Advances in Computer Science and its Applications (ACSA)
Vol. 2, No. 3, 2013, ISSN 2166-2924

www.worldsciencepublisher.org

The generation extinction time and the estimation of the mean using Monte Carlo method in Galton-Watson geometric offspring distribution

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Abstract- One of interesting matter issues in branching process is the computation of probability and the time of process generation extinction. In this article we initially obtain the time of process extinction with this assumption that the offspring distribution is geometric. Since this distribution depends on the mean of offspring, eventually we represent estimation for the mean by Monte-Carlo Method.

Keywords- Branching process, Estimation, Extinction time, Geometric offspring distribution, Monte Carlo simulation.

1. Introduction

In branching process, the random number of the initial particles produces the higher number of the same or different kind of particles. They in turn create the second generation and the process of reproduction continues as the same way. The number of offspring of each generation is not previously obvious and will be determined randomly. We indicate the number of existing particles in each generation of process by $Z_i, i=0,1,2,...$ and suppose that each particle based on probability mass function $\{q_k, k=0,1,2,...\}$ reproduces independent from themselves and parent particles. More specifically, if consider the random variable X as the number of offspring of each particle, then $P(X=k)=q_k, k=0,1,...$. Suppose that $E(X)=m, Var(X)=\sigma^2$. In Galton- Watson branching process it is supposed that the life time of each particle is a time unit that this unit can be hour, day, week, year or every instant period of time and at the end of each particle lifetime, the reproduction occurs.

In investigation of a branching process, whether extinction happens or not, knowing that how long it takes to be a generation extinct, for example, how long does it take to be an infectious disease extinct before becoming an epidemic disease? or how long it takes to a royal family extinct?

The branching process theory is of specific important for mathematics, biologists and statisticians in terms of the estimation of the number of particles in each generation make it possible to determine the final extinction probability and the time of generation extinction.

2. The generation extinction time of Galton-Watson process with geometric offspring distribution

If we have $Z_n=0$ for a generation like n, but $Z_{n-1}\neq 0$, then the result of this event is the extinction of initial current generation. Now if α_n is the extinction probability of $n-th$ generation, then given the probability generating function, we have $\alpha_n=P(Z_n=0)=G_n(0)$, so that α is the final extinction probability of the chain of the smallest non-negative root of equation $G(x)=x$ and $G(x)$ is considered as the probability generating function of offspring distribution [1]. Suppose that T is the exact time of extinction, the distribution of T is the number of previous generation before extinction, it means $T=n$ when generation n is the first generation with 0 member**.** In other words we have $T=n \Leftrightarrow Z_n=0, Z_{n-1}>0$

So, the distribution T is the exact time of extinction that can be obtained from following relation:

$$P(T=n)=P(Z_n=0\cap Z_{n-1}>0)=G_n(0)-G_{n-1}(0)=\alpha_n-\alpha_{n-1} \tag{1.2}$$

Because according to partitioning rule we have:

$$P(X_n = 0) = P(X_n = 0 \cap X_{n-1} > 0) + P(X_n = 0 \cap X_{n-1} = 0) \quad (2.2)$$

But the event $\{Z_n = 0 \cap Z_{n-1} = 0\}$ is an event that the process extinct by $(n-1)$ -th generation and nth. Nevertheless we know that if the process be extinct in $(n-1)$ - th generation, it will also be extinct in $n-$ th generation. So $Z_n = 0$ is a redundant term, thus: $P(Z_n = 0 \cap Z_{n-1} = 0) = P(Z_{n-1} = 0) = G_{n-1}(0)$
From equation (2.2) we get,

$$P(T = n) = P(Z_n = 0 \cap Z_{n-1} > 0) = P(Z_n = 0) - P(Z_n = 0 \cap Z_{n-1} = 0)$$
$$= G_n(0) - G_{n-1}(0) = \alpha_n - \alpha_{n-1}$$

from [4] we have $G_n(S) = G_{(n)}(S) = G(G(...(S)...))$. In general, it is not possible to find a closed form expression of $G_n(s)$. the only non-trivial family size distribution that allows us to find a closed-form expression for $G_n(s)$ is the geometric distribution. Following term state this closed-form.

Theorem (2.1): let $\{Z_n : n \geq 0\}$ be a branching process with family size distribution with parameter p means $X \sim Geometric(p)$, then the probability generating function is obtained as follows:

$$G_n(s) = E(s^{Z_n}) = \begin{cases} \dfrac{n-(n-1)s}{n+1-ns} & p = q = \dfrac{1}{2} \\ \dfrac{(m^n - 1) - m(m^{n-1} - 1)s}{(m^{n+1} - 1) - m(m^n - 1)s} & p \neq q ; m = \dfrac{q}{p} \end{cases}$$

Proof: The proof for both $p = q$ and $p \neq q$ proceed by mathematical induction. We will give a sketch of the proof when $p = q = 1/2$. The proof for $p \neq q$ works in the same way but is trickier. Suppose that $p = q = 1/2$, then

$$G(s) = \sum_{x=0}^{\infty} s^x pq^x = p\sum_{x=0}^{\infty} (qs)^x = \frac{p}{1-qs} = \frac{\frac{1}{2}}{1-\frac{s}{2}} = \frac{1}{2-s},$$

Using the branching process recursion formula

$$G_2(s) = G(G(s)) = \frac{1}{2-G(s)} = \frac{1}{2-\frac{1}{2-s}} = \frac{2-s}{2(2-s)-1} = \frac{2-s}{3-2s},$$

The inductive hypothesis is that: $G(s) = \dfrac{n-(n-1)s}{n+1-ns}$,

Then we obtain that:

$$G_{n+1}(s) = G_n(G(s)) = \frac{n-(n-1)G(s)}{n+1-nG(s)} = \frac{n-(n-1)(\frac{1}{2-s})}{n+1-n(\frac{1}{2-s})}$$

$$= \frac{(2-s)n-(n-1)}{(2-s)(n+1)-n}$$

$$= \frac{n+1-ns}{n+2-(n+1)s}$$

So the judgment of theorem is satisfied for $n+1$. Now using mentioned theorem and the relation (1.2) we obtain the time of generation extinction for branching process with geometric offspring distribution.
While we have $m = 1$,

$$P(T = n) = G_n(0) - G_{n-1}(0) = \frac{n}{n+1} - \frac{n-1}{n} = \frac{n^2 - n^2 + 1}{n(n+1)} = \frac{1}{n(n+1)} \quad (3.2)$$

Let $m \neq 1$, then

$$P(T = n) = \frac{m^n - 1}{m^{n+1} - 1} - \frac{m^{n-1} - 1}{m^n - 1} = \frac{(m^n - 1)^2 - (m^{n+1} - 1)(m^{n-1} - 1)}{(m^{n+1} - 1)(m^n - 1)}$$
$$= \frac{m^{n+1} - 2m^n - m^{n-1}}{(m^{n+1} - 1)(m^n - 1)} \quad (4.2)$$

On the other hand *j*-th moment of T is obtained from the relation $E(T^j) = \sum_{n=0}^{\infty} \left[(n+1)^j - n^j\right](1-\alpha)$,[3].

So, $E(T) = \sum_{n=0}^{\infty}(1-\alpha_n)$ is the mean of extinction time, so that

1) Finite if $m < 1$,
2) Infinite if $m = 1$ (despite extinction being definite) , if σ^2 is finite,
3) Infinite if $m > 1$ (because with positive probability, extinction never happens).

3. Estimation m using Monte Carlo method

Since the extinction time of T depends on m value, we introduce a graphic method to estimate m . From Martingale convergence theorem we have $Z_n / m^n \xrightarrow{a.s.} W$, where W is a random variable [4]. So by taking the logarithm we obtain: $Z_n = Wm^n \Rightarrow \ln Z_n = \ln W + n \ln m$.
Therefore the chart $\ln Z_n$ tends to a straight line after several stages where for large n, $\ln m = \dfrac{\ln Z_n}{n}$, is estimation for the slope of line. If the offspring distribution is a geometric process with parameter $p = 0.3$, figure 1 shows an example of a simulation of a branching process in 10 generations.

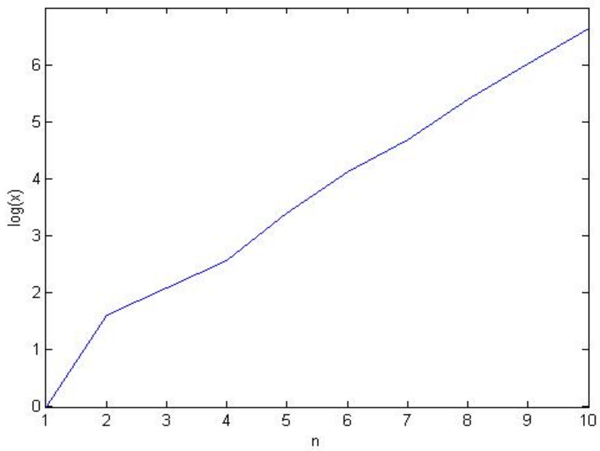
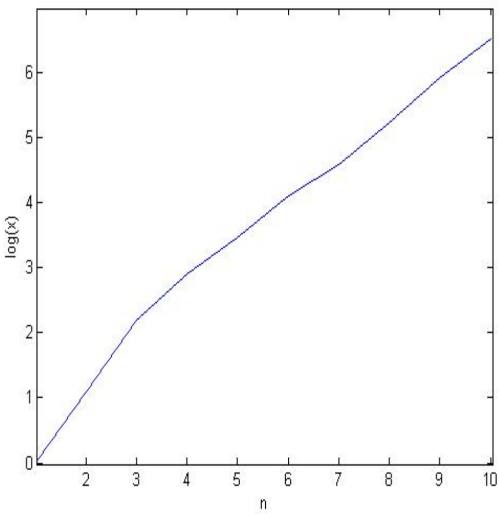


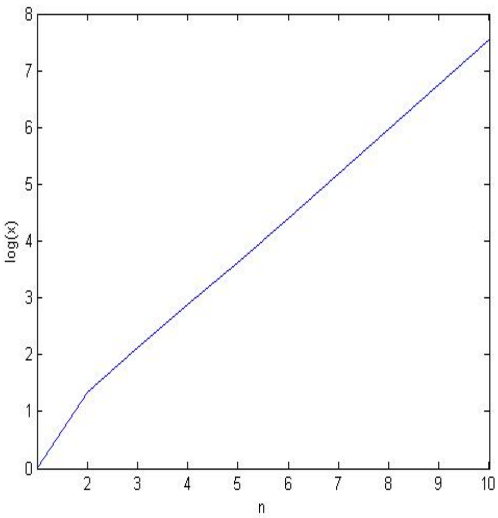
Figure1. A simulation of Galton-Watson branching processes when the offspring distribution is Geometric

Now to improve the estimation in lower generation using Monte Carlo method, we have plotted charts by repeating algorithm and drawing figure and averaging of generations

In figure 2, the charts (b) and (c) indicate algorithm performing 10 and 100 times respectively.



(b)



(c)

Figure2. A simulation of Galton-Watson branching processes when the offspring distribution is Geometric using Monte Carlo simulation

The algorithm of Monte Carlo method to stimulate the branching process with geometric offspring distribution is as follows:

1. Get m for the number of interactions of the algorithm.
2. Get n for estimate of Z_n.
3. Set k=1.
4. Set j=1 and $Z_0 = 1$.
5. Set $Z_j = 0$.
6. Set $i = 1$.
7. Generate X according to Geometric distribution with the parameter p $X \equiv \left[\frac{\log(U)}{\log(q)}\right]+1$ .where U is a random number.
8. Set $Z_j \leftarrow Z_j + X$.
9. Set $i \leftarrow i+1$, if $i \le Z_{j-1}$ go to step 7.
10. Set $j \leftarrow j+1$, if $j \le n$ go to step 5.
11. Set $sum = sum + Z_j, k \leftarrow k+1$ and if $k \le m$ go to step 4.
12. Obtain sum/k as Monte Carlo simulation Z_n .
13. As the end of algorithm, show the value of $\ln Z_n$.

As it can be seen by increasing algorithm performing, the better result for $\ln m$ estimation is obtained through the chart and the slope of charts in lower generation is calculable. Now if this function becomes exponential we obtain:

$$e^{\ln m} = e^{\frac{\ln Z_n}{n}} \Rightarrow m_n^* = \sqrt[n]{Z_n}$$

Corollary: Using relation (3.2) and (3.4) we can obtain the extinction time of a branching process by geometric offspring distribution and since this distribution depends on the offspring mean, we present estimation of m using graphical method and stimulation of process. However, given to a time stimulation, the chart tends to a straight line in higher generation and we can provide better results to estimate m using Monte Carlo method and algorithm repeating.

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