A Numerical Algorithm for Newton-Cotes Open and Closed Integration Formulae Associated with Eleven Equally Spaced Points

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Abstract
We know that the algorithm plays a very important role in the solution of any problem and without algorithm it is very difficult to write the program. So in this paper we have designed a new algorithm for the solution of definite integral which can't be solved by means of Newton-Cotes closed integration formulae(Trapezoidal rule, Simpson's 1/3 rule, Simpson's 3/8 rule, Boole's rule and Weddle's rule). In this paper we have consider two integrals and solved it with the help of proposed algorithm and existing five Newton-Cotes open integration formulae, a result we have shown that the proposed algorithm is much better than the existing five Newton-Cotes open integration formulae.

Keywords and Phrases: Algorithm, Integrals, Newton-Gregory forward difference formula.

1. Introduction

Numerical analysis naturally finds applications in all fields of engineering and the physical sciences, but in the 21st century, the life sciences and even the arts have adopted elements of scientific computations. Integrations appear in the movement of heavenly bodies planets, stars and galaxies.Before the advent of modern computers numerical methods often depended on hand calculation in large printed tables. Since the mid 20th century, computers calculate the required functions instead. So far as the techniques of the numerical integration are concerned, the following five Newton-Cotes open integration formulae (1.1) to (1.5) in simplest forms, are fairly well known in the literature of numerical analysis.

$$\int_{x_0}^{x_2} f(x)\mathrm{d}x = 2hf_1 \tag{1.1}$$

$$\int_{x_0}^{x_3} f(x)\mathrm{d}x = \frac{3h}{2}(f_1 + f_2) \tag{1.2}$$

$$\int_{x_0}^{x_4} f(x)\mathrm{d}x = \frac{4h}{3}(2f_1 - f_2 + 2f_3) \tag{1.3}$$

$$\int_{x_0}^{x_5} f(x)\mathrm{d}x = \frac{5h}{24}(11f_1 + f_2 + f_3 + 11f_4) \tag{1.4}$$

$$\int_{x_0}^{x_6} f(x)\mathrm{d}x = \frac{3h}{10}(11f_1 - 14f_2 + 26f_3 - 14f_4 + 11f_5) \tag{1.5}$$

In the continuation of earlier work, we obtain one more quadrature formula for 11 equally spaced points in the following form

$$\int_{x_0}^{x_{10}} f(x)\mathrm{d}x \approx \frac{5h}{4536}[4045(f_1+f_9)-11690(f_2+f_8)+33340(f_3+f_7)-55070(f_4+f_6)+67822f_5] \tag{1.6}$$

In (1.6), $f(x)$ is defined on the open intervel (x_0, x_{10}) i.e., $f(x)$ is not defined at $x = x_0$ & $x = x_{10}$. When the integrand is defined on the intervals $(x_0, x_{10}]$ or $[x_0, x_{10})$, in each case we can apply (1.6).
When the integrand is well defined on the closed interval $[x_0, x_{10}]$ then we can derive

$$\int_{x_0}^{x_{10}} f(x)\mathrm{d}x \approx \frac{5h}{299376}[16067(f_0+f_{10})+106300(f_1+f_9)-48525(f_2+f_8)+272400(f_3+f_7)$$
$$-260550(f_4+f_6)+427368f_5] \tag{1.7}$$

Here the integrand is defined at all $x \in [x_0, x_{10}]$.

2. Derivation of (1.6) and (1.7)

Case first:

Suppose the integrand $f(x)$ is undefined at $x = x_0$ and $x = x_{10}$ i.e., $f(x)$ is defined on (x_0, x_{10}). Consider the integral

$$I = \int_{x_0}^{x_{10}} f(x)\mathrm{d}x \tag{2.1}$$

where $x_{10} = x_0 + 10h$.

$$I = h\int_{-1}^{9} f(x_1 + ph)\mathrm{d}p \tag{2.2}$$

Now using the well known Newton-Gregory forward difference formula for nine equally spaced points, we get

$$I \approx h\int_{-1}^{9}\Big\{f_1+p\Delta f_1+\frac{p(p-1)}{2!}\Delta^2 f_1+\frac{p(p-1)(p-2)}{3!}\Delta^3 f_1+\frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 f_1+$$
$$+\frac{p(p-1)(p-2)(p-3)(p-4)}{5!}\Delta^5 f_1+\frac{p(p-1)(p-2)(p-3)(p-4)(p-5)}{6!}\Delta^6 f_1+$$
$$+\frac{p(p-1)(p-2)(p-3)(p-4)(p-5)(p-6)}{7!}\Delta^7 f_1+$$
$$+\frac{p(p-1)(p-2)(p-3)(p-4)(p-5)(p-6)(p-7)}{8!}\Delta^8 f_1\Big\}\mathrm{d}p$$

$$I = h\Big\{10f_1+40(f_2-f_1)+\frac{305}{3}(f_3-2f_2+f_1)+165(f_4-3f_3+3f_2-f_1)+\frac{3305}{18}(f_5-4f_4+6f_3-4f_2+f_1)+$$

$$+\frac{1250}{9}(f_6-5f_5+10f_4-10f_3+5f_2-f_1)+\frac{53975}{756}(f_7-6f_6+15f_5-20f_4+15f_3-6f_2+f_1)+$$

$$+\frac{17225}{756}(f_8-7f_7+21f_6-35f_5+35f_4-21f_3+7f_2-f_1)+$$

$$+\frac{20225}{4536}(f_9-8f_8+28f_7-56f_6+70f_5-56f_4+28f_3-8f_2+f_1)\Big\}$$

After simplification we get (1.6).

Case second:

When $f(x)$ is undefined at $x = x_0$ i.e., $f(x)$ is defined on $(x_0, x_{10}]$ then
Consider the integral

$$I = \int_{x_0}^{x_{10}} f(x)\mathrm{d}x = h\int_{-1}^{9} f(x_1+ph)\mathrm{d}p$$

$$\approx h\int_{-1}^{9}\Big\{f_1+p\Delta f_1+\frac{p(p-1)}{2!}\Delta^2 f_1+\frac{p(p-1)(p-2)}{3!}\Delta^3 f_1+\frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 f_1+$$

$$+\frac{p(p-1)(p-2)(p-3)(p-4)}{5!}\Delta^5 f_1+\frac{p(p-1)(p-2)(p-3)(p-4)(p-5)}{6!}\Delta^6 f_1+$$

$$+\frac{p(p-1)(p-2)(p-3)(p-4)(p-5)(p-6)}{7!}\Delta^7 f_1+$$

$$+\frac{p(p-1)(p-2)(p-3)(p-4)(p-5)(p-6)(p-7)}{8!}\Delta^8 f_1+$$

$$+\frac{p(p-1)(p-2)(p-3)(p-4)(p-5)(p-6)(p-7)(p-8)}{9!}\Delta^9 f_1\Big\}\mathrm{d}p \qquad (2.3)$$

After simplification, we get (1.6)

Case third:

When $f(x)$ is undefined at $x = x_{10}$ i.e., $f(x)$ is defined on $[x_0, x_{10})$ then
Consider the integral

$$I = \int_{x_0}^{x_{10}} f(x)\mathrm{d}x = h\int_{0}^{10} f(x_0+ph)\mathrm{d}p$$

$$\approx h\int_{0}^{10}\Big\{f_0+p\Delta f_0+\frac{p(p-1)}{2!}\Delta^2 f_0+\frac{p(p-1)(p-2)}{3!}\Delta^3 f_0+\frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 f_0+$$

$$+\frac{p(p-1)(p-2)(p-3)(p-4)}{5!}\Delta^5 f_0+\frac{p(p-1)(p-2)(p-3)(p-4)(p-5)}{6!}\Delta^6 f_0+$$

$$+\frac{p(p-1)(p-2)(p-3)(p-4)(p-5)(p-6)}{7!}\Delta^7 f_0+$$

$$+\frac{p(p-1)(p-2)(p-3)(p-4)(p-5)(p-6)(p-7)}{8!}\Delta^8 f_0+$$

$$+\frac{p(p-1)(p-2)(p-3)(p-4)(p-5)(p-6)(p-7)(p-8)}{9!}\Delta^9 f_0\Big\}\mathrm{d}p \tag{2.4}$$

After simplification, we get (1.6)

Case fourth:

When $f(x)$ is defined on $[x_0, x_{10}]$ then
Consider the integral

$$I = \int_{x_0}^{x_{10}} f(x)\mathrm{d}x = h\int_0^{10} f(x_0+ph)\mathrm{d}p$$

$$\approx h\int_0^{10}\Big\{f_0+p\Delta f_0+\frac{p(p-1)}{2!}\Delta^2 f_0+\frac{p(p-1)(p-2)}{3!}\Delta^3 f_0+\frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 f_0+$$

$$+\frac{p(p-1)(p-2)(p-3)(p-4)}{5!}\Delta^5 f_0+\frac{p(p-1)(p-2)(p-3)(p-4)(p-5)}{6!}\Delta^6 f_0+$$

$$+\frac{p(p-1)(p-2)(p-3)(p-4)(p-5)(p-6)}{7!}\Delta^7 f_0+$$

$$+\frac{p(p-1)(p-2)(p-3)(p-4)(p-5)(p-6)(p-7)}{8!}\Delta^8 f_0+$$

$$+\frac{p(p-1)(p-2)(p-3)(p-4)(p-5)(p-6)(p-7)(p-8)}{9!}\Delta^9 f_0+$$

$$+\frac{p(p-1)(p-2)(p-3)(p-4)(p-5)(p-6)(p-7)(p-8)(p-9)}{10!}\Delta^{10} f_0\Big\}\mathrm{d}p \tag{2.5}$$

After simplification, we get (1.7)

3. Composite Formula

$$I = \int_{x_0}^{x_{10n}} f(x)\mathrm{d}x = \int_{x_0}^{x_{10}} f(x)\mathrm{d}x + \int_{x_{10}}^{x_{20}} f(x)\mathrm{d}x + \int_{x_{20}}^{x_{30}} f(x)\mathrm{d}x + \int_{x_{30}}^{x_{40}} f(x)\mathrm{d}x + \cdots +$$

$$+\int_{x_{10(n-4)}}^{x_{10(n-3)}} f(x)\mathrm{d}x + \int_{x_{10(n-3)}}^{x_{10(n-2)}} f(x)\mathrm{d}x + \int_{x_{10(n-2)}}^{x_{10(n-1)}} f(x)\mathrm{d}x + \int_{x_{10(n-1)}}^{x_{10n}} f(x)\mathrm{d}x$$

4. Algorithm

Suppose the integrand $f(x)$ is defined on the open interval (x_0, x_{10}) or on the semi open intervals $(x_0, x_{10}]$ or on $[x_0, x_{10})$.

Step 1: Enter the value of upper limit and lower limit in UL and LL respectively.

Step 2: Compute h i.e. $h = \frac{UL-LL}{10}$.

Step 3: Find the values between UL and LL with the difference h and call it $x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8$ and x_9.

Step 4: For each value of $x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9$, calculate the value of given integrand and store the results in $f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8$ and f_9 respectively.

Step 5: Calculate the value of the integrand of any defined integral with the help of (1.6)

Step 6: Exit.

In case of the integrand $f(x)$ is defined on the closed interval $[x_0, x_{10}]$. In this case follow the same procedure to calculate all $x_0, x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9$ and x_{10} and the corresponding $f_0, f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8, f_9$ and f_{10} respectively, and evaluate the integral using (1.7).

3. Comparative Study

Consider the following problems:

Problem I

$$\int_0^1 \frac{\ln x}{(1-x)} \mathrm{d}x = \int_0^1 \frac{\ln(1-x)}{x} \mathrm{d}x$$

The exact value of the above definite integral is $-\frac{\pi^2}{6} = -1.644934067$.

(i) Using Newton-Cotes open integration formula (1.1), the value of the above integral is -1.386294361. The difference between the exact value and the value by Newton-Cotes open integration (1.1) is 0.258639706.

(ii) Using Newton-Cotes open integration formula (1.2), the value of the above integral is -1.432156879. The difference between the exact value and the value by Newton-Cotes open integration (1.2) is 0.212777188.

(iii) Using Newton-Cotes open integration formula (1.3), the value of the above integral is -1.537315727. The difference between the exact value and the value by Newton-Cotes open integration (1.3) is 0.10761834.

(iv) Using Newton-Cotes open integration formula (1.4), the value of the above integral is -1.550286746. The difference between the exact value and the value by Newton-Cotes open integration (1.4) is 0.094647321.

(v) Using Newton-Cotes open integration formula (1.5), the value of the above integral is -1.58138543. The difference between the exact value and the value by Newton-Cotes open integration (1.5) is 0.063548637.

(vi) Using our algorithm(1.6), the value of the above integral is -1.61211667. The difference between the exact value and the value by our algorithm is 0.032817397.

Problem II

$$\int_0^1 \frac{\ln x}{\sqrt{(1-x^2)}}\mathrm{d}x$$

The exact value of the above definite integral is $-\frac{\pi}{2}\ln 2 = -1.088793045$.

(i) Using Newton-Cotes open integration formula (1.1), the value of the above integral is -0.800377423. The difference between the exact value and the value by Newton-Cotes open integration (1.1) is 0.2884156222.

(ii) Using Newton-Cotes open integration formula (1.2), the value of the above integral is -0.854621413. The difference between the exact value and the value by Newton-Cotes open integration (1.2) is 0.234171632.

(iii) Using Newton-Cotes open integration formula (1.3), the value of the above integral is -0.977669573. The difference between the exact value and the value by Newton-Cotes open integration (1.3) is 0.111123472.

(iv) Using Newton-Cotes open integration formula (1.4), the value of the above integral is -0.991588954. The difference between the exact value and the value by Newton-Cotes open integration (1.4) is 0.097204091.

(v) Using Newton-Cotes open integration formula (1.5), the value of the above integral is -1.02487500. The difference between the exact value and the value by Newton-Cotes open integration (1.5) is 0.063918045.

(vi) Using our algorithm(1.6), the value of the above integral is -1.0561703046. The difference between the exact value and the value by our algorithm is 0.0326227404.

4. Conclusion

The magnitude of the difference between exact value and the value obtained by existing five Newton-Cotes open integration formula, is minimum in the case of our algorithm. From the problem (i) and (ii), it is clear that our algorithm is more accurate than the existing five Newton-Cotes open integration formula.

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