psi_{si} = \sum_{j=1}^{x} \mu_{S_{ij}} \qquad i \in m \qquad (7)$$

Note that in the probabilistic case $F(\infty) = 1$ while in the possibilistic case $1 < \psi(\infty) < \Lambda$.
also $\mu_S(\infty) > 1$ is an important issue to be solved. This is an interpretation problem since the definition of a normalized fuzzy set determines that $max_{x\in S}\, \mu_S(x) = 1$ and the above definition dose not have this property. To solve it, an easy way to normalize S(x) is dividing it by Λ, obtaining the following definition:

$$\psi_{si} = \frac{1}{\Lambda_i} \sum_{j=1}^{m} \mu_{S_{ij}}, \qquad i \in m. \qquad (8)$$

Remark 1 (Relation between $\psi_S(x)$ and $|S|$). Recall that $\psi_S(\infty) = \Lambda$.

Theorem 3.1. Let $\bar{P}$ a fuzzy Markov chain with elements $p_{ij} = \mu_{S_{ij}}$, then $\bar{P}$ can be converted to a classic Markov chain namely P, using the following linear transformation:
$P = \Lambda^{-1}\bar{P}$.

Proof. First, the scaler cardinality of S_i defined as (6), so, Λ, Λ^{-1} and $\bar{P}$ are defined as:

$$\Lambda = \begin{bmatrix} \Lambda_1 & 0 & \cdots & 0 \\ 0 & \Lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \Lambda_m \end{bmatrix};$$

$$\Lambda^{-1} = \begin{bmatrix} \Lambda_1^{-1} & 0 & \cdots & 0 \\ 0 & \Lambda_2^{-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \Lambda_m^{-1} \end{bmatrix};$$

$$\bar{P} = \begin{bmatrix} \mu_{S_{11}} & \mu_{S_{12}} & \cdots & \mu_{S_{1m}} \\ \mu_{S_{21}} & \mu_{S_{22}} & \cdots & \mu_{S_{2m}} \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{S_{m1}} & \mu_{S_{m2}} & \cdots & \mu_{S_{mm}} \end{bmatrix}$$

Finally we have:

$$P = \begin{bmatrix} \Lambda_1^{-1} & 0 & \cdots & 0 \\ 0 & \Lambda_2^{-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \Lambda_m^{-1} \end{bmatrix} * \begin{bmatrix} \mu_{S_{11}} & \mu_{S_{12}} & \cdots & \mu_{S_{1m}} \\ \mu_{S_{21}} & \mu_{S_{22}} & \cdots & \mu_{S_{2m}} \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{S_{m1}} & \mu_{S_{m2}} & \cdots & \mu_{S_{mm}} \end{bmatrix}$$

,

which is $P = \Lambda^{-1}\bar{P}$. Now, all their components result in the following matrix:

$$P = \begin{bmatrix} \frac{\mu_{S_{11}}}{\Lambda_1} & \frac{\mu_{S_{12}}}{\Lambda_1} & \cdots & \frac{\mu_{S_{1m}}}{\Lambda_1} \\ \frac{\mu_{S_{21}}}{\Lambda_2} & \frac{\mu_{S_{22}}}{\Lambda_2} & \cdots & \frac{\mu_{S_{2m}}}{\Lambda_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\mu_{S_{m1}}}{\Lambda_m} & \frac{\mu_{S_{m2}}}{\Lambda_m} & \cdots & \frac{\mu_{S_{mm}}}{\Lambda_m} \end{bmatrix}.$$

This transformation obtains a matrix P whose elements satisfy all basic properties of a stochastic classic distribution function, i.e.

$$i)\ 0 \le p(x_{ij}) \le 1$$

$$ii)\ \sum_{j \in S}^{\infty} p(x_{ij}) = 1, \quad i \in S$$

$$iii)\ P(X \le x_{ij}) = \sum_{i \in S} p(x_{ij}) = F(x), \quad i \in S$$

In this way, it is possible to show that all elements of P obtained from $\bar{P}$ by using $P = \Lambda^{-1}\bar{P}$ conforms a classic distribution function, agree to the Markovian property of a stochastic transition matrix. To that effect, their properties are shown next.

The property presented in $i)$ refers to the domain of i, j, and as $\mu_{S_{ij}} < \Lambda_i$, $i, j \in m$, then it asserts that $\frac{\mu_{S_{ij}}}{\Lambda_i} < 1, i, j \in m$ [4].

4. Simulation

We present a simulation on fuzzy Markov chains to identify some characteristics about their behavior based on matrix analysis.

Algorithm: Now, using the linear operation given in section 3, we simulate the fuzzy Markov chains.

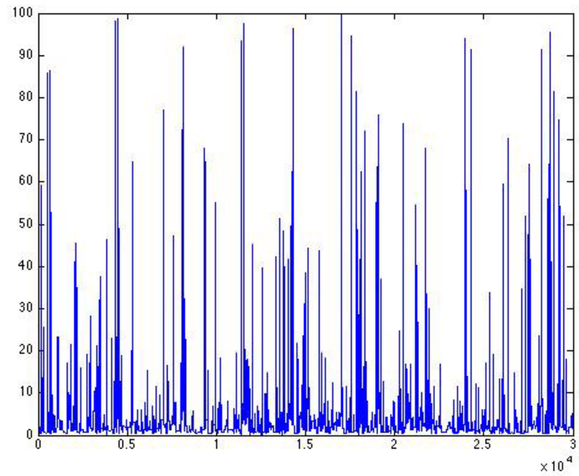


Figure 1: Behavior of Fuzzy Markov Chains using RAND function of MATLAB.

Size of the fuzzy Markov chains: The size of $\bar{P}$ denoted by m, $m = \{5, 10, 50, 100\}$.

Random number generator: All the entries $\{p_{ij}\} = \{\mu_{ij}\}$ of the matrix $\bar{P}$ are obtained by using the Rand function of MATLAB(Table 1).

Number of runs: 100 runs are simulated for per each size of $\bar{P}$.

Size	Strong ergodicity	Weak ergodicity	Periodic	Total
$m = 5$	8	11	81	100
$m = 10$	5	8	87	100
$m = 50$	1	2	97	100
$m = 100$	-	2	98	100

Table 1: Total number of strong ergodicity, weak ergodicity and periodic chains using the Rand function of MATLAB

The Table 1 shows the total number of fuzzy Markov chains which has either a strong ergodic, weak ergodic or periodic behavior per each size of P.

5. Conclusion

Based on our presented results and former related results we conclude that the fuzzy Markov chains generated by MATLAB Rand function are usually periodic.

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