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FURTHER RESULTS ON APPROXIMATE INERTIAL MANIFOLDS FOR THE FITZHUGH-NAGUMO MODEL

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ABSTRACT. For two particular choices of the three parameters in the FitzHugh-Nagumo model the equilibrium points are found. The corresponding phase portrait around them is graphically represented allowing us to delimit an absorbing domain. Then the Jolly-Rosa-Temam numerical method is applied in order to study the approximate inertial manifold for the model. To this aim the own numerical code of the first author is used.

1. Introduction

Consider the abstract evolution equation

$$\frac{du}{dt} + Au = f(u), \tag{1}$$

with the initial condition $u(0) = u_0$. With it we can associate the semigroup $\{S(t)\}_{t\geq 0}$ on a Banach space E, where $S(t) : u_0 \to u(t)$, $u(\cdot)$ is the solution of (1), with $u(0) = u_0$, A is a linear operator and f is a linear mapping.

An *inertial manifold* [1] $\mathcal{M}$ is a finite-dimensional Lipschitz manifold, positively invariant (i.e. $S(t)\mathcal{M} \subset \mathcal{M}$, $t \geq 0$) and which exponentially attracts all orbits of (1).

The concept of inertial manifold has been introduced in 1985 by C. Foias, G. R. Sell and R. Temam in the context of dissipative evolution equations [2, 3, 4, 5, 6].

Remind several facts in the theory of Navier-Stokes partial differential equations, used by these specialists, which are the premises leading to this concept. In the following we give three of them.

The first is the Galerkin-Faedo-Hopf method applied to the generalized form of these equations in order to get existence and nonlinear stability results [7]. Accordingly, the equations were approximated by those whose solutions were linear combinations of the first n eigenvectors of the linear operator A. Further the projection of the approximate equation on the space spanned by these linear combinations reduces the problem to a system of n ordinary differential equations in the time-dependent Fourier coefficients in those combinations. The key point in this approach is the fact that, given the good properties of A, its eigenvectors formed a total set in the space of the given equations. Remark the occurrence of finite dimension spaces in the infinite-dimension given problem.

The second fact is related to the Lyapunov-Perron method and its Lyapunov-Schmidt variant used in stability theory. According to this method, the Banach space of the problem and its image through A are splitted into direct sums of finite-dimensional and infinite-dimensional subspaces. The finite-dimensional subspaces in these sums are the ker$A$ and corange$A$. Then one projection is defined on ker$A$ to split the solution and one projection of equation, on corange$A$ and range$A$ to obtain a finite-dimensional equation, and an infinite-dimensional equation defined by a contraction for which a fixed point theorem holds [8, 9]. Again all these are possible if A is good, e.g. it is a Fredholm operator. Remark also the involvement of finite-dimensional spaces and the corresponding projections on them.

A third fact occurs in dynamical systems theory and concerns the Axiom A attractors defined by hyperbolic sets, i.e. sets contracting the phase space in some directions and dilating it in some others. The simplest hyperbolic set is the saddle point. As a consequence, the asymptotic dynamics for large time is approaching that one along the unstable manifold W^u of the saddle. In this sense, the asymptotic dynamics reduces to that on W^u and the phase spaces is flattened along W^u. In the case of infinite-dimensional phase spaces if W^u is finite-dimensional, the reduction is drastic: the infinite-dimensional dynamics generated by partial differential equations is reduced to the finite-dimensional dynamics generated by an equation on the manifold W^u. This result is very important for numerical analysis.

All these facts occur in the definition of the inertial manifold and its construction. In addition, the inertial manifold enjoys the property that for every point of the phase space of the given dynamical system there exists a point on this manifold such that the distance between their trajectories decreases exponentially to zero.

We mention also that the Lipschitzianity of the inertial manifold and the fact that it is the fixed point of the mapping defining the dynamics on this manifold occurs in the reduction principle presented as early as 1964 in [10], of course in another context and without defining and emphasizing the importance of the inertial manifold.

Many articles have been devoted to the construction of inertial manifolds. Most of them provide existence results for these manifolds under the restrictive hypothesis of a large gap in the spectrum of A. More exactly, the space E can be split into a direct sum $E = PE \oplus QE$, where P is a spectral projector and $Q = I - P$. Then the inertial manifold is defined as the graph of a Lipschitz function $\Phi : PE \to QE$, determined by the nonlinear term f which must be globally Lipschitz continuous and possesing the Lipschitz constant small if compared with the spectral gap. This condition is called the spectral gap condition. In general, in applications, f is not globally Lipschitz, but the equation possesses an absorbing set, and f can be modified outside the absorbing set to become globally Lipschitz. In this case, the inertial manifold is obtained as the graph of a function $\Phi : B \subset PE \to QE$, where B is an open set.

The existence theories for inertial manifolds are not constructive and even when an inertial manifold is known to exist, it might be not known its representation. This is why useful substitutes to inertial manifolds, namely the approximate inertial manifolds and exponential attractors, were formulated and constructive methods for them were advanced.

An *approximate inertial manifold* (AIM) is a smooth finite dimensional manifold of the phase space which attracts all orbits to a thin neighborhood of it in a finite time uniformly for initial conditions in a given bounded set. This neighborhood contains the global attractor.

The global attractor is contained in any inertial manifold.

For the FitzHugh-Nagumo (FN) model, we truncate the equations to a ball of radius ρ, to obtain the so-called prepared equation. Inside this ball, the dynamics is the same with that of the given equation. For two cases, by means of a numerical method we study the AIM for this model.

2. Jolly-Rosa-Temam algorithm

In [11] and [12] it was developed an algorithm for the computation of an inertial manifold as a limit of a converging sequence of AIMs. This algorithm is based on a variant of the Lyapunov-Perron method. It provides the sequence of AIMs with the dimension of the manifold kept fixed.

2.1. Hypotheses. The assumptions presented below guarantee the existence of an inertial manifold and also the convergence of the algorithm.

Consider the Cauchy problem $u(0) = u_0$ for equation (1).

A1. The nonlinear term f is globally Lipschitz continuous from E into the Banach space F, $E \subset F \subset \mathcal{E}$, the injections being continuous, each space dense in the following one, and $\mathcal{E}$ is a Banach space. It follows that $|f(u)|_F \leq M_0 + M_1|u|_E$, for $M_0 \geq 0$.

A2. The linear operator $-A$ generates a strongly continuous semigroup $\{e^{-tA}\}_{t\geq 0}$ of bounded operators on $\mathcal{E}$ such that $e^{-tA}F \subset E$ for all $t > 0$.

A3. There exist two sequences of numbers $\{\lambda_n\}_{n=n_0}^{n_1}$, $\{\Lambda_n\}_{n=n_0}^{n_1}$, $n_0 \in \mathbb{N}$, $n_1 \in \mathbb{N} \cup \infty$ such that $0 < \lambda_n \leq \Lambda_n$, for all $n_0 \leq n \leq n_1$, and a sequence of finite-dimensional projectors $\{P_n\}_{n=n_0}^{n_1}$ such that $P_n\mathcal{E}$ is invariant under e^{-tA} for $t \geq 0$, and $\{e^{-tA}|P_n\mathcal{E}\}_{t\geq 0}$ can be extended up to a strongly continuous semigroup $\{e^{-tA}P_n\}_{t\in\mathbb{R}}$ of bounded operators on $P_n\mathcal{E}$ with $\|e^{-tA}P_n\|_{\mathcal{L}(E)} \leq K_1 e^{-\lambda_n t}$, $t \leq 0$, $\|e^{-tA}P_n\|_{\mathcal{L}(F,E)} \leq K_1\lambda_n^\alpha e^{-\lambda_n t}$, $t \leq 0$, $Q_n\mathcal{E}$ is positively invariant under the operators e^{-tA} for $t \geq 0$, with $\|e^{-tA}Q_n\|_{\mathcal{L}(E)} \leq K_2 e^{-\Lambda_n t}$, $t \geq 0$, $\|e^{-tA}Q_n\|_{\mathcal{L}(F,E)} \leq K_2(t^{-\alpha} + \Lambda_n^\alpha)e^{-\Lambda_n t}$, $t > 0$, where $K_1, K_2 \geq 1$ and $0 \leq \alpha < 1$.

A4. The equation (1) has a continuous semiflow $\{S(t)\}_{t\geq 0}$ in E.

A5. There exists $K_3 \geq 0$ independent of n such that $\|AP_n\|_{\mathcal{L}(E)} \leq K_3\lambda_n$.

A6. A is invertible.

A7. The spectral gap condition $\Lambda_n - \lambda_n > 3M_1K_1K_2[\lambda_n^\alpha + (1+\gamma_\alpha)\Lambda_n^\alpha]$, holds for some $n \in \mathbb{N}$, where $\gamma_\alpha = \begin{cases} \int_0^\infty e^{-r}r^{-\alpha}dr, & \text{if } 0 < \alpha < 1, \\ 0, & \text{if } \alpha = 0. \end{cases}$

2.2. Approximate inertial manifolds. According to the authors in [11, 12], a single trajectory on an invariant manifold can be found as the fixed point $\varphi = \varphi(p)$ of a mapping $\mathcal{T}(\cdot, p)$. The inertial manifold is the set of such trajectories $\mathcal{M} = graph\Phi$, where $\Phi : PE \to QE$ is defined by $\Phi(p) = Q\varphi(p)(0)$, for all $p \in PE$.

For the sequence of AIMs, the initial guess $\varphi_0(p_0)(t) = p_0$ is considered, for all $p_0 \in P_nE$. By the Picard iteration, $\varphi_j(p_0)$ is obtained [11]. The AIMs are $\mathcal{M}_j = graph\Phi_j$, where $\Phi_j(p_0) = Q_n\varphi_j(p_0)(0)$, for all p_0 in P_nE.

3. FitzHugh-Nagumo system

The Cauchy problem $x(0) = x_0$, $y(0) = y_0$ for the FN system is modelling the evolution of the electrical potential at the sinatrial point of the heart. The system is [13, 14]:

$$\dot{x} = c(x + y - x^3/3), \quad \dot{y} = -(x - a + by)/c, \tag{2}$$

where x, y represents the electrical potential of cell membrane and the excitability respectively, a, b are real parameters depending on the number of channels of the cell membrane which are open for the ions of K^+ and Ca^{++} and $c > 0$ is the relaxation parameter.

As in [15], we modify (2) in order to have it of the form (1) where the operator A is defined by a diagonal matrix and to have the hypotheses A1-A7 satisfied.

The system (2) is of the form (1), where $A = \begin{pmatrix} -c & -c \\ 1/c & b/c \end{pmatrix}$, $\mathbf{f}(x, y) = \begin{pmatrix} -cx^3/3 \\ a/c \end{pmatrix}$. With these, (2) reads $\dot{\mathbf{x}} + A\mathbf{x} = \mathbf{f}(\mathbf{x})$, where $\mathbf{x} = (x, y)$. The eigenvalues of A are $\lambda_1 = \frac{b-c^2-d}{2c}$, $\lambda_2 = \frac{b-c^2+d}{2c}$, while the corresponding eigenvectors read $\mathbf{v}_1 = (1, -\frac{c+\lambda_1}{c})$, $\mathbf{v}_2 = (1, -\frac{c+\lambda_2}{c})$. Here $d = \sqrt{(c^2 + b)^2 - 4c^2}$.

In order to diagonalize A, we make the change of variables $\mathbf{x} = T\mathbf{u}$, where $\mathbf{u} = (u_1, u_2)$ and T is defined by the eigenvectors of A, namely $T = \begin{pmatrix} 1 & 1 \\ -\frac{c+\lambda_1}{c} & -\frac{c+\lambda_2}{c} \end{pmatrix}$. Denote $B = T^{-1}AT$ and $\mathbf{g}(\mathbf{u}) = T^{-1}\mathbf{f}(T\mathbf{u})$ to obtain the modified FitzHugh-Nagumo system

$$\dot{\mathbf{u}} + B\mathbf{u} = \mathbf{g}(\mathbf{u}), \tag{3}$$

which will be studied further, where the operator B is defined by the diagonal matrix $\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$, and

$$\mathbf{g}(\mathbf{u}) = \begin{pmatrix} -\frac{(c^2+b+d)(u_1+u_2)^3c}{6d} + \frac{ca}{d} \\ \frac{(c^2+b-d)(u_1+u_2)^3c}{6d} - \frac{ca}{d} \end{pmatrix} = \begin{pmatrix} -\frac{c^2}{3d}(c + \lambda_2)(u_1 + u_2)^3 + \frac{ca}{d} \\ -\frac{c^2}{3d}(c + \lambda_1)(u_1 + u_2)^3 - \frac{ca}{d} \end{pmatrix}.$$

The eigenvectors of B are $\mathbf{w}_1 = \mathbf{e}_1$ and $\mathbf{w}_2 = \mathbf{e}_2$, where $(\mathbf{e}_1, \mathbf{e}_2)$ is the canonical basis of $\mathbb{R}^2$.

3.1. The inertial form. In the case of the FN model we have $E = \mathbb{R}^2$. Consider the projectors $P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $Q = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, such that PE and QE are the vector spaces spanned by $\mathbf{e}_1$ and $\mathbf{e}_2$ respectively. Therefore PE and QE are the Ox-axis and Oy-axis respectively. Denote $\mathbf{p} = P\mathbf{x}$ and $\mathbf{q} = Q\mathbf{x}$. It follows that $\mathbf{p}$ and $\mathbf{q}$ have the forms $\mathbf{p} = (p_1, 0)$, $\mathbf{q} = (0, q_2)$. Then, the system (2), in projection on Ox and Oy-axis, becomes

$$\dot{p}_1 - cp_1 - cq_2 = -c\frac{p_1^3}{3}, \tag{4}$$

$$\dot{q}_2 + \frac{1}{c}p_1 + \frac{b}{c}q_2 = \frac{a}{c}. \tag{5}$$

For any given $q_2(0)$, the equation (5) has the unique solution

$$q_2(t) = q_2(0)e^{-\frac{b}{c}t} + \frac{ac}{b}(1 - e^{-\frac{b}{c}t}) - \frac{1}{c}\int_0^t p_1(\tau)d\tau.$$

Using it in (4), we obtain the inertial form

$$\dot{p_1} - cp_1 - cq_2(0)e^{-\frac{b}{c}t} + \frac{ac}{b}(1 - e^{-\frac{b}{c}t}) - \frac{1}{c}\int_0^t p_1(\tau)d\tau = -c\frac{p_1^3}{3}.$$

It is an integro-differential equation with cubic nonlinearities and coefficients depending on time at an exponential rate and, also, on the initial value $q_2(0)$.

Remark that $(Q\mathbf{u})(t)$ is the difference at time t between two points of the trajectories from E and PE respectively, if at $t = 0$ this difference was $q_2(0)$.

Due to the particular form of the eigenvectors w_1 and w_2 of B, denoting by the subscript 0 the initial value, we have $\mathbf{u}_0 = \mathbf{p}_0 + \mathbf{q}_0 = (u_{10}, u_{20}) = (p_{10}, 0) + (0, q_{20})$. Therefore $p_{10} = u_{10}$ and $q_{20} = u_{20}$, which means that $\mathbf{p}_0$ belongs to the Ou_1-axis and $\mathbf{q}_0$ to the Ou_2-axis.

For the form (2) of the FN system we do not succeded to fulfill A1-A7., but we were successful in the case of system (3), in spite of its more complicated form. Thus, the equations in projections of (3) read

$$\dot{u}_1 = -\lambda_1 u_1 - \frac{c^2(c+\lambda_2)}{3d}(u_1+u_2)^3 + \frac{ac}{d}, \tag{6}$$

$$\dot{u}_2 = -\lambda_2 u_2 + \frac{c^2(c+\lambda_1)}{3d}(u_1+u_2)^3 - \frac{ac}{d}. \tag{7}$$

Eliminating the third order terms between (6) and (7) and multiplying the obtained equation by $e^{\lambda_2 t}$ we get

$$(c+\lambda_2)\left(u_2 e^{\lambda_2 t}\right)^{\cdot} = -(c+\lambda_1)\left(u_1 e^{\lambda_1 t}\right)^{\cdot} e^{(-\lambda_1+\lambda_2)t} - ae^{\lambda_2 t} = 0,$$

leading, by integration over $[0, t]$, to the expression

$$u_2(t) = u_{20}e^{-\lambda_2 t} + u_{10}\frac{c+\lambda_1}{c+\lambda_2}e^{-\lambda_2 t} - \frac{c+\lambda_1}{c+\lambda_2}u_1(t) -$$

$$\frac{c+\lambda_1}{c+\lambda_2}(\lambda_1-\lambda_2)e^{-\lambda_2 t}\int_0^t u_1(\tau)e^{\lambda_2 t}d\tau + \frac{a}{\lambda_2(c+\lambda_2)}e^{-\lambda_2 t} - \frac{a}{\lambda_2(c+\lambda_2)}.$$

showing that $u_2(t)$ is an affine function of $u_1(\tau)$. Introducing it into (6), the inertial form is obtained.

3.2. Phase portraits and absorbing domains. For the values $a = 0.01$, $b = 5$ and $c = 1$ of the parameters, there exist three equilibria:
$(x^{(1)}, y^{(1)}) = (-1.5479418, 0.3115884)$, corresponding to the eigenvalues $\lambda_1 = -1.6990686$ and $\lambda_2 = -4.6970552$, hence $(x^{(1)}, y^{(1)})$ is an attractive node;
$(x^{(2)}, y^{(2)}) = (1.5504418, -0.3080884)$, with negative eigenvalues $\lambda_1 = -1.7075995$ and $\lambda_2 = -4.6962702$, hence $(x^{(2)}, y^{(2)})$ is also an attractive node;
$(x^{(3)}, y^{(3)}) = (-0.0025, 0.0025)$, which corresponds to the eigenvalues $\lambda_1 = 0.8284207 > 0$ and $\lambda_2 = -4.8284269 < 0$, so $(x^{(3)}, y^{(3)})$ is a saddle with the stable manifold close to

the Oy-axis and the unstable one close to the Ox-axis. The phase portrait for these values of the parameters is represented in fig. 1.

The two branches of the unstable manifold unite the saddle and one node, while the stable manifold separate the basins of attraction of the two nodes. In order to apply the Jolly-Rosa-Temam algorithm we choose as phase space the basin of attraction for each node. For this space, each node becomes the global attractor.

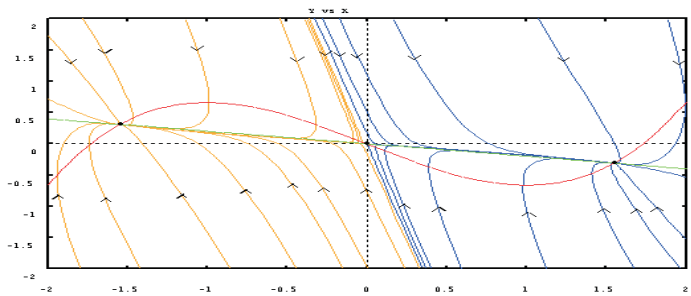

Fig. 1. Phase portrait for the FN model, for a=0.01, b=5, c=1.

For a=0.01, b=0.9, c=0.1, the dynamical system associated to the FN model has only one equilibrium $(\bar{x}, \bar{y}) = (0.0972415, -0.096935)$.
Indeed, in this case, the equilibrium equation reads as $x^3 + px + q = 0$, where $p = \frac{3}{b} - 3 = 0.3333333$ and $q = -\frac{3a}{b} = -0.3333333$. Let us denote $D = (\frac{p}{3})^3 + (\frac{q}{2})^2$ $P = \sqrt[3]{-\frac{q}{2} + \sqrt{D}} = 0.3854814$ and $Q = \sqrt[3]{-\frac{q}{2} - \sqrt{D}} = -0.2882399$. We find $D = 0.0016495 > 0$, so, this equation has only one real root, namely $\bar{x} = P + Q \approx 0.0972415$. Then $\bar{y} = (a - x)/b = -0.096935$ follows. The eigenvalues are $\lambda_1 = -0.0122086 < 0$ are $\lambda_2 = -8.88874 < 0$, therefore $(\bar{x}, \bar{y})$ is an attractive node. The phase portrait for these values of the parameters is represented in fig. 2.

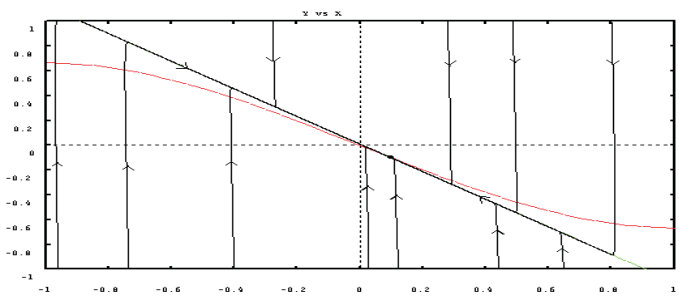

Fig. 2. Phase portrait for the FN model, for a=0.01, b=0.9, c=0.1.

There is an absorbing domain, namely the disk of radius ρ, centered at $(\bar{x}, \bar{y})$.

The phase portraits were drawn by using the software WINPP, created by Professor B. Ermentrout from Pittsburg University for numerical simulations of the dynamics and bifurcations [16].

3.3. Phase portraits for the FitzHugh-Nagumo system. Two linearly independent eigenvectors of B are $\mathbf{w}_1 = (1,0) = \mathbf{e}_1$ and $\mathbf{w}_2 = (0,1) == \mathbf{e}_2$.

Let us represent phase portraits for the system (3) for the values of the parameters from Section 3.2.

For $a = 0.01$, $b = 5$ and $c = 1$, this system becomes

$$\begin{cases} \dot{u_1} = 0.8284271250 \cdot u_1 - 0.3434433618 \cdot (u_1 + u_2)^3 + 0.001767766952, \\ \dot{u2} = -4.828427122 \cdot u_2 + 0.01011002867 \cdot (u_1 + u_2)^3 - 0.001767766952. \end{cases} \tag{8}$$

Its phase portrait is represented in fig. 3.

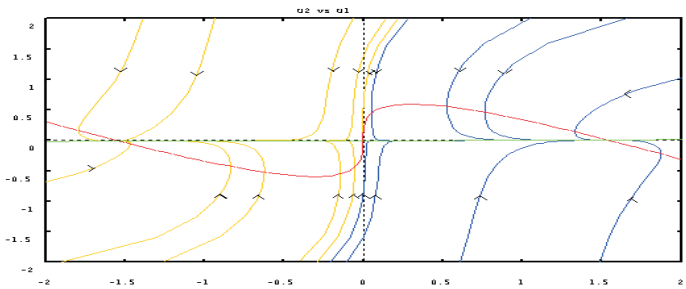

Fig. 3. Phase portrait for the system (3), for a=0.01, b=5, c=1.

The system (8) has three eqilibria, corresponding to three eqilibria of the initial system: $(u_1^{(1)}, u_2^{(1)}) = (-1.53981, -0.00813235)$, with eigenvalues $\lambda_1 = -1.69651$ and $\lambda_2 = -4.69716$, therefore $(u_1^{(1)}, u_2^{(1)})$ is an attractive node;
$(u_1^{(2)}, u_2^{(2)}) = (1.543, 0.00743781)$, with the eigenvalues $\lambda_1 = -1.71018$ and $\lambda_2 = -4.69616$, therefore $(u_1^{(2)}, u_2^{(2)})$ is an attractive node;
$(u_1^{(3)}, u_2^{(3)}) = (-0.00213389, -0.000366117)$, with the eigenvalues $\lambda_1 = 0.828421 > 0$ and $\lambda_2 = -4.82843 < 0$, therefore $(u_1^{(3)}, u_2^{(3)})$ is a saddle.

For $a = 0.01$, $b = 0.9$ and $c = 0.1$, the system becomes

$$\begin{cases} \dot{u_1} = -0.01125017617 \cdot u_1 - 0.03375105672 \cdot (u_1 + u_2)^3 + 0.001126443300, \\ \dot{u2} = -8.888749825 \cdot u_2 + 0.0004177233847 \cdot (u_1 + u_2)^3 - 0.001126443300. \end{cases} \tag{9}$$

Phase portrait for this system is represented in fig. 4.

The system (9) has only one echilibrium $(\bar{u_1}, \bar{u_2}) = (0.892699, -0.00376404)$, with the eigenvalues $\lambda_1 = -0.0913503 < 0$ and $\lambda_2 = -8.8875 < 0$, therefore $(\bar{u_1}, \bar{u_2})$ is an attractive node.

3.4. Hypotheses of the algorithm for the modified FitzHugh-Nagumo model. In [15] we prove that this model satisfies all hypotheses of the Jolly-Rosa-Temam algorithm.

Thus, the Lipschitz constant for each component of $\mathbf{g} = (g_1, g_2)$ is computed:
$|g_1(\mathbf{u}) - g_1(\mathbf{v})| \leq c|\frac{c^2+b+d}{d}| \cdot 6r^2\|\mathbf{u} - \mathbf{v}\|$, $|g_2(\mathbf{u}) - g_2(\mathbf{v})| \leq c|\frac{c^2+b-d}{d}| \cdot 6r^2\|\mathbf{u} - \mathbf{v}\|$.
We conclude that the first condition $\|\mathbf{g}(\mathbf{u}) - \mathbf{g}(\mathbf{v})\| \leq M_r\|\mathbf{u} - \mathbf{v}\|$ is satisfied, where $M_r = \frac{6cr^2}{d}\max\{|c^2 + b + d|, |c^2 + b - d|\}$.
By direct computation, the following inequality is obtained

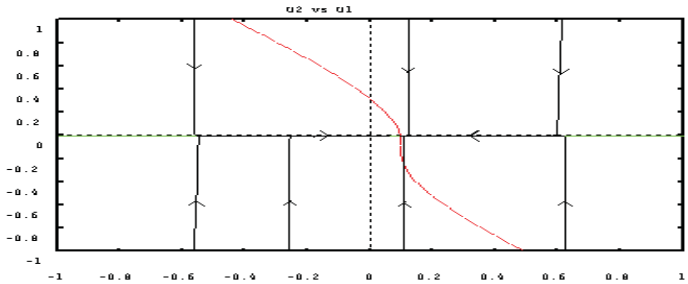

Fig. 4. Phase portrait for system (3), for a=0.01, b=0.9, c=0.1.

$\|\mathbf{g}(\mathbf{u})\| \leq \frac{1}{d}[4cr^3 \max\{|c^2+b+d|, |c^2+b-d|\} + c|a|]$.
The prepared equation is

$$\frac{d\mathbf{u}}{dt} + B\mathbf{u} = \mathbf{g}_\rho(\mathbf{u}),$$

where $\mathbf{g}_\rho(\mathbf{u}) = \chi_\rho(r)\mathbf{g}(\mathbf{u})$, $\chi_\rho(r) = \chi(\frac{r^2}{\rho^2})$, $\chi \in \mathcal{C}^1(\mathbb{R}_+)$, $\chi/_{[0,1]} = 1, \chi/_{[2,\infty)} = 0, 0 \leq \chi(s) \leq 1, \forall s \in [1,2]$. Thus, we obtain $\|\mathbf{g}_\rho(\mathbf{u}) - \mathbf{g}_\rho(\mathbf{v})\| \leq M_\rho \|\mathbf{u} - \mathbf{v}\|$, where

$$M_\rho = \frac{\max\{|c^2+b+d|, |c^2+b-d|\}}{d} \cdot (48\sqrt{2}+12)c\rho^2 + \frac{6c|a|}{\rho d}. \tag{10}$$

Hence, the prepared equation satisfies the first condition.

For the third assumption, let us choose the projectors $P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $Q = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.
We have $\|e^{-tB}P\| = e^{-\lambda_1 t}$ and $\|e^{-tB}Q\| = e^{-\lambda_2 t}$. In order to satisfy the conditions A3, we have to choose $0 < \lambda_n \leq \Lambda_n$. According to the eigenvalues of A are distinguish three cases.
For $0 < \lambda_1 \leq \lambda_2$, we obtain
$\|e^{-tB}P\| = e^{-\lambda_1 t} \leq 1e^{-\lambda_1 t}$, $\forall t \leq 0$, $\|e^{-tB}Q\| = e^{-\lambda_2 t} \leq 1e^{-\lambda_2 t}$, $\forall t \geq 0$. Therefore, we can choose $\lambda_n = \lambda_1$, $\Lambda_n = \lambda_2$, $K_1 = 1, K_2 = 1$ and $\alpha = 0$.
For $\lambda_1 \leq 0 < \lambda_2$,
$\|e^{-tB}P\| = e^{-\lambda_1 t} \leq e^0 < 1e^{-10^{-1}t}$, $\forall t \leq 0$, $\|e^{-tB}Q\| = e^{-\lambda_2 t} \leq 1e^{-\lambda_2 t}$, $\forall t \geq 0$. For $\lambda_n = 10^{-1}$, $\Lambda_n = \lambda_2$, $K_1 = 1, K_2 = 1$ and $\alpha = 0$, we have A3 satisfied if $\lambda_2 \geq 10^{-1}$.
Finally, for $\lambda_1 < \lambda_2 \leq 0$, we can not have the conditions A3 satisfied, thus, the algorithm can not be applied.

For the fifth condition, there is obtained $\|BP\| = |\lambda_1|$. In the first case, $\lambda_1 > 0$, hence $\|BP\| = \lambda_1$, $\lambda_n = \lambda_1$, and $K_3 = 1$. In the second case $\lambda_1 < 0$ and we must have $\|BP\| = -\lambda_1 \leq K_3\lambda_n$, where $\lambda_n = \frac{1}{10}$. In conclusion, there exists $K_3 \geq 0$ independent of n such that $\|BP\| \leq K_3\lambda_n$, for λ_n defined as above, and, so, condition A5 holds.

The seventh condition (the spectral gap condition) reads
$\Lambda_n - \lambda_n > 3M_\rho K_1 K_2[\lambda_n^\alpha + (1+\gamma_\alpha)\Lambda_n^\alpha]$. For $\alpha = 0$, we have $\gamma_\alpha = 0$, then, this condition becomes

$$\Lambda_n - \lambda_n > 6M_\rho, \tag{11}$$

with M_ρ defined in (10).

3.5. AIMs for the prepared equation. As a phase space we will choose, the absorbing basin for each node, like in Section 3.3, namely disks of radius ρ for $a = 0.01, b = 5, c = 1$ and $\mathbb{R}^2$ for $a = 0.01, b = 0.9$ and $c = 0.1$.

Using a program implemented by the first author in Scilab [17], let us construct the AIMs. These manifolds are collections of trajectories $\mathcal{M}_j = \text{graph}\Phi_j$, where $\Phi_j : P\mathbb{R}^2 \to Q\mathbb{R}^2$, i.e. $\Phi_j : \mathbb{R} \to \mathbb{R}$, $\Phi_j(p_0) = Q\varphi^j(p_0)(0)$.

Let us choose $\tau_j = c_1^{-j}$ and $N_j = \frac{c_2}{c_1} j 2^j$, where $c_1 = c_2 = 0.01$.

Although, for $a = 0.01, b = 5, c = 1$, all hypotheses are satisfied for all three equilibria, our interest concerns only the two attractive nodes. Thus, all further computations will be done only for these two points.

For the attractive node $(1.543, 0.00743781)$, the eigenvalues of B are: $\lambda_1 = 1.71018$ and $\lambda_2 = 4.69616$. For the third condition, we are in the first case, thus $\lambda_n = \lambda_1$ and $\Lambda_n = \lambda_2$. For the seventh condition, let us choose $\rho = 1/20$, and, then $M_\rho = 0.1931370848$, therefore (11) reads: $2.98598 > 1.1588$, which is obviously satisfied.

Since $\rho = 1/20$, we choose the initial point $(1.5, 0)$, belonging to the absorbing domain intersected to Ox, i.e. to the space $P\mathbb{R}^2$. The corresponding initial conditions for the given system (2) are $(1.5, -4.5 + 3\sqrt{2})$.

In fig. 5, we represent graphically $Q\varphi^j$ as a function of time, with the initial conditions $u_1(0) = 1.5$, $u_2(0) = 0$, for 7 iterations.

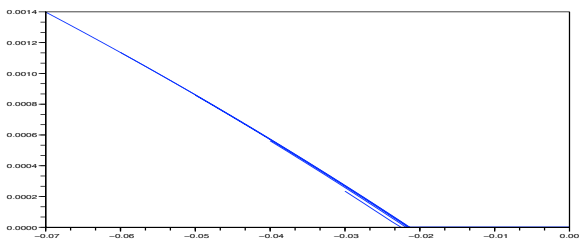

Fig. 5. Graphical representation, for the FN model, of $q_2 = |Qu|$ on the AIM as a function of t, for 7 iterations (the graph of the last function is the upper curve) for a=0.01, b=5, c=1; $u_1(0) = 1.5$, $u_2(0) = 0$.

The second attractive node is $(-1.53981, -0.00813235)$ and the eigenvalues of B are: $\lambda_1 = -1.69651$ and $\lambda_2 = 4.69716$. We are also in the first case, therefore $\lambda_n = \lambda_1$ and $\Lambda_n = \lambda_2$. For the seventh condition, let $\rho = 1/20$, thus $M_\rho = 0.1931370848$ and (11) is satisfied because: $3.00065 > 1.1588$.

Since $\rho = 1/20$, the initial point can be $(-1.5, 0)$, belonging to the absorbing domain of the second node. and also on the Ox axis. The corresponding initial conditions for (2) are $(-1.5, 4.5 - 3\sqrt{2})$.

In fig. 6, there are the graphical representations for the same parameters, but for the initial conditions $u_1(0) = -1.5, u_2(0) = 0$, for 7 iterations.

For $a = 0.01, b = 0.9$ and $c = 0.1$, the eigenvalues of B are positive (first case), $\lambda_n = \lambda_1 = 0.0913503$ and $\Lambda_n = \lambda_2 = 8.8875$. Let $\rho = 1/3$, then $M_\rho = 0.1931370848$, and the spectral gap condition is satisfied (it reads: $8.7961497 > 4.1551$). The graphical

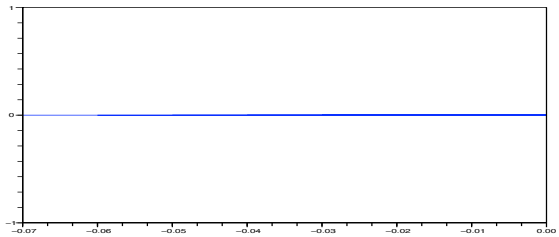

Fig. 6. Graphical representation, for the FN model, of $q_2 = |Qu|$ on the AIM as a function of t, for 7 iterations for a=0.01, b=5, c=1; $u_1(0) = -1.5$, $u_2(0) = 0$.

representation for $u_1(0) = 1$, $u_2(0) = 0$ are to be found in fig. 7. The corresponding initial conditions for (2) are $(1, -89.8874982)$.

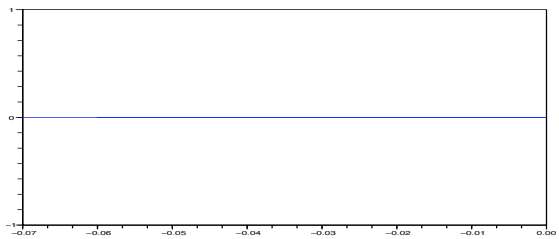

Fig. 7. Graphical representation, for the FN model, of $q_2 = |Qu|$ on the AIM as a function of t, for 7 iterations for a=0.01, b=0.9, c=0.1; $u_1(0) = 1$, $u_2(0) = 0$.

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