DOI: 10.1478/C1A1002004

AAPP | Atti della Accademia Peloritana dei Pericolanti
Classe di Scienze Fisiche, Matematiche e Naturali
ISSN 1825-1242

Vol. LXXXVIII, No. 2, C1A1002004 (2010)

BUFFON TYPE PROBLEMS WITH MULTIPLE INTERSECTIONS FOR REGULAR LATTICES

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ABSTRACT. In this paper we study Buffon type problems with multiple intersections for lattices of equilateral triangles and a circle as test body.

1. Introduction

In papers [1] and [2] A. Duma and M. Stoka studied Buffon type problems with multiple intersections for lattices of the Euclidian plane $\mathbf{E}_2$, with a parallelogram $\mathcal{P}$ and an equilateral triangle τ, elementary tile respectively, and a segment of constant length.

In this paper, we consider two lattices, $\mathcal{R}_1$ and $\mathcal{R}_2$, with the same fundamental cell, an equilateral triangle τ of side a. Hence, we determine the probability of multiple intersections of the test body, a circle of constant radius r with the sides of the lattices $\mathcal{R}_1$ and $\mathcal{R}_2$, respectively.

2. Geometric probability of multiple intersections for the lattice $\mathcal{R}_1$

Considering the lattice $\mathcal{R}_1$, we denote by p_{1i}, $(i = 1, 2, \ldots, 6)$ the probability that the body test intersects the sides of the lattice i-times.

Theorem 1. *If $r < a\frac{\sqrt{3}}{6}$, the probabilities that a circle **C** of constant radius r, uniformly distributed in a bounded region of the plane, intersects, the sides of the lattice $\mathcal{R}_1$, i-times, (i=1,..., 6), are respectively*

$$p_{11} = p_{13} = p_{15} = 0, \tag{1}$$

$$p_{12} = \frac{12}{\sqrt{3}}\frac{r}{a} - 20\left(\frac{r}{a}\right)^2, \tag{2}$$

$$p_{14} = p_{16} = 4\frac{r^2}{a^2}. \tag{3}$$

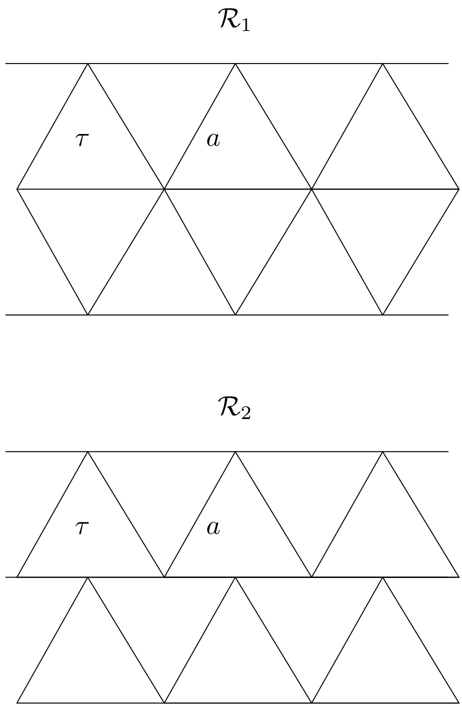

FIGURE 1.

Proof. We denote by $\mathcal{M}$ the set of circles of radius r with center $C(x, y)$ which belongs to τ. Then, as in figure 2, the test body intersects the sides of the triangles τ i-times, (i=1,..., 6), (i. e. the sides of the lattice $\mathcal{R}_1$), if and only if, $C \in \tau_{1i}, i = 1, \dots, 6$. Hence, putting $\mathcal{N}_{1i} = \{(x, y) \in \tau_{1i}\}$, (i=1,..., 6), we have

$$p_{1i=}\frac{\mu(\mathcal{N}_{1i})}{\mu(\mathcal{M})},\ (i = 1, ..., 6), \tag{4}$$

where μ is the Lebesgue measure in the Euclidean plane. We can compute the previous measures using the elementary kinematic measure of Poincaré ([4])

$$dK = dx \wedge dy \wedge d\varphi;$$

where x and y are the coordinates of the center of **C** (or the components of a translation) and φ is the angle of rotation. We have

$$\mu(\mathcal{M}) = area\,\tau = \frac{a^2\sqrt{3}}{4}. \tag{5}$$

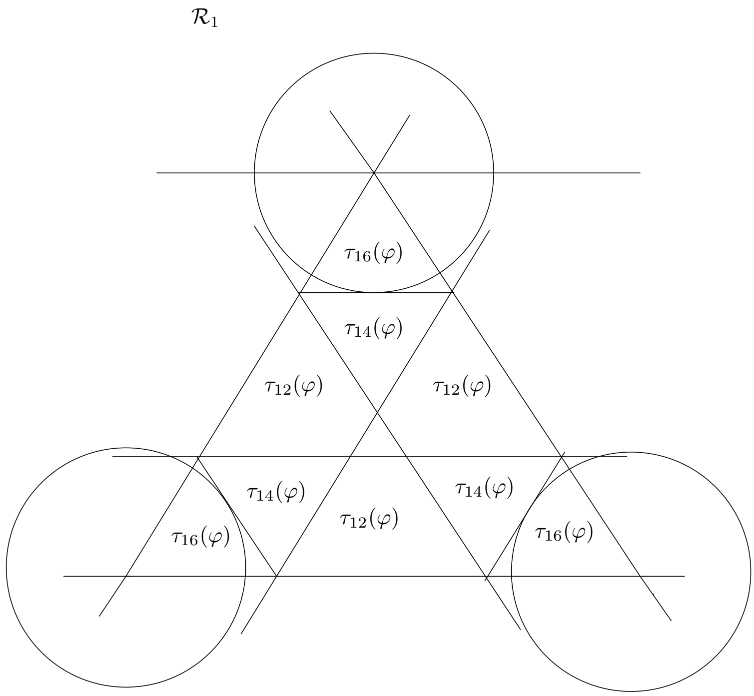

FIGURE 2.

We compute $\mu(\mathcal{N}_{12})$ observing that τ_{12} is union of three congruent trapeziums with bases of lengths $a - \frac{4r}{\sqrt{3}}$ and $a - \frac{6r}{\sqrt{3}}$ and height r. Then

$$\mu(\mathcal{N}_{12}) = area\,\tau_{12} = 3ar - \frac{15}{\sqrt{3}}r^2. \tag{6}$$

The sets τ_{14} and τ_{16} are the union of three congruent equilateral triangles of side $\frac{2r}{\sqrt{3}}$, therefore

$$\mu(\mathcal{N}_{14}) = \mu(\mathcal{N}_{16}) = area\,\tau_{14} = area\,\tau_{16} = r^2\sqrt{3}. \tag{7}$$

We observe that the circle never intersects once, three times or five times the sides of the lattice $\mathcal{R}_1$. From formulas (4), (5), (6) and (7) we have the probabilities (1), (2) and (3). □

Considering the lattice $\mathcal{R}_1$, we denote by pl_{1i}, $(i = 1, 2, \ldots, 6)$ the probability that the test body intersects i (i=1,...,6) sides of the lattice.

Theorem 2. *If $r < a\frac{\sqrt{3}}{6}$, the probabilities that a circle **C** of constant radius r, uniformly distributed in a bounded region of the plane, intersects, i sides (i=1,. . . , 6), of the lattice $\mathcal{R}_1$ are respectively*

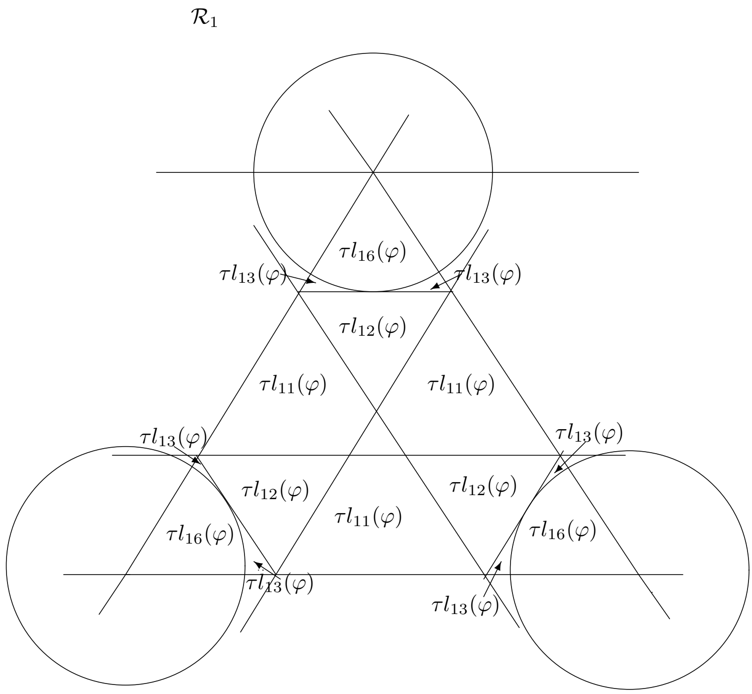

FIGURE 3.

$$pl_{11} = \frac{12}{\sqrt{3}}\frac{r}{a} - 20\left(\frac{r}{a}\right)^2, \tag{8}$$

$$pl_{12} = 4\frac{r^2}{a^2} \tag{9}$$

$$pl_{13} = \left(4 - \frac{2\pi}{\sqrt{3}}\right)\left(\frac{r}{a}\right)^2 \tag{10}$$

$$pl_{14} = pl_{15} = 0 \tag{11}$$

$$pl_{16} = \frac{2\pi}{\sqrt{3}}\frac{r^2}{a^2}. \tag{12}$$

Proof. With the same notations as in the previous theorem, as in figure 3, the test body intersects i sides (i=1,...,6) of the lattice $\mathcal{R}_1$ if, and only if, $C \in \tau l_{1i}, i = 1, \ldots, 6$. Hence, putting $\mathcal{N}_{1i} = \{(x, y) \in \tau l_{1i}\}$, (i=1,..., 6), we have

$$pl_{1i=}\frac{\mu(\mathcal{N}_{1i})}{\mu(\mathcal{M})},\ (i = 1, ..., 6). \tag{13}$$

We compute $\mu(\mathcal{N}_{13})$ observing that τl_{13} is the union of three congruent surfaces, given as the difference between the area of an equilateral triangle of side $\frac{2r}{\sqrt{3}}$ and the area of the circular sector of radius r and angle $\frac{\pi}{3}$, therefore

$$\mu(\mathcal{N}_{13}) = area\, \tau l_{13} = r^2\sqrt{3} - \frac{\pi}{2}r^2. \tag{14}$$

The sets τl_{16} are the union of three congruent circular sectors of radius r and angle $\frac{\pi}{3}$. Then

$$\mu(\mathcal{N}_{16}) = area\, \tau l_{16} = \frac{\pi}{2}r^2. \tag{15}$$

We observe that the circle never intersects four or five sides of the lattice $\mathcal{R}_1$, hence (11) follows. Since $pl_{11} = p_{12}$, $pl_{12} = p_{14}$ and from formulas (13), (14) and (15), we have the probabilities (8), (9), (10), (11) and (12). □

Corollary 3. *The probability that a circle* **C** *of constant radius $r < a\frac{\sqrt{3}}{6}$ intersects one of the sides of the lattice $\mathcal{R}_1$ is*

$$p = \frac{12}{\sqrt{3}}\frac{r}{a} - 12\left(\frac{r}{a}\right)^2. \tag{16}$$

Proof. Taking into account that $p = pl_{11} + pl_{12} + pl_{13} + pl_{14} + pl_{15} + pl_{16}$, formulas (8), (9), (10), (11) and (12) give the probability (16). □

Remark Applying formula

$$p_{3;a,\alpha} = 4\frac{1+\cos\alpha}{\sin\alpha}\frac{r}{a} - 4\left(\frac{1+\cos\alpha}{\sin\alpha}\right)^2\left(\frac{r}{a}\right)^2 \tag{17}$$

of the probability that a circle of constant radius r, uniformly distributed in a bounded region of the plane, intersects a straight line of the lattice $\mathcal{R}_{3;a,\alpha}$ of lines, having an isosceles triangle as elementary tile with basis of length a and angles α, α and $\pi - 2\alpha$, [3] with $\alpha = \frac{\pi}{3}$, we obtain (16).

3. Geometric probability of multiple intersections for the lattice $\mathcal{R}_2$

Now we consider the lattice $\mathcal{R}_2$ and we denote by p_{2i}, (i=1,2,3,4) the probability that the test body intersects the sides of the lattice i-times .

Theorem 4. *If $r < a\frac{\sqrt{3}}{6}$, the probabilities that a circle* **C** *of constant radius r, uniformly distributed in a bounded region of the plane, intersects, the sides of the lattice $\mathcal{R}_2$, i-times, (i=1,2,3,4), are respectively*

$$p_{21} = p_{23} = 0, \tag{18}$$

$$p_{22} = 4\sqrt{3}\frac{r}{a} - \left(20 + \frac{2\sqrt{3}\pi}{3}\right)\left(\frac{r}{a}\right)^2 \tag{19}$$

$$p_{24} = \left(8 + \frac{2\pi\sqrt{3}}{3}\right)\left(\frac{r}{a}\right)^2 \tag{20}$$

Proof. With the same notations as theorem 1, as in figure 4, the test body intersects the sides of the lattice $\mathcal{R}_2$ i-times (i=1,...,4) if, and only if, $C \in \tau_{2i}, i = 1, \ldots, 4$. We compute $\mu(\mathcal{N}_{22})$ observing that τ_{22} is the union of two congruent trapeziums with bases of lengths $a - \frac{4r}{\sqrt{3}}$ and $a - \frac{6r}{\sqrt{3}}$ and height r and a surface given as the difference between the area

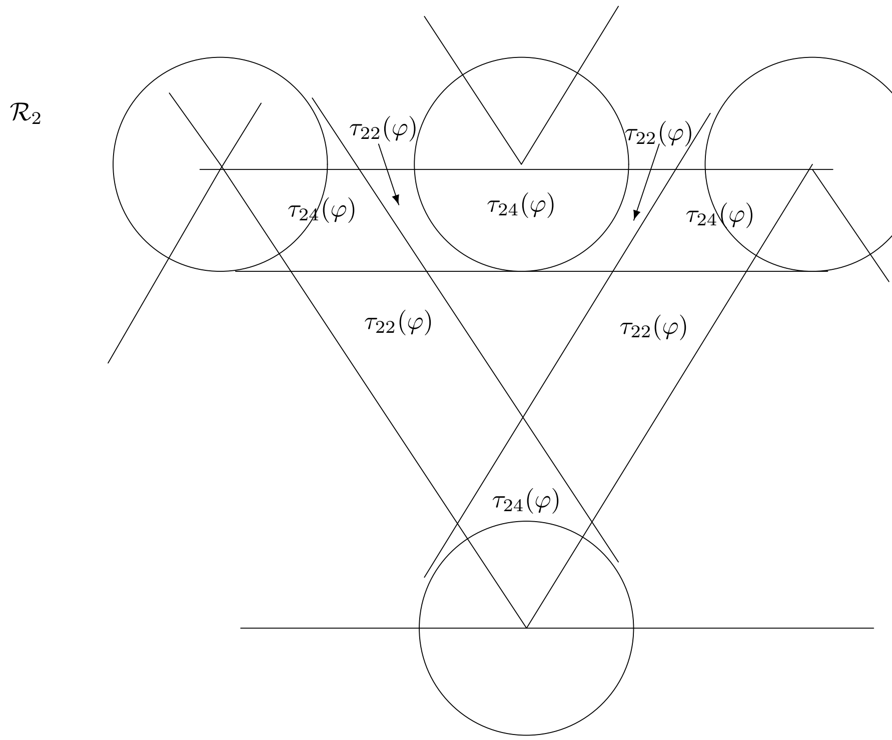

FIGURE 4.

of a trapezium with bases of lengths $a - \frac{4r}{\sqrt{3}}$ and $a - \frac{6r}{\sqrt{3}}$ and height r, and the area of a semicircle of radius r and angle $\frac{\pi}{3}$. Then

$$\mu(\mathcal{N}_{22}) = area\, \tau_{22} = 3ar - \frac{15}{\sqrt{3}}r^2 - \frac{\pi}{2}r^2. \tag{21}$$

The sets τ_{24} is the union of three congruent rhombs of side $\frac{2r}{\sqrt{3}}$ and a semicircle of radius r. Then

$$\mu(\mathcal{N}_{24}) = area\, \tau_{24} = \left(2\sqrt{3} + \frac{\pi}{2}\right) r^2. \tag{22}$$

We observe that the circle never intersects the sides of the lattice $\mathcal{R}_2$ once or three times. From formulas (4), (5), (21) and (22) we have the probabilities (18), (19) and (20). $\square$

Considering the lattice $\mathcal{R}_2$, we denote by pl_{2i}, (i=1,2,3,4) the probability that the test body intersects i sides of the lattice.

Theorem 5. *If $r < a\frac{\sqrt{3}}{6}$, the probabilities that a circle **C** of constant radius r, uniformly distributed in a bounded region of the plane, intersects, i sides (i=1,2,3,4) of the lattice*

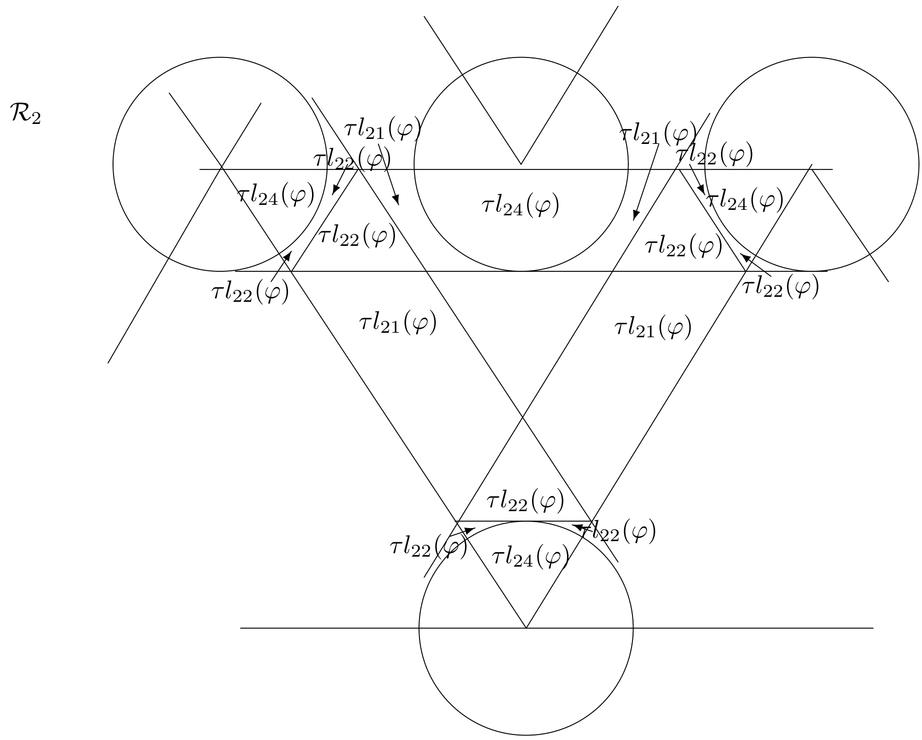

FIGURE 5.

$\mathcal{R}_2$, *are respectively*

$$pl_{21} = 4\sqrt{3}\frac{r}{a} - \left(20 + \frac{2\sqrt{3}\pi}{3}\right)\left(\frac{r}{a}\right)^2, \tag{23}$$

$$pl_{22} = \left(8 - \frac{2\pi\sqrt{3}}{3}\right)\left(\frac{r}{a}\right)^2, \tag{24}$$

$$pl_{23} = 0, \tag{25}$$

$$pl_{24} = \frac{4\sqrt{3}\pi}{3}\left(\frac{r}{a}\right)^2. \tag{26}$$

Proof. With the same notations as theorem 1, as in figure 5, the test body intersects i sides (i=1,...,4) of the lattice $\mathcal{R}_2$ if, and only if, $C \in \tau l_{2i}, i = 1, \dots, 4$. We compute $\mu(\mathcal{N}_{21})$ observing that τl_{21} is equal to τ_{22}. The set τl_{22} is the union of six congruent equilateral triangles of sides $\frac{2r}{\sqrt{3}}$ minus three circular sector of radius r and angle $\frac{\pi}{3}$. Then

$$\mu(\mathcal{N}_{22}) = area\,\tau l_{22} = \left(2\sqrt{3} - \frac{\pi}{2}\right) r^2. \tag{27}$$

Finally, the set τl_{24} is the union of three congruent circular sectors of radius r and angle $\frac{\pi}{3}$, and the area of a semicircle of radius r. Therefore

$$\mu(\mathcal{N}_{24}) = area\, \tau l_{24} = \pi r^2. \tag{28}$$

We observe that the circle never intersects the sides of the lattice $\mathcal{R}_2$ once or three times. From formulas (4), (5), (27) and (28) we have the probabilities (23), (24), (25) and (26). □

Corollary 6. *The probability that a circle* **C** *of constant radius $r < a\frac{\sqrt{3}}{6}$ intersects one of the sides of the lattice $\mathcal{R}_2$ is*

$$p = \frac{12}{\sqrt{3}}\frac{r}{a} - 12\left(\frac{r}{a}\right)^2. \tag{29}$$

Proof. Taking into account that $p = pl_{11} + pl_{12} + pl_{13} + pl_{16}$, formulas (23), (24) and (26) give the probability (29). □

References

[1] A. Duma and M. Stoka, "Problemi di tipo Buffon con intersezioni multiple per reticoli di triangoli equilateri", *Pub. Ist. Stat. Univ. Paris* **51**, Fasc. III, 3-15 (2007).
[2] A. Duma and M. Stoka, "Problems of Buffon type with multiple intersections for lattices of parallelograms", *Rend. Circ. Mat. Palermo*, serie II, Tomo **LV**, 241-248 (2006).
[3] M. Pettineo, "Problemi di probabilità geometriche per reticoli triangolari nel piano euclideo", *Rend. Circ. Mat. Palermo*, serie II, Suppl. **38**, 89-96 (1995).
[4] H. Poincaré, "Calcul des probabilités" (Carré, Paris, 1912), ed. 2.

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Presented 25 November 2009; published online 20 September 2010