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CONSTRUCTIONS FOR LARGE SETS OF DISJOINT COMPATIBLE PACKINGS ON $6k+5$ POINTS

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ABSTRACT. In this paper, we give two methods to construct large sets of disjoint compatible packings ($LMP(1^vC_4)$) on $6k+5$ points. As a result, we prove that there exists an $LMP(1^vC_4)$ for $v = 7^iu^j$ where $i \geq 0, j \geq 1$ and $u \in \{13, 19, 25, 31, 43, 67, 139, 163, 211, 283, 307, 331, 379\}$.

1. Introduction

Let X be a set of v points. A $(2,3)$-*packing* on X is a pair $(X, \mathcal{A})$, where $\mathcal{A}$ is a set of 3-subsets (called *triples*) of X, such that every 2-subset of X appears in at most one triple. The *edge-leave* of a $(2,3)$-packing $(X, \mathcal{A})$ is a graph (X, E), where E consists of all the pairs which do not appear in any triple of $\mathcal{A}$.

A $(2,3)$-packing $(X, \mathcal{A})$ is said to be *degenerate* if there exist points that occur in no triple of $\mathcal{A}$. A degenerate $(2,3)$-packing on v points is actually a $(2,3)$-packing on v' points for some $v' < v$. Throughout this paper we restrict our attention to non-degenerate $(2,3)$-packings.

Two $(2,3)$-packings $(X, \mathcal{A})$ and $(X, \mathcal{B})$ are called *disjoint* if $\mathcal{A} \bigcap \mathcal{B} = \phi$. Two $(2,3)$-packings are called *compatible* if they have the same edge-leave. A set of more than two $(2,3)$-packings is called *disjoint* (*compatible*, respectively) if each pair of them is disjoint (compatible, respectively).

A $(2,3)$-packing $(X, \mathcal{A})$ is called *maximum* if there does not exist any $(2,3)$-packing $(X, \mathcal{B})$ with $|\mathcal{A}| < |\mathcal{B}|$. A maximum $(2,3)$-packing with edge-leave (X, E) is denoted by $(2,3)$-$MP(E)$ in this paper. We usually denote $(2,3)$-$MP(E)$ briefly by $MP(E)$. When the edge-leave (X, E) is a graph without any edge, i.e. v isolated vertices, $MP(E)$ is denoted by $MP(1^v)$. Similarly, an $MP(1^{v-4}C_4)$ denotes an maximum $(2,3)$-packing with edge-leave of $v-4$ isolated vertices and a cycle of length four. An $MP(1^v)$ is actually a Steiner triple system of order v. It is well known that an $MP(1^v)$ exists if and only if $v \equiv 1, 3 \pmod 6$. When $v \equiv 5 \pmod 6$, an $MP(1^{v-4}C_4)$ exists in [16, 17].

Denote by $M(v)$ the maximum number of disjoint compatible packings on v points. Determination of the number $M(v)$ is related to the construction of perfect threshold schemes (see, for example, [7, 15]). The upper bound on $M(v)$ is proved in [15].

Theorem 1.1. ([15]) *$M(v) \le v-2$ for $v \equiv 1,3 \pmod 6$; $M(v) \le v-4$ for $v \equiv 0,2,5 \pmod 6$; and $M(v) \le v-6$ for $v \equiv 4 \pmod 6$. Further, except when $v \equiv 4 \pmod 6$, the upper bound is attained only if the packings are maximum.*

Values of v for which $M(v)$ meets the upper bound are summarized as follows.

Lemma 1.2. (1) *For $v \equiv 1,3 \pmod 6$ and $v \neq 7$, $M(v) = v-2$. Also $M(7) = 3$* ([13, 14, 18]).

(2) *For $v \equiv 0,2 \pmod 6$, $M(v) = v-4$* ([8, 5, 12]).

(3) *For $(v-4)/2 \notin \{12, 36, 48, 144\} \cup \{n > 0 : n = 6m, m \equiv 1, 5$ (mod 6)$\}$, $M(v) = v-6$* ([2, 3, 4]).

(4) *For $v \in \{7^k t + 4 : \ k \ge 0, t = 1, 7, 13, 19, 25, 31, 43, 67\} \cup \{11, 17, 23\}$, $M(v) = v-4$* ([6, 7, 15]).

In the literature, there are several methods in constructing sets of disjoint packings which are not required to be compatible in [9, 10]. Such structures have applications to the construction of constant-weight codes [1].

For $v \equiv 5 \pmod 6$, there exists an $MP(1^{v-4}C_4)$. If there exists a set of $v-4$ disjoint compatible $MP(1^{v-4}C_4)$, then $M(v) = v-4$. A set of $v-4$ disjoint compatible $MP(1^{v-4}C_4)$ is thus called a *large set* and denoted by $LMP(1^{v-4}C_4)$.

Suppose that $v \equiv 1 \pmod 6$. Let $I_v = \{1, 2, \cdots, v\}$ and $X = I_v \cup \{\infty_1, \infty_2, \infty_3, \infty_4\}$. An ${}^*LMP(1^v C_4)$ is an $LMP(1^v C_4) = \{(X, \mathcal{B}_i) : \ i \in I_v\}$ which satisfies the following conditions:

(1) Each $(X, \mathcal{B}_i)$, $i \in I_v$, has the edge-leave $(\infty_1\ \infty_2\ \infty_3\ \infty_4)$.

(2) $\{\infty_1, \infty_3, i\}, \{\infty_2, \infty_4, i\} \in \mathcal{B}_i$ for any $i \in I_v$.

We summarize the known results on ${}^*LMP(1^v C_4)$ as follows.

Lemma 1.3. *There exists an ${}^*LMP(1^v C_4)$ for $v \in \{7, 13, 19, 25, 31, 43, 67\}$.*

2. A direct way to construct ${}^*\mathbf{LMP(1^q C_4)}$ with prime power $\mathbf{q \equiv 1 \pmod 6}$

Let $GF(q)$ be a finite field with q elements where $q \equiv 1 \pmod 6$. Let $GF(q)^* = GF(q) \setminus \{0\}$. Let α be an element in $GF(q)$. An α-partition of $GF(q)$ is a partition $GF(q)^* = Y \cup Z$ such that

(a) x is never in the same class as αx, and

(b) x is never in the same class as $-x$.

Lemma 2.1. *Let $GF(q)$ be a finite field and t be the multiplicative order of α in $GF(q)^*$. Then $GF(q)$ has an α-partition if and only if $t \equiv 2 \pmod 4$.*

Proof Suppose that $GF(q)$ has an α-partition $GF(q)^* = Y \cup Z$. Without loss of generality, let $1 \in Y$. By condition (a) of α-partition, we have $\alpha^1 \in Z$, $\alpha^2 \in Y$, $\cdots$, $\alpha^{2i-1} \in Z$, $\alpha^{2i} \in Y, \cdots$.

Since $\alpha^t = 1 \in Y$, it implies that t is even. Let $t = 2s$. Note that $\alpha^s = -1$. By condition (b) of α-partition, $\alpha^s \in Z$, which implies that s is odd. Thus, $t \equiv 2 \pmod 4$.

If $t \equiv 2 \pmod 4$, let $\langle \alpha \rangle$ be the multiplicative sub-group of $GF(q)^*$ generated by α, and let $h_0, h_1, \cdots, h_{\frac{q-1}{t}-1}$ be all the representative elements of coset classes. Define

$$Y = \{h_j \alpha^{2i} : i = 0, 1, \cdots, t/2-1; j = 0, 1, \cdots, (q-1)/t - 1\};$$

$$Z = \{h_j\alpha^{2i+1} : i = 0, 1, \cdots, t/2-1; j = 0, 1, \cdots, (q-1)/t-1\}.$$

It is readily checked that $GF(q)^* = Y \cup Z$ is an α-partition of $GF(q)$. □

Let g be a primitive root of $GF(q)$. Define $\log_g \beta = a$ if $g^a = \beta$. We usually write $\log_g \beta = a$ as $\log \beta = a$. In this section, we always denote $n = q-1$. For any unordered pair $\{\lambda, \mu\} \subseteq Z_n \setminus \{0, n/2\}$ and $\lambda \neq \mu$, define a set $\Delta\{\lambda, \mu\}$ as follows:

$$\Delta\{\lambda, \mu\} = \left\{\pm\log\frac{g^\lambda - 1}{g^\mu - 1},\ \pm\log\frac{g^\lambda - 1}{g^\lambda - g^\mu},\ \pm\log\frac{g^\mu - 1}{g^\mu - g^\lambda}\right\}.$$

It is easy to see that $\Delta\{\lambda, \mu\} = \Delta\{\mu, \lambda\}$ and $0 \notin \Delta\{\lambda, \mu\}$. Let $\{\lambda_j, \mu_j\} \subseteq Z_n \setminus \{0, n/2\}$, $j = 1, 2, \cdots, \frac{n}{6}-1$, denote the $\frac{n}{6}-1$ unordered pairs which satisfy the following conditions:

Con 1. All elements $\pm\lambda_j, \pm\mu_j, \pm(\lambda_j - \mu_j), j = 1, 2, \cdots, \frac{n}{6}-1$, are distinct. Let x and y be in Z_n such that

$$\left\{\pm\lambda_j, \pm\mu_j, \pm(\lambda_j - \mu_j) :\ j = 1, 2, \cdots, \frac{n}{6} - 1\right\} = Z_n \setminus \left\{0, \frac{n}{2}, \pm x, \pm y\right\}.$$

Con 2. $n/\gcd(n, x) \equiv n/\gcd(n, y) \equiv 2 \pmod 4$.

Con 3. The six element in each $\Delta\{\lambda_j, \mu_j\}$ are distinct and different from $0, n/2$ for any $j = 1, 2, \cdots, n/6-1$; Any two $\Delta\{\lambda_j, \mu_j\}, \Delta\{\lambda_k, \mu_k\}$ are disjoint for $j \neq k \in [1, \frac{n}{6}-1]$.

Theorem 2.2. *Let q be a prime power and $q \equiv 1 \pmod 6$. If $\{\lambda_j, \mu_j\} \subseteq Z_n \setminus \{0, n/2\}$, $j = 1, 2, \cdots, \frac{n}{6}-1$ satisfy Con 1-3, then there is an ${}^*LMP(1^qC_4)$.*

Construction: Let $X = GF(q) \cup \{\infty_1, \infty_2, \infty_3, \infty_4\}$. By the assumption of Con 2, the order of g^x and g^y in $GF(q)^*$ is $n/\gcd(n, x) \equiv 2 \pmod 4$ and $n/\gcd(n, y) \equiv 2 \pmod 4$, respectively. By Lemma **2.1**, there exist a g^x-partition $GF(q)^* = Y_1 \cup Z_1$ and a g^y-partition $GF(q)^* = Y_2 \cup Z_2$. We will construct q $MP(1^qC_4)$ $(X, \mathcal{B}_i)$ $(i \in GF(q))$ with the same edge-leave of 4-cycle $(\infty_1\ \infty_2\ \infty_3\ \infty_4)$ where $\mathcal{B}_i = \mathcal{B}_0 + i$ and $\mathcal{B}_0$ consists of the following triples:

Part 1. $\{\infty_1, \infty_3, 0\}$, $\{\infty_2, \infty_4, 0\}$;

Part 2. $\{\infty_1, z, g^x z\}$ where $z \in Y_1$, $\{\infty_2, z, g^x z\}$ where $z \in Z_1$, $\{\infty_3, z, g^y z\}$ where $z \in Y_2$, $\{\infty_4, z, g^y z\}$ where $z \in Z_2$;

Part 3. $\{0, g^k, -g^k\}$ for $k = 0, 1, \cdots, n/2-1$;

Part 4. $\{g^k, g^{k+\lambda_j}, g^{k+\mu_j}\}$ for $k \in Z_n$ and $j = 1, 2, \cdots, n/6-1$.

Proof By Con 1 and Con 2, it is easy to check that each $(X, \mathcal{B}_i)$ is an $MP(1^qC_4)$ with edge leave $C_4 = (\infty_1\ \infty_2\ \infty_3\ \infty_4)$ for $i \in GF(q)$. Next we should prove that $\mathcal{B}_i$ and $\mathcal{B}_j$ are disjoint for $i \neq j$. It is enough to show that if $T \in \mathcal{B}_0 \cap \mathcal{B}_i$ then $i = 0$. We consider four cases below.

Case 1. $T = \{\infty_1, \infty_3, 0\}$, or $\{\infty_2, \infty_4, 0\}$. It is easy to see that $i = 0$.

Case 2. $T = \{\infty_1, z, g^x z\}$ where $z \in Y_1$. Then there exists $z' \in Y_1$ such that $T = \{\infty_1, z' + i, g^x z' + i\} \in \mathcal{B}_i$, which implies that $\{z, g^x z\} = \{z' + i, g^x z' + i\}$. So, $\pm(g^x z - z) = (g^x z' + i) - (z' + i)$ and hence $z = \pm z'$ since $g^x \neq 1$. By $z, z' \in Y_1$ and $Y_1 \cup Z_1$ is a g^x-partition of $GF(q)$, then $z \neq -z'$. Hence $z = z'$ which is actually $i = 0$. The proof is similar for the cases $T = \{\infty_2, z, g^x z\}$ where $z \in Z_1$, or $T = \{\infty_3, z, g^y z\}$ where $z \in Y_2$, or $T = \{\infty_4, z, g^y z\}$ where $z \in Z_2$.

Case 3. $T = \{0, g^k, -g^k\}$ where $k = 0, 1, \cdots, n/2-1$. Then there exists $k' \in \{0, 1, \cdots, n/2-1\}$ such that $T = \{i, g^{k'} + i, -g^{k'} + i\}$, or $T = \{g^{k'} + i, g^{k'+\lambda_j} + i, g^{k'+\mu_j} + i\}$ where $j \in \{1, 2, \cdots, \frac{n}{6} - 1\}$. If $T = \{g^{k'} + i, g^{k'+\lambda_j} + i, g^{k'+\mu_j} + i\}$ where $j \in \{1, 2, \cdots, \frac{n}{6} - 1\}$. Without loss of generality we can assume that $g^{k'} + i = 0$, then $i = -g^{k'}$. So, we have $(g^{k'+\lambda_j} + i)/(g^{k'+\mu_j} + i) = g^k/(-g^k) = -1$, which implies that $\log[(g^{\lambda_j} - 1)/(g^{\mu_j} - 1)] = n/2$. Hence, $n/2 \in \Delta\{\lambda_j, \mu_j\}$ which is impossible by Con 3.

Hence we must have $T = \{i, g^{k'} + i, -g^{k'} + i\}$, summing the elements in both sides gives $3i = 0$ and so $i = 0$.

Case 4. $T = \{g^k, g^{k+\lambda}, g^{k+\mu}\}$ where $k \in Z_n$ and $\{\lambda, \mu\}$ is a pair among $\{\{\lambda_j, \mu_j\} : j = 1, 2, \cdots, \frac{n}{6} - 1\}$. Then there exist $k' \in Z_n$ and a pair $\{a, b\}$ belonging to $\{\{\lambda_j, \mu_j\} : j = 1, 2, \cdots, \frac{n}{6} - 1\}$ which satisfy that

$$\{g^k, g^{k+\lambda}, g^{k+\mu}\} = \{g^{k'} + i, g^{k'+a} + i, g^{k'+b} + i\}. \tag{2.1}$$

Without loss of generality we can assume that $g^k = g^{k'} + i$. Then the second and third elements minus the first one in both-sides of (**2.1**) gives $\{g^k(g^\lambda - 1), g^k(g^\mu - 1)\} = \{g^{k'}(g^a - 1), g^{k'}(g^b - 1)\}$. So, $\log[(g^\lambda - 1)/(g^\mu - 1)] = \pm \log[(g^a - 1)/(g^b - 1)]$. By the hypothesis of Con. 3 we have $\{\lambda, \mu\} = \{a, b\}$ and then (**2.1**) becomes

$$\{g^k, g^{k+\lambda}, g^{k+\mu}\} = \{g^{k'} + i, g^{k'+\lambda} + i, g^{k'+\mu} + i\}. \tag{2.2}$$

Note that the sum of the 2nd and 3rd-elements minus 2 times of the first one should be equal in both-sides of (**2.2**). Simplification gives $g^k(g^\lambda + g^\mu - 2) = g^{k'}(g^\lambda + g^\mu - 2)$. Since $\log[(g^\lambda - 1)/(g^\mu - 1)] \neq n/2$, we can deduce that $g^\lambda + g^\mu - 2 \neq 0$. So, $g^k = g^{k'}$. Summing the 3 elements in both-sides of (**2.2**) gives $3i = 0$ and hence $i = 0$.

Therefore, $\{(X, \mathcal{B}_i :\ i \in GF(q)\}$ forms an $^*LMP(1^qC_4)$. This completes the proof.
□

Lemma 2.3. *There exists an* $^*LMP(1^qC_4)$ *for* $q = 139, 163, 211, 283, 307,\ 331, 379$.

Proof Let g be a primitive root in $GF(q)$. For each value of q, with the aid of computer, we found $n/6 - 1$ pairs $\{\lambda_j, \mu_j\}$, $j = 1, 2, \cdots, n/6 - 1$, and x, y for which Con 1-3 hold. By Theorem 2.2, there exists an $^*LMP(1^qC_4)$.

$q = 139$: $g = 2$, $x = 1$, $y = 67$, $\{\lambda_j, \mu_j\}$ for $j = 1, 2, \cdots, 22$ are

$\{2,5\}$	$\{4,10\}$	$\{7,16\}$	$\{8,19\}$	$\{12,25\}$
$\{14,29\}$	$\{17,35\}$	$\{20,41\}$	$\{22,62\}$	$\{23,72\}$
$\{24,83\}$	$\{26,90\}$	$\{27,87\}$	$\{28,73\}$	$\{30,91\}$
$\{31,84\}$	$\{32,88\}$	$\{33,101\}$	$\{34,80\}$	$\{36,75\}$
$\{38,81\}$	$\{42,86\}$			

$q = 163$: $g = 2$, $x = 1$, $y = 3$, $\{\lambda_j, \mu_j\}$ for $j = 1, 2, \cdots, 26$ are

$\{19,40\}$	$\{22,46\}$	$\{23,49\}$	$\{27,55\}$	$\{29,59\}$
$\{32,65\}$	$\{35,71\}$	$\{37,75\}$	$\{39,80\}$	$\{42,89\}$
$\{43,88\}$	$\{44,100\}$	$\{48,111\}$	$\{14,25\}$	$\{15,76\}$
$\{16,34\}$	$\{17,85\}$	$\{20,92\}$	$\{31,95\}$	$\{50,104\}$
$\{52,57\}$	$\{53,155\}$	$\{66,150\}$	$\{69,79\}$	$\{149,158\}$
$\{154,156\}$				

$q = 211$: $g = 2$, $x = 1$, $y = 9$, $\{\lambda_j, \mu_j\}$ for $j = 1, 2, \cdots, 34$ are

$\{22, 45\}$	$\{24, 50\}$	$\{27, 55\}$	$\{29, 61\}$	$\{31, 64\}$
$\{34, 69\}$	$\{37, 77\}$	$\{38, 79\}$	$\{42, 86\}$	$\{43, 89\}$
$\{47, 96\}$	$\{48, 110\}$	$\{51, 122\}$	$\{52, 126\}$	$\{53, 111\}$
$\{54, 129\}$	$\{56, 132\}$	$\{57, 127\}$	$\{59, 144\}$	$\{60, 203\}$
$\{21, 115\}$	$\{25, 36\}$	$\{30, 147\}$	$\{39, 119\}$	$\{65, 73\}$
$\{68, 206\}$	$\{82, 190\}$	$\{87, 97\}$	$\{90, 104\}$	$\{92, 193\}$
$\{98, 205\}$	$\{191, 204\}$	$\{192, 208\}$	$\{195, 198\}$	

$q = 283$: $g = 3$, $x = 1$, $y = 11$, $\{\lambda_j, \mu_j\}$ for $j = 1, 2, \cdots, 46$ are

$\{24, 50\}$	$\{27, 56\}$	$\{28, 59\}$	$\{32, 65\}$	$\{34, 70\}$
$\{37, 75\}$	$\{39, 80\}$	$\{42, 85\}$	$\{44, 90\}$	$\{47, 96\}$
$\{48, 99\}$	$\{52, 105\}$	$\{54, 109\}$	$\{57, 115\}$	$\{60, 123\}$
$\{61, 125\}$	$\{62, 129\}$	$\{66, 134\}$	$\{69, 143\}$	$\{71, 171\}$
$\{72, 179\}$	$\{73, 164\}$	$\{76, 174\}$	$\{77, 198\}$	$\{78, 172\}$
$\{79, 180\}$	$\{81, 169\}$	$\{82, 196\}$	$\{83, 178\}$	$\{87, 193\}$
$\{92, 252\}$	$\{35, 277\}$	$\{45, 189\}$	$\{97, 267\}$	$\{116, 261\}$
$\{117, 280\}$	$\{120, 132\}$	$\{124, 131\}$	$\{126, 142\}$	$\{127, 136\}$
$\{128, 147\}$	$\{130, 133\}$	$\{257, 274\}$	$\{259, 269\}$	$\{260, 278\}$
$\{262, 268\}$				

$q = 307$: $g = 5$, $x = 1$, $y = 5$, $\{\lambda_j, \mu_j\}$ for $j = 1, 2, \cdots, 50$ are

$\{24, 50\}$	$\{27, 56\}$	$\{28, 60\}$	$\{31, 67\}$	$\{33, 70\}$
$\{34, 72\}$	$\{39, 81\}$	$\{41, 84\}$	$\{44, 92\}$	$\{46, 99\}$
$\{47, 98\}$	$\{49, 101\}$	$\{54, 112\}$	$\{55, 118\}$	$\{57, 116\}$
$\{61, 125\}$	$\{62, 127\}$	$\{66, 143\}$	$\{68, 137\}$	$\{71, 145\}$
$\{73, 148\}$	$\{76, 154\}$	$\{79, 172\}$	$\{80, 186\}$	$\{82, 176\}$
$\{83, 197\}$	$\{85, 206\}$	$\{86, 201\}$	$\{87, 204\}$	$\{88, 196\}$
$\{89, 184\}$	$\{90, 187\}$	$\{91, 261\}$	$\{35, 138\}$	$\{40, 195\}$
$\{96, 276\}$	$\{104, 113\}$	$\{107, 128\}$	$\{123, 141\}$	$\{124, 283\}$
$\{129, 146\}$	$\{131, 302\}$	$\{132, 296\}$	$\{133, 300\}$	$\{140, 284\}$
$\{149, 299\}$	$\{281, 295\}$	$\{286, 298\}$	$\{287, 290\}$	$\{291, 293\}$

$q = 331$: $g = 3$, $x = 1$, $y = 7$, $\{\lambda_j, \mu_j\}$ for $j = 1, 2, \cdots, 54$ are

$\{23, 49\}$	$\{27, 55\}$	$\{29, 62\}$	$\{31, 63\}$	$\{34, 70\}$
$\{37, 75\}$	$\{39, 79\}$	$\{42, 86\}$	$\{43, 90\}$	$\{46, 94\}$
$\{50, 101\}$	$\{52, 106\}$	$\{53, 111\}$	$\{56, 113\}$	$\{59, 123\}$
$\{60, 121\}$	$\{65, 131\}$	$\{67, 139\}$	$\{68, 137\}$	$\{71, 145\}$
$\{73, 150\}$	$\{76, 154\}$	$\{80, 162\}$	$\{81, 164\}$	$\{84, 171\}$
$\{85, 183\}$	$\{88, 184\}$	$\{89, 194\}$	$\{91, 206\}$	$\{92, 218\}$
$\{93, 212\}$	$\{95, 223\}$	$\{97, 201\}$	$\{99, 221\}$	$\{100, 216\}$
$\{102, 210\}$	$\{103, 285\}$	$\{41, 158\}$	$\{110, 130\}$	$\{125, 140\}$
$\{127, 314\}$	$\{132, 135\}$	$\{133, 306\}$	$\{134, 138\}$	$\{141, 316\}$
$\{142, 320\}$	$\{144, 149\}$	$\{151, 160\}$	$\{153, 309\}$	$\{161, 163\}$
$\{295, 308\}$	$\{300, 311\}$	$\{305, 322\}$	$\{312, 318\}$	

$q = 379$: $g = 2$, $x = 1$, $y = 9$, $\{\lambda_j, \mu_j\}$ for $j = 1, 2, \cdots, 62$ are

$\{24,50\}$	$\{27,55\}$	$\{29,60\}$	$\{32,66\}$	$\{33,68\}$
$\{37,75\}$	$\{40,81\}$	$\{42,86\}$	$\{43,89\}$	$\{47,95\}$
$\{49,100\}$	$\{52,105\}$	$\{54,110\}$	$\{57,116\}$	$\{58,119\}$
$\{62,127\}$	$\{63,130\}$	$\{64,136\}$	$\{69,139\}$	$\{71,147\}$
$\{73,151\}$	$\{74,154\}$	$\{77,160\}$	$\{79,161\}$	$\{84,169\}$
$\{87,175\}$	$\{90,181\}$	$\{92,188\}$	$\{93,191\}$	$\{94,208\}$
$\{97,199\}$	$\{99,207\}$	$\{101,223\}$	$\{103,234\}$	$\{104,238\}$
$\{106,241\}$	$\{107,253\}$	$\{109,237\}$	$\{111,240\}$	$\{112,255\}$
$\{113,230\}$	$\{115,260\}$	$\{120,252\}$	$\{121,245\}$	$\{45,201\}$
$\{142,159\}$	$\{149,172\}$	$\{150,371\}$	$\{152,366\}$	$\{153,192\}$
$\{158,374\}$	$\{163,173\}$	$\{165,195\}$	$\{166,360\}$	$\{167,365\}$
$\{168,204\}$	$\{176,196\}$	$\{178,193\}$	$\{353,375\}$	$\{357,376\}$
$\{362,367\}$	$\{364,372\}$			

□

Combining Lemma **1.3** and Lemma **2.3** we have the the following results.

Theorem 2.4. *There exists an* $^*LMP(1^vC_4)$ *for* $v \in \{7, 13, 19, 25, 31, 43, 67, 139, 163, 211, 283, 307, 331, 379\}$.

3. A Construction of $^*\mathbf{LMP(1^vC_4)}$ via 3-designs

A *3-wise balanced design* is a pair $(X, \mathcal{B})$, where X is a finite set and $\mathcal{B}$ is a set of subsets of X, called *blocks* with the property that every 3-subset of X is contained in a unique block. If $|X| = v$ and K is the set of block sizes, we denote it by $S(3, K, v)$. Let $(X \cup \{\infty\}, \mathcal{B})$ be an $S(3, K_0 \cup K_1, v + 1)$ where $|X| = v$. $(X \cup \{\infty\}, \mathcal{B})$ is denoted by $S(3, K_0, K_1, v + 1)$ if $|B| \in K_0$ for any $\infty \notin B \in \mathcal{B}$; and $|B| \in K_1$ for any $\infty \in B \in \mathcal{B}$.

An $S(3, \{k\}, v)$ is denoted by $S(3, k, v)$. An $S(3, 4, v)$ is usually called a Steiner quadruple system of order v. The following results can be found in [11].

Lemma 3.1. (1) *There exists an* $S(3, q+1, q^n+1)$ *for any prime power q and any integer* $n \geq 2$.

(2) *There exists an* $S(3, 4, v)$ *if and only if* $v \equiv 2, 4 \pmod 6$.

Theorem 3.2. *If there exists an* $S(3, K_0, K_1, v + 1)$ *and there exists an* $^*LMP(1^{k-1}C_4)$ *for any* $k \in K_1$*, and* $k \equiv 2, 4 \pmod 6$ *for any* $k \in K_0$*, then there exists an* $^*LMP(1^vC_4)$.

Construction: Let $(X \bigcup \{\infty_1\}, \mathcal{B})$ be an $S(3, K_0, K_1, v + 1)$. We will construct an $^*LMP(1^vC_4)$ on $X \cup \{\infty_1, \infty_2, \infty_3, \infty_4\}$ by the following two steps.

Step 1. For any $B \in \mathcal{B}$, $\infty_1 \in B$ (i.e., $|B| \in K_1$), by the hypothesis, there exists an

$$^*LMP(1^{|B|-1}C_4) = \{(B \cup \{\infty_2, \infty_3, \infty_4\}, \mathcal{A}_B(x)) : x \in B \setminus \{\infty_1\}\}$$

such that each $\mathcal{A}_B(x)$ have edge-leave $(\infty_1\ \infty_2\ \infty_3\ \infty_4)$ and $\{\infty_1, \infty_3, x\}$, $\{\infty_2, \infty_4, x\} \in \mathcal{A}_B(x)$ for $x \in B \setminus \{\infty_1\}$.

Step 2. For any $B \in \mathcal{B}$, $\infty_1 \notin B$ (i.e., $|B| \in K_0$), there exists an $S(3, 4, |B|)$ $(B, \mathcal{A}_B)$ by (2) of Lemma **3.1**. Let $\mathcal{A}_B(x) = \{C \setminus \{x\} : x \in C \in \mathcal{A}_B\}$.

For any $x \in X$, define

$$\mathcal{B}_x = \bigcup_{x \in B \in \mathcal{B}} \mathcal{A}_B(x).$$

Then it is readily checked that the collection $\{(X\bigcup\{\infty_1,\infty_2,\infty_3,\infty_4\},\mathcal{B}_x):\ x\in X\}$ is an ${}^*LMP(1^vC_4)$. □

Corollary 3.3. *There exists an* ${}^*LMP(1^vC_4)$ *for* $v=u^j$ *where* $j\geq 1$ *and* $u\in\{7,13,19,25,31,43,67,139,163,211,283,307,331,379\}$.

Proof It follows immediately from Lemma **3.1** and Theorem **3.2**. □

4. A direct product construction

In this section, we will give a direct product construction, which is actually a generalization of Theorem 3.1 of [6]. Firstly, we introduce some definitions.

Assume that $v\equiv 1 \pmod 6$. Let $I_v=\{1,2,\cdots,v\}$ and $X=I_v\bigcup\{\infty_1,\infty_2,\infty_3,\infty_4\}$. Let $\{(X,\mathcal{B}_i):\ i\in I_v\}$ be an $LMP(1^vC_4)$. The *triple-leave* of the $LMP(1^vC_4)$ is the set of $\binom{I_v}{3}\setminus(\bigcup\limits_{i\in I_v}\mathcal{B}_i)$ and denoted by $L_T(v)$. A simple counting shows that $|L_T(v)|=v(v-1)/3$.

Let ${}^*LMP(1^vC_4)=\{(I_v\bigcup\{\infty_1,\infty_2,\infty_3,\infty_4\},\mathcal{B}_i):\ i\in I_v\}$ with $\{\infty_1,\infty_3,i\}$, $\{\infty_2,\infty_4,i\}\in\mathcal{B}_i$ and each $\mathcal{B}_i$ have the edge-leave $C_4=(\infty_1\ \infty_2\ \infty_3\ \infty_4)$ and triple-leave $L_T(v)$. Let $E_i=\{\{a,b\}:\ a,b\in I_v,\{\infty_l,a,b\}\in\mathcal{B}_i,l=1,2,3,4\}$. A partitioned ${}^*LMP(1^vC_4)$ is an ${}^*LMP(1^vC_4)$ if the following conditions hold:

(1) E_i can be partitioned into E_i^1, E_i^2, such that $(I_v\setminus\{i\},E_i^1)$ and $(I_v\setminus\{i\},E_i^2)$ are all 2-regular graphs for $i\in I_v$.

(2) Given a direction for each cycle of E_i^1, we obtain a directed graph $\bar{E}_i^1$, such that $\bigcup\limits_{i\in I_v}\bar{E}_i^1=DK_v$ (DK_v is the complete digraph of order v) (i.e., for any ordered pair (a,b) $a\neq b\in I_v$, there is a unique i such that $(a,b)\in\bar{E}_i^1$).

(3) There exists a partition $\{P_1,P_2,\cdots,P_v\}$ of the triple-leave $L_T(v)$, such that $|P_i|=|P_j|$, $i\neq j\in I_v$, and P_i cover E_i^2 (i.e., for any $\{a,b\}\in E_i^2$, there exists one block $B\in P_i$ such that $\{a,b\}\subset B$. Note: $|P_i|=\frac{v-1}{3}$ and $|E_i^2|=v-1$).

Lemma 4.1. [6] *There exists a partitioned* ${}^*LMP(1^7C_4)$.

Theorem 4.2. *If there exist both a partitioned* ${}^*LMP(1^vC_4)$ *and an* $LMP(1^uC_4)$ *(or a* ${}^*LMP(1^uC_4)$*), then there exists an* $LMP(1^{uv}C_4)$ *(or a* ${}^*LMP(1^{uv}C_4)$*).*

Construction: Let $\{(I_v\bigcup\{\infty_1,\infty_2,\infty_3,\infty_4\},\mathcal{B}_i)\ :\ i\in I_v\}$ be a partitioned ${}^*LMP(1^vC_4)$. The symbols E_i^1, E_i^2, P_i, $i\in I_v$, are the same as in the definition. And let $LMP(1^uC_4)=\{(Z_u\bigcup\{\infty_1,\infty_2,\infty_3,\infty_4\},\mathcal{A}_j)\ :\ j\in Z_u\}$. We will construct uv $MP(1^{uv}C_4)$s $(X,\mathcal{C}_{ij})$, $i\in I_v$, $j\in Z_u$ on $X=(Z_u\times I_v)\bigcup\{\infty_1,\infty_2,\infty_3,\infty_4\}$ where $\mathcal{C}_{ij}$ consists of the following triples:

Part 1. $\{(x,i),(y,i),(z,i)\}$, where $\{x,y,z\}\in\mathcal{A}_j$ and $(\infty_l,i)=\infty_l$, $l=1,2,3,4$.

Part 2. $\{(x,k_1),(y,k_2),(z,k_3)\}$, where $\{k_1,k_2,k_3\}\in\mathcal{B}_i$, $k_1<k_2<k_3$, $k_1,k_2,k_3\in I_v$, $x+y+z=j$ (mod u).

Part 3. $\{(x,k_1),(y,k_1),(\frac{x+y}{2}+j,k_2)\}$, where $(k_1,k_2)\in\bar{E}_i^1$, $x\neq y\in Z_u$, $x<y$.

Part 4. $\{(x,k_1),(x-y,k_2),(x+y+j,k_3)\}$, where $\{k_1,k_2,k_3\}\in P_i$, $k_1<k_2<k_3$, $x\in Z_u$, $y\in Z_u\setminus\{j\}$.

Part 5. $\{(x,k_1),(x+j,k_2),\infty_l\}$, where $(k_1,k_2)\in\bar{E}_i^1$ and $\{k_1,k_2,\infty_l\}\in\mathcal{B}_i$, $x\in Z_u$.

Part 6. $\{(x_1,k_1),(x_2,k_2),\infty_l\}$, where $\{k_1,k_2\}\in E_i^2$, $k_1<k_2$ and $\{k_1,k_2,\infty_l\}\in\mathcal{B}_i$. Let $\{k_1,k_2,k_3\}\in P_i$, then

$x_1=x$, $x_2=x-j$ if $k_1<k_2<k_3$, $x\in Z_u$;
$x_1=x$, $x_2=x+2j$ if $k_1<k_3<k_2$, $x\in Z_u$;
$x_1=x-j$, $x_2=x+2j$ if $k_3<k_1<k_2$, $x\in Z_u$.

Proof (1) Each $(X,\mathcal{C}_{ij})$, $i\in I_v, j\in Z_u$, is an $MP(1^{uv}C_4)$.

In fact, there are exactly

$$\frac{u^2+7u+4}{6}+\frac{u^2(v-1)(v-4)}{6}+\frac{u(u-1)(v-1)}{2}+\frac{u(u-1)(v-1)}{3}+2u(v-1)\\=\frac{u^2v^2+7uv+4}{6}$$

blocks in each $\mathcal{C}_{ij}$ ($i\in I_v$, $j\in Z_u$). Thus, we only need to show that any 2-subset $P\in(X\times X)\setminus C_4$ is contained in a block of $\mathcal{C}_{ij}$. All the possibilities of P are exhausted as follows:

(a). $P=\{\infty_1,\infty_3\}$ or $\{\infty_2,\infty_4\}$, then P is contained in one block of Part 1 of $\mathcal{C}_{ij}$.

(b). $P=\{(x,h),\infty_l\}$, $x\in Z_u, h\in I_v$. If $h=i$, since pair $\{x,\infty_l\}$ is contained in exactly one block B of $\mathcal{A}_j$, P is contained in one block of Part 1 of $\mathcal{C}_{ij}$. If $h\neq i$, there is an $s\in I_v$ such that $\{h,s,\infty_l\}\in\mathcal{B}_i$. When $\{h,s\}\in E_i^1$, P is contained in one block of Part 5 of $\mathcal{C}_{ij}$. When $\{h,s\}\in E_i^2$, P is contained in one block of Part 6 of $\mathcal{C}_{ij}$.

(c). $P=\{(x,h),(y,h)\}$, $x\neq y\in Z_u, h\in I_v$. If $h=i$, then P is contained in one block of Part 1 of $\mathcal{C}_{ij}$. If $h\neq i$, then P is contained in one block of Part 3 of $\mathcal{C}_{ij}$.

(d). $P=\{(x,h),(y,s)\}$, $x,y\in Z_u, h\neq s\in I_v$. There is a $t\in I_v\cup\{\infty_1,\infty_2,\infty_3,\infty\}$ such that $\{h,s,t\}\in\mathcal{B}_i$. If $t\in I_v$, then P is contained in one block of Part 2 of $\mathcal{C}_{ij}$. If $t=\infty_l$, then when $\{h,s\}\in E_i^1$, P is contained in one block of Part 3 or Part 5; when $\{h,s\}\in E_i^2$, P is contained in one block of Part 4 or Part 6 of $\mathcal{C}_{ij}$.

Thus, each $(X,\mathcal{C}_{ij})$, $i\in I_v, j\in Z_u$, is an $MP(1^{uv}C_4)$.

(2). For any $(i,j)\neq(s,t)$, $i,s\in I_v, j,t\in Z_u$, $\mathcal{C}_{ij}$ and $\mathcal{C}_{st}$ are disjoint.

(a). $i\neq s$. Since $\mathcal{B}_i\cap\mathcal{B}_s=\phi$, $E_i^n\cap E_s^n=\phi$, $(n=1,2)$, $P_i\cap P_s=\phi$, and $(P_i\cup P_s)\cap(\mathcal{B}_i\cup\mathcal{B}_s)=\phi$, we have $\mathcal{C}_{ij}\cap\mathcal{C}_{st}=\phi$.

(b). If $i=s$, then $j\neq t$. Note that $\mathcal{A}_j\bigcap\mathcal{A}_t=\phi$, $P_i\bigcap\mathcal{B}_i=\phi$,

$$\{\{x,y,z\}:\ x+y+z=j,x,y,z\in Z_u\}\cap\{\{x,y,z\}:\ x+y+z=t,x,y,z\in Z_u\}=\phi,$$

and

$$\{(x,x-y,x+y+j):x\in Z_u,y\in Z_u\setminus\{j\}\}\cap\{(x,x-y,x+y+t):x\in Z_u,y\in Z_u\setminus\{t\}\}=\phi.$$

It is not difficult to check that $\mathcal{C}_{ij}\bigcap\mathcal{C}_{st}=\phi$.

Remark: If $LMP(1^uC_4)=\{(Z_u\bigcup\{\infty_1,\infty_2,\infty_3,\infty_4\},\mathcal{A}_j):\ j\in Z_u\}$ is an $^*LMP$ (1^uC_4), i.e. the blocks $\{\infty_1,\infty_3,j\}$, $\{\infty_2,\infty_4,j\}\in\mathcal{A}_j$. Then by the construction of Theorem 4.2, we have the blocks $\{\infty_1,\infty_3,(i,j)\}$, $\{\infty_2,\infty_4,(i,j)\}\in\mathcal{C}_{ij}$. Thus we get an $^*LMP(1^{uv}C_4)$. □

5. Conclusion

Combining Corollary **2.4**, Corollary **3.3**, Lemma **4.1** and Theorem **4.2**, we obtain the following results:

Theorem 5.1. *There exists an $LMP(1^vC_4)$ for $v = 7^iu^j$ where $i \geq 0, j \geq 0$ and $u \in \{13, 19, 25, 31, 43, 67, 139, 163, 211, 283, 307, 331, 379\}$.*

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