DOI: 10.1478/C1A1001003

AAPP | Atti della Accademia Peloritana dei Pericolanti
Classe di Scienze Fisiche, Matematiche e Naturali
ISSN 1825-1242

Vol. LXXXVIII, No. 1, C1A1001003 (2010)

AN ALGORITHM FOR PAYOFF SPACE IN C^1-GAMES

DAVID CARFÌ [a*] AND ANGELA RICCIARDELLO [b]

ABSTRACT. In this paper we present an algorithm implemented by MATLAB, and several examples completely realized by this algorithm, based on a method developed by one of the authors to determine the payoff-space of certain normal-form C^1-games. Specifically, our study is based on a method able to determine the payoff space of normal form C^1-games in n dimensions, that is for n-players normal form games whose payoff functions are defined on compact intervals of the real line and of class at least C^1. In this paper we will determine the payoff space of such normal form C^1-games in the particular case of two dimensions. The implementation of the algorithm gives the parametric form of the critical zone of a game in the bistrategy space and in the payoff space and their graphical representations. Moreover, we obtain the parametric form of the transformation of the topological boundary of the bistrategy space and of the transformation of the critical zone. The final aim of the program is to plot the entire payoff space of the considered games. One of the main motivations of our paper is that the mixed extension of a bimatrix game - the most used in the application of Game Theory - is a game of the type considered. For this reason we realized an algorithm that produces the payoff space and the critical zone of a game in normal form supported by a finite family of compact intervals of the real line. Resuming in details, the algorithm returns: the parametric form of the critical zone; the parametric form of the transformation of the topological boundary of the bistrategy space; the parametric form of the transformation of the critical zone. All of them are graphically represented. To prove the efficiency of the algorithm, we show several examples. Our final goal is to provide a valuable tool to study simply but completely normal form C^1-games in two dimensions.

1. Introduction

Often in Game Theory the study of a normal-form game consists principally in the determination of the Nash equilibria in mixed strategies and in the analysis of their various stability properties (see, for instance, Refs. [1], [2], and [3]). Others feel the need to know the entire set of possibilities (consequences) of the players actions, what we call the payoff space of the game; moreover, they introduce other forms of non-cooperative solutions such as the pairs of conservative strategies [4, 5]. Nevertheless, only recently Carfì proposed a method to determine analytically the topological boundary of the payoff space and consequently to handle more consciously and precisely the entire payoff space [6]. This method gives a complete and global view of the game, since, for instance, it allows to know the positions of the payoff profiles corresponding to the Nash equilibria in the payoff space of the game or the position of the conservative n-value of the game. The knowledge of these positions requires, indeed, the knowledge of the entire payoff space. Moreover, the knowledge of the entire payoff space becomes indispensable when the problem to solve in the game is a bargaining one: in fact, the determination of a bargaining solution (or of compromise solutions) needs the analytical determination of the Pareto boundaries or at least of the topological one [7]. In Ref. [6] Carfì presented a general method to find an explicit expression of the topological boundary of the payoff-space of a Game and this latter boundary contains the two Pareto boundaries of the game.

2. Preliminaries and notations

For the ease of the reader we recall some basic notions of Game Theory. We shall consider n-person games in normal form. The form of definition we will give is particularly interesting since it is nothing but the definition of a specific differentiable parametric ordered submanifold of the Euclidean space.

Games in normal form. Let $E = (E_i)_{i=1}^n$ be a finite ordered family of non-empty sets. We call n-person game in normal form upon the support E each pair $G = (f, \mathcal{R})$, where f is a function of the cartesian product ${}^\times E$ of the family E into the Euclidean space $\mathbb{R}^n$ and $\mathcal{R}$ is one of the two natural orders ($\leq$ or $\geq$) of the real n-dimensional Euclidean space $\mathbb{R}^n$. By ${}^\times E$ we mean the cartesian product $\times_{i=1}^n E_i$ of the finite family E. The set E_i is called the strategy set of player i, for every index i of the family E, and the product ${}^\times E$ is called the strategy profile space, or the n-strategy space, of the game. The set $\{i\}_{i=1}^n$ of the first n positive integers is said the set of players of the game G; each element of the cartesian product ${}^\times E$ is said a strategy profile of the game; the image of the function f, i.e., the set of all real n-vectors of type $f(x)$, with x in the strategy profile space ${}^\times E$, is called the n-payoff space, or simply the payoff space, of the game f.

Pareto boundaries. The Pareto maximal boundary of a game $G = (f, \mathcal{R})$ is the subset of the n-strategy space of those n-strategies x such that the corresponding payoff $f(x)$ is maximal in the n-payoff space, with respect to the usual order $\mathcal{R}$ of the euclidean n-space $\mathbb{R}^n$. We shall denote the maximal boundary of the n-payoff space by $\overline{\partial} f(S)$ and the maximal boundary of the game by $\overline{\partial}_f(S)$ or by $\overline{\partial}(G)$. In other terms, the maximal boundary $\overline{\partial}_f(S)$ of the game is the reciprocal image (by the function f) of the maximal boundary of the payoff space $f(S)$. We shall use analogous terminologies and notations for the minimal Pareto boundary (for an introduction to Pareto boundaries see Ref. [8]).

The method. We deal with a type of normal form game $G = (f, \mathcal{R})$ defined on the product of n compact non-degenerate intervals of the real line $\mathbb{R}$, and such that the payoff function f is the restriction to the n-strategy space of a C^1 -function defined on an open set of the Euclidean space $\mathbb{R}^n$ containing the n-strategy space S (that, in this case, is a compact non-degenerate n-interval of the Euclidean n-space $\mathbb{R}^n$). We recall that the topological boundary of a subset S of a topological space (X, τ) is the set defined by the following three equivalent propositions:

(1) it is the closure of S without the interior of S: $\partial(S) = \mathrm{cl}(S) \backslash \mathrm{int}(S)$;
(2) it is the intersection of the closure of S with the closure of its complement $\partial(S) = \mathrm{cl}(S) \cap \mathrm{cl}(X \backslash S)$;

(3) it is the set of those points x of X such that any neighborhood of x contains at least one point of S and at least one point in the complement of S.

The key theorem of the method proposed by Carfì is the following one.

Theorem 2.1. *teo1 Let f be a C^1 function defined upon an open set O of the euclidean space $\mathbb{R}^n$ and with values in $\mathbb{R}^n$. Then, for every part S of the open O, the topological boundary of the image of S by the function f is contained in the union $f(\partial S) \cup f(C)$, where C is the critical set of the function f in S, that is the set of all points x of S such that the Jacobian matrix $J_f(x)$ is not invertible.*

3. Algorithm

In this section we present the algorithm that we used to determine numerically the payoff space of normal form C^1-games in 2 dimensions.

Let A, B, C, D be the vertices of the initial rectangular domain. The inputs are the coordinates of such vertices and the functions $f1(\cdot)$ and $f2(\cdot)$ that define the game f, so that $f(P) = (f1(P), f2(P))$ con $P \in \mathbb{R}^2$.
Denote by x_{min} and x_{max} (y_{min} and y_{max}) the minimum and maximum of vertex abscissae (ordinates).

STEP 1. TRANSFORMATION OF THE TOPOLOGICAL BOUNDARY.

Then the initial domain is the rectangle

$$R_{ABCD} = \{(x, y) : x_{min} \leq x \leq x_{max} \text{ and } y_{min} \leq y \leq y_{max}\}.$$

The transformation of the topological boundary, is a new quadrilateral, of vertices (A', B', C', D'), where a point P' is the image of a point P, by means the following transformation

$$P' = (f1(P), f2(P)) \, \forall P \in \{A,\ B,\ C,\ D\}$$

STEP 2. PAYOFF SPACE AND CRITICAL ZONE.

We evaluate the Jacobian determinant of the game f.
If it is zero, then the payoff space agrees the transformation of topological boundary.
Otherwise, the Jacobian determinant is a function of x or y.
In this case, we solve the jabobian in the depending variable. The critical zone is defined by all the points of this transformation, that are in the initial domain R_{ABCD}, too.

The payoff space is the area delimited by the topological boundary and the critical zone. Note that, if the critical zone is void, the payoff space agrees the transformation of the topological boundary.

STEP 3. PLOTS

The outputs of the algorithm are the graphics of transformation of topological boundary,of critical zone (if it exists), of payoff space.

4. First game

Description of the game. We consider a loss-game $G = (f, <)$, with strategy sets $E = F = [0,1]$ and biloss (disutility) function defined by

$$f(x,y) = (-4xy, x+y)$$

for every bistrategy (x,y) of the game.

The critical space of the game. In the following, we shall denote by A, B, C and D the vertices of the square $E \times F$, starting from the origin and going anticlockwise.

Jacobian matrix. The Jacobian matrix is

$$J_f = \begin{pmatrix} -4y & -4x \\ 1 & 1 \end{pmatrix},$$

for every bistrategy (x,y). The Jacobian determinant is

$$detJ_f(x,y) = -4y + 4x$$

for every pair (x,y). The **critical zone** is the subset of the bistrategy space of those bistrategies verifying the equality $-y + x = 0$. In symbols, the critical zone is the segment

$$\mathcal{C}(f) = \left\{(x,y) \in [0,1]^2 : x = y\right\} = [A, C]$$

that is graphically represented in Fig. 1.

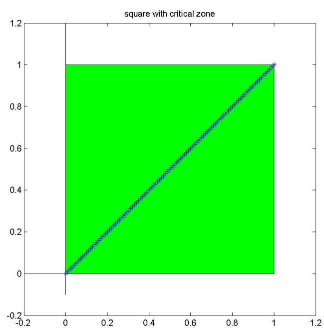

FIGURE 1. Bistrategy square with critical zone.

Transformation of the critical space. Let us determine the image $f([A,C])$. The value of the biloss function upon the generic point (y,y) of the segment $[A,C]$, is

$$f(y,y) = \left(-4y^2, 2y\right).$$

The image of the critical zone is represented in Fig. 2.

The biloss (disutility) space. Transformation of the topological boundary of the bistrategy space. We start from the image $f([A,B])$. The segment $[A,B]$ is defined by

$$\begin{cases} y = 0 \\ x \in [0,1] \end{cases}$$

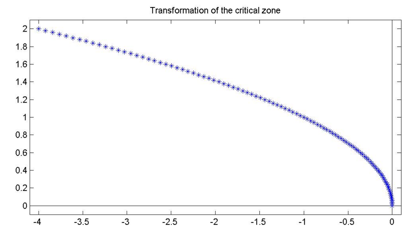

FIGURE 2. Plot of the transformation of the critical zone.

The value of the biloss function upon the generic point is $f(x, 0) = (0, x)$, that is graphically represented in Fig. 3.

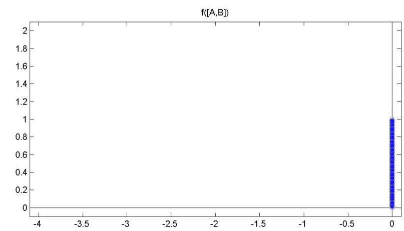

FIGURE 3. Plot of the transformation of the segment [A,B].

We now consider the image $f([D, C])$. The segment $[D, C]$ is defined by

$$\begin{cases} y = 1 \\ x \in [0, 1] \end{cases}$$

The value of the biloss function upon the generic point is $f(x, 1) = (-4x, x + 1)$ that is graphically represented in Fig. 4.

Let us determine the image $f([C, B])$. The segment $[C, B]$ is defined by

$$\begin{cases} x = 1 \\ y \in [0, 1] \end{cases}$$

The value of the biloss function upon the generic point is $f(1, y) = (-4y, 1 + y)$, that is graphically represented in Fig. 5.

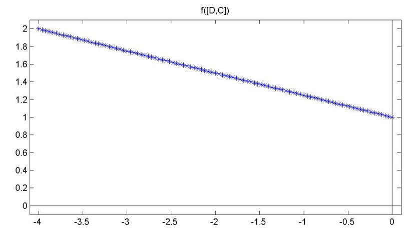

FIGURE 4. Plot of the transformation of the segment [D,C].

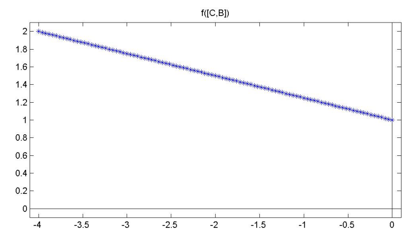

FIGURE 5. Plot of the transformation of the segment [C,B].

Let us finally determine the image $f([A,D])$. The segment $[A,D]$ is defined by

$$\begin{cases} x = 0 \\ y \in [0,1] \end{cases}$$

The value of the biloss function upon the generic point is $f(0,y) = (0,y)$, that is graphically represented in Fig. 6. The image of the transformation of the topological boundary of the bistrategy space is plotted in Fig. 7, while the resulting payoff space is shown in Fig. 8.

5. Second game

Description of the game. We consider a loss-game $G = (f, <)$, with strategy sets $E = F = [0,1]$ and biloss (disutility) function defined by

$$f(x,y) = \left(x - \frac{1}{2}xy, y - \frac{1}{2}xy\right)$$

for every bistrategy (x,y) of the game.

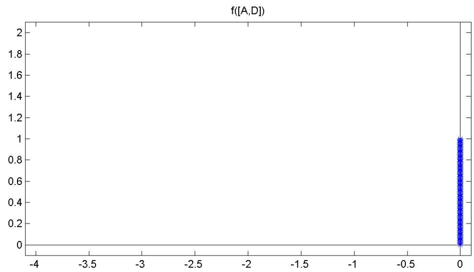

FIGURE 6. Plot of the transformation of the segment [A,D].

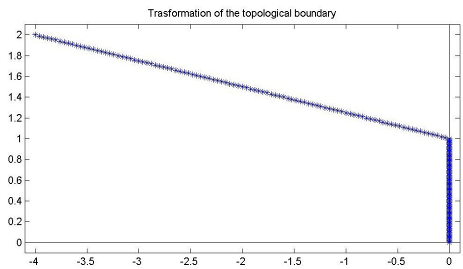

FIGURE 7. Plot of the transformation of the topological boundary of the bistrategy space.

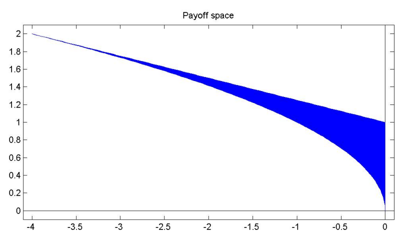

FIGURE 8. Payoff space.

The critical space of the game. In the following we shall denote by $A,\ B,\ C$ and D the vertices of the square $E\times F$, starting from the origin and going anticlockwise.

Jacobian matrix. The Jacobian matrix is

$$J_f = \begin{pmatrix} 1-\frac{1}{2}y & -\frac{1}{2}x \\ -\frac{1}{2}y & 1-\frac{1}{2}x \end{pmatrix}$$

for every bistrategy (x,y). The Jacobian determinant is

$$detJ_f\,(x,y) = 1-\frac{1}{2}x-\frac{1}{2}y$$

for every pair (x,y).

The **critical zone** is the subset of the bistrategy space of those bistrategies verifying the equality $y=2-x$, i.e.:

$$\mathcal{C}(f) = \left\{(x,y)\in[0,1]^2 : y=2-x\right\} = (C)$$

that is graphically represented in Fig. 9.

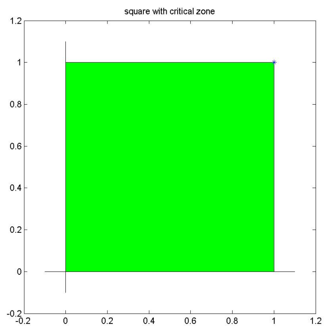

FIGURE 9. Bistrategy square with critical zone.

Transformation of the critical space. Let us determine the transformation of the critical zone. It is defined by the relations

$$\begin{cases} y=2-x \\ x\in[0,1] \end{cases}$$

The value of the biloss function upon he generic point $(x,2-x)$ is

$$f\,(x,2-x) = \left(x-x^2, 2-x-x^2\right).$$

The image of the critical zone is pictured in Fig. 10.

The biloss (disutility) space. Transformation of the topological boundary of the bistrategy space. We start from the image $f\,([A,B])$. The segment $[A,B]$ is defined by

$$\begin{cases} y=0 \\ x\in[0,1] \end{cases}$$

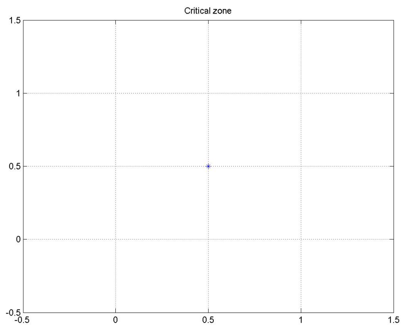

FIGURE 10. Plot of the transformation of the critical zone.

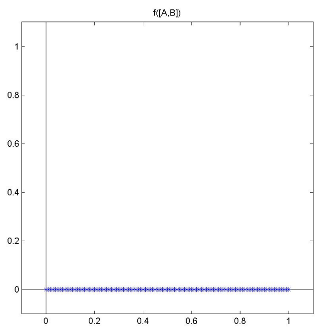

FIGURE 11. Plot of the transformation of the segment [A,B].

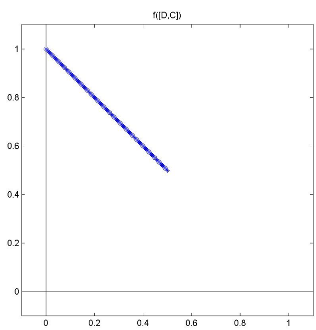

FIGURE 12. Plot of the transformation of the segment [D,C].

The value of the biloss function upon the generic point is $f\,(x, 0) = (x, 0)$, that is graphically represented in Fig. 11.

We now consider the image $f([D, C])$. The segment $[D, C]$ is defined by

$$\begin{cases} y = 1 \\ x \in [0, 1] \end{cases}$$

The value of the biloss function upon the generic point is $f(x, 1) = \left(\frac{1}{2}x, 1 - \frac{1}{2}x\right)$, that is graphically represented in Fig. 12. Let us determine the image $f([C, B])$. The segment $[C, B]$ is defined by

$$\begin{cases} x = 1 \\ y \in [0, 1] \end{cases}$$

The value of the biloss function upon the generic point is $f(1, y) = \left(1 - \frac{1}{2}y, \frac{1}{2}y\right)$, that is graphically represented in Fig. 13.

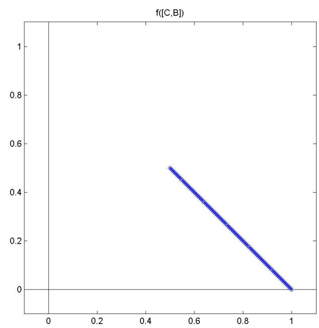

FIGURE 13. Plot of the transformation of the segment [C,B].

Finally, let us determine the image $f([A, D])$. The segment $[A, D]$ is defined by

$$\begin{cases} x = 0 \\ y \in [0, 1] \end{cases}$$

The value of the biloss function upon the generic point is $f(0, y) = (0, y)$, that is graphically represented in Fig. 14. The image of the transformation of the topological boundary of the bistrategy space is pictured in Fig. 15, and the resulting payoff space is shown in Fig. 16.

6. Third game

Description of the game. We consider a loss-game $G = (f, <)$, with strategy sets $E = F = [0, 1]$ and biloss (disutility) function defined by

$$f(x, y) = (x, y + xy)$$

for every bistrategy (x, y) of the game.

The critical space of the game. In the following we shall denote by $A,\ B,\ C$ and D the vertices of the square $E \times F$, starting from the origin and going anticlockwise.

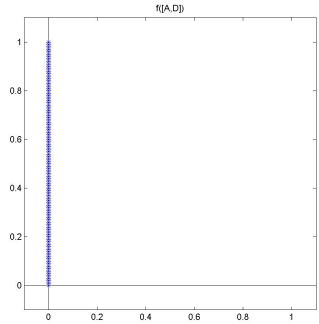

FIGURE 14. Plot of the transformation of the segment [A,D].

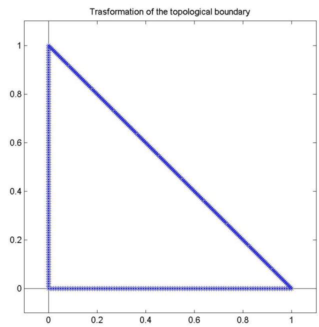

FIGURE 15. Plot of the transformation of the topological boundary of the bistrategy space.

Jacobian matrix. The Jacobian matrix is

$$J_f = \begin{pmatrix} 1 & 0 \\ y & 1+x \end{pmatrix},$$

for every bistrategy (x, y). The Jacobian determinant is

$$detJ_f\,(x, y) = 1 + x$$

for every pair (x, y).

The **critical zone** is the subset of the bistrategy space of those bistrategies verifying the equality $x = -1$. So there are not points of the critical zone in the strategy sets (see Fig. 17).

The biloss (disutility) space. Transformation of the topological boundary of the bistrategy space. We start from the image $f\,([A, B])$. The segment $[A, B]$ is defined by $y = 0$ and $x \in [0, 1]$. The value of the biloss function upon the generic point is $f\,(x, 0) = (x, 0)$ (see Fig. 18).

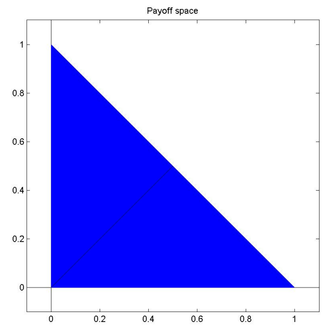

FIGURE 16. Payoff space.

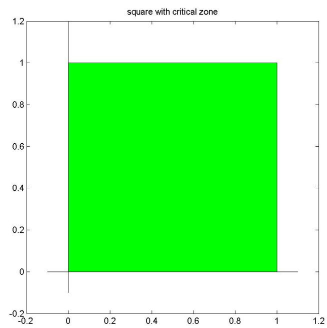

FIGURE 17. Bistrategy square with critical zone.

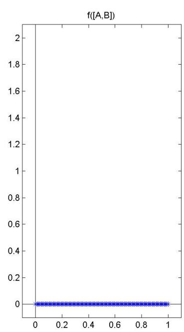

FIGURE 18. Plot of the transformation of the segment [A,B].

We now consider the image $f([D,C])$. The segment $[D,C]$ is defined by $y = 1$ and $x \in [0,1]$. The value of the biloss function upon the generic point is $f(x,1) = (x, 1+x)$ that is graphically represented in Fig. 19.

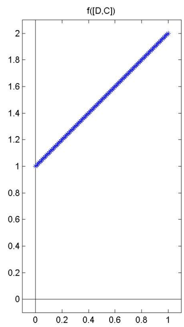

FIGURE 19. Plot of the transformation of the segment [D,C].

Let us determine the image $f\left([C,B]\right)$. The segment $[C,B]$ is defined by $x = 1$ and $y \in [0,1]$. The value of the biloss function upon the generic point is $f\left(1,y\right) = (1,2y)$. Setting $X = 1$ and $Y = 2y$, we have $X = 1$ and $Y \in [0,2]$ (see Fig. 20).

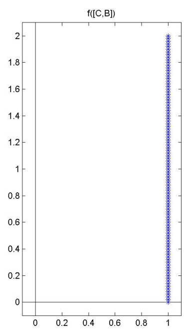

FIGURE 20. Plot of the transformation of the segment [C,B].

Finally, let us determine the image $f\left([A,D]\right)$. The segment $[A,D]$ is defined by $x = 0$ and $y \in [0,1]$. The value of the biloss function upon the generic point is $f\left(0,y\right) = (0,y)$ that is graphically represented in Fig. 21. The image of the transformation of the topological boundary of the bistrategy space is pictured in Fig. 22, while the resulting payoff space is shown in Fig. 23.

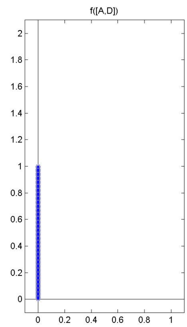

FIGURE 21. Plot of the transformation of the segment [A,D].

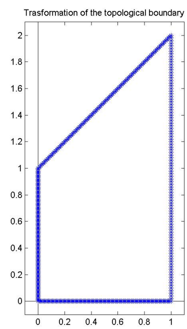

FIGURE 22. Plot of the transformation of the topological boundary of the bistrategy space.

7. Fourth game

Description of the game. We consider a loss-game $G = (f, <)$, with strategy sets $E = F = [0, 1]$ and biloss (disutility) function defined by

$$f(x, y) = \left(x - \frac{3}{4}xy, y - \frac{3}{4}xy\right)$$

for every bistrategy (x, y) of the game.

The critical space of the game. In the following we shall denote by A, B, C and D the vertices of the square $E \times F$, starting from the origin and going anticlockwise.

Jacobian matrix. The Jacobian matrix is

$$J_f = \begin{pmatrix} 1 - \frac{3}{4}y & -\frac{3}{4}x \\ -\frac{3}{4}y & 1 - \frac{3}{4}x \end{pmatrix},$$

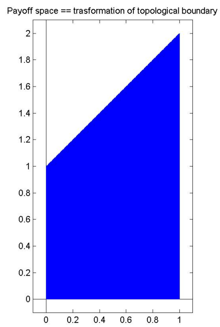

FIGURE 23. Payoff space.

for every bistrategy (x, y). The Jacobian determinant is

$$detJ_f(x, y) = 1 - \frac{3}{4}x - \frac{3}{4}y$$

for every pair (x, y).

The **critical zone** is the subset of the bistrategy space of those bistrategies verifying the equality $y = \frac{4}{3} - x$. In symbols, the critical zone is

$$\mathcal{C}(f) = \left\{(x, y) \in [0, 1]^2 : y = \frac{4}{3} - x\right\}$$

that is graphically represented in Fig. 24.

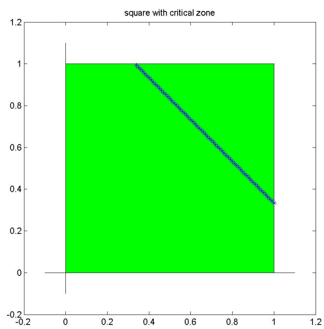

FIGURE 24. Bistrategy square with critical zone.

Transformation of the critical space. Let us determine the transformation of the critical zone. It is defined by the relations:

$$\begin{cases} y = \frac{4}{3} - x \\ x \in [0, 1] \end{cases}$$

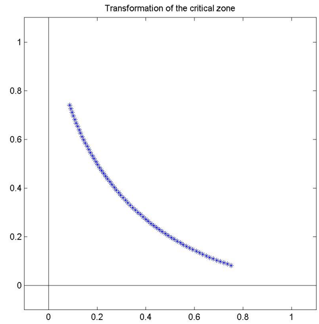

FIGURE 25. Plot of the transformation of the critical zone.

The value of the biloss function upon the generic point $\left(x, \frac{4}{3} - x\right)$ is

$$f\left(x, \frac{4}{3} - x\right) = \left(x - x^2, \frac{4}{3} - x - x^2\right).$$

The image of the critical zone is pictured in Fig. 25.

The biloss (disutility) space. Transformation of the topological boundary of the bistrategy space. We start from the image $f([A, B])$. The segment $[A, B]$ is defined by $y = 0$ and $x \in [0, 1]$. The value of the biloss function upon the generic point is $f(x, 0) = (x, 0)$, as shown in Fig. 26.

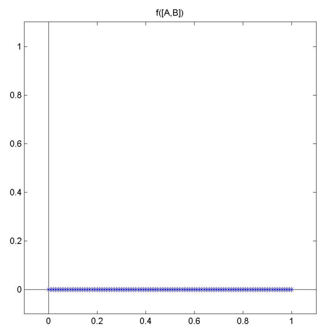

FIGURE 26. Plot of the transformation of the segment [A,B].

We now consider the image $f([D, C])$. The segment $[D, C]$ is defined by $y = 1$ and $x \in [0, 1]$. The value of the biloss function upon the generic point is $f(x, 1) = \left(\frac{1}{4}x, 1 - \frac{3}{4}x\right)$ that is graphically represented in Fig. 27. Let us determine the image $f([C, B])$. The segment $[C, B]$ is defined by $x = 1$ and $y \in [0, 1]$. The value of the biloss function upon the generic point is $f(1, y) = \left(1 - \frac{3}{4}y, \frac{1}{4}y\right)$ (see Fig. 28).

Let us finally determine the image $f([A, D])$. The segment $[A, D]$ is defined by $x = 0$ and $y \in [0, 1]$. The value of the biloss function upon the generic point is $f(0, y) = (0, y)$

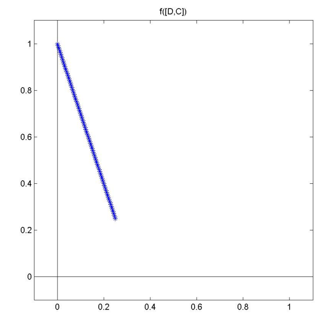

FIGURE 27. Plot of the transformation of the segment [D,C].

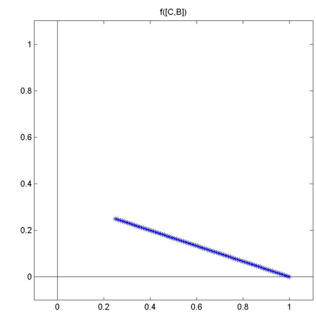

FIGURE 28. Plot of the transformation of the segment [C,B].

(see Fig. 29). The image of the transformation of the topological boundary of the bistrategy space is pictured in Fig. 30, and the resulting payoff space is shown in Fig. 31.

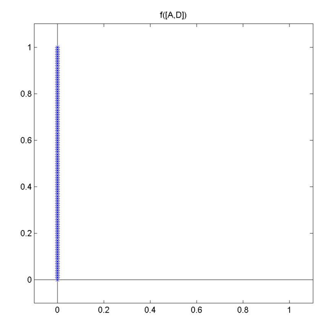

FIGURE 29. Plot of the transformation of the segment [A,D].

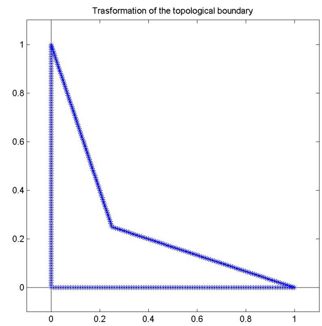

FIGURE 30. Plot of the transformation of the topological boundary of the bistrategy space.

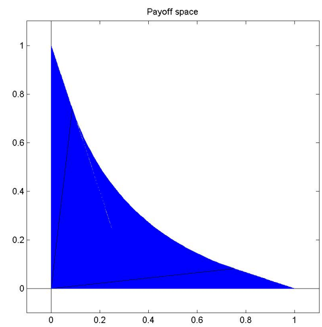

FIGURE 31. Payoff space.

References

[1] M. J. Osborne and A. Rubinstein, *A Course in Game Theory* (Academic Press, 2001).

[2] G. Owen, *Game Theory* (Academic Press, 2001).

[3] R. B. Myerson, *Game Theory* (Harvard University Press, 1991).

[4] J. Aubin, *Mathematical Methods of Game and Economic Theory* (North-Holland, Amsterdam, 1980).

[5] J. Aubin, *Optima and Equilibria*, second edition (Springer Verlag, 1998).

[6] D. Carfì, "Payoff space of C^1 Games", *Applied Sciences* **11**, 35-47 (2009).

[7] D. Carfì, "Differentiable game complete analysis for tourism firm decisions", in *Proceedings of the 2009 International Conference on Tourism and Workshop on "Sustainable Tourism within High Risk Areas of Environmental Crisis"*, Messina, Italy, 2009.

[8] D. Carfì, "Optimal boundaries for decisions", *Atti Accad. Pelorit. Pericol. Cl. Sci. Fis. Mat. Nat.* **86**, C1A0801002 (2008).

[a] Università degli Studi di Messina
Dipartimento di Economia, Statistica, Matematica e Sociologia "W. Pareto" (DESMAS)
Via dei Verdi, 75
98122 Messina, Italy

[b] Università degli Studi di Messina
Dipartimento di Matematica
Contrada Papardo,
98166, Messina, Italy

* To whom correspondence should be addressed | e-mail: davidcarfi71@yahoo.it

Presented 18 May 2009; published online 19 February 2010