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COHEN MACAULAY FIBRES OF A MORPHISM

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ABSTRACT. We present the main results about morphisms whose fibres are Cohen - Macaulay and about rings having Cohen-Macaulay formal fibers. We give new proofs for the localization theorems for morphisms with Cohen-Macaulay or (S_n) fibres and we point out some open questions in this direction.

1. Introduction

The study of the fibres of a ring morphism was initiated by Grothendieck [8]. For a given property **P** of noetherian local rings (e.g. regular, normal, reduced, Cohen-Macaulay etc.) he defined the notion of a **P**-morphism and of a **P**-ring. Of course, the property **P** must fulfill some initial conditions, like for example stability to taking fractions, a good behaviour to flat morphism, etc. The theory developed by Grothendieck was very fruitful, the main notion that came out from it being the notion of excellent ring, notion corresponding to the property **P**=regular. The aim of the present paper is to present the results and problems concerning the notion derived from Grothendieck's work corresponding to the property **P**=Cohen-Macaulay and the connected property **P**=(S_n), which was introduced by Serre.

All rings considered will be commutative and with unit. All morphisms between local rings are supposed to be local morphisms. A noetherian local ring with maximal ideal $\mathfrak{m}$ and residue field k, will be denoted by $(A, \mathfrak{m}, k)$. If A is a ring and $\mathfrak{p}$ is a prime ideal of A, we denote by $k(\mathfrak{p})$ the residue field of the local ring $A_{\mathfrak{p}}$. We generally use the notations from [14], [6] and [8].

2. The properties Cohen Macaulay and (S_n)

We start by reminding the definitions and main properties of the Cohen-Macaulay and (S_n) rings.

Definition 2.1. A noetherian local ring A is called a Cohen-Macaulay ring if $\mathrm{depth(A)} = \mathrm{dim(A)}$.

Definition 2.2. Let n be a natural number. A noetherian local ring A is said to have the property (S_n), if for any prime ideal $\mathfrak{p}$ of A, $\mathrm{depth(A_{\mathfrak{p}})} \geq \min\{\mathrm{n}, \mathrm{ht}(\mathfrak{p})\}$.

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Remark **2.3**. It is well-known and obvious to prove that a ring A is Cohen-Macaulay if and only if A has the property (S_n), for all natural numbers n.

Remark **2.4**. The property (S_0) is trivially verified in any noetherian ring. The property (S_1) means that the ideal (0) of A has not embedded primes, while the property (S_2) means that not only (0), but also any principal ideal of A generated by a non-zero-divisor has not embedded primes.

Remark **2.5**. It is well-known that any reduced ring has the property (S_1) and that any normal ring has the property (S_2).

The following two results are summarizing the main properties of Cohen-Macaulay and (S_n) rings.

Proposition 2.6. a) Any regular local ring is Cohen-Macaulay and consequently has also the property (S_n), for any natural number n.

b) If A is a Cohen-Macaulay ring and $\mathfrak{p}$ is a prime ideal of A, then $A_{\mathfrak{p}}$ is a Cohen-Macaulay ring.

c) If A has the property (S_n) and $\mathfrak{p}$ is a prime ideal of A, then $A_{\mathfrak{p}}$ has the property (S_n).

d) Let $u : A \to B$ be a flat morphism of noetherian local rings. If B is Cohen-Macaulay, then A is also Cohen-Macaulay.

e) Let $u : A \to B$ be a flat morphism of noetherian local rings. If B has the property (S_n), then A has also the property (S_n).

f) Let $u : A \to B$ be a flat morphism of noetherian local rings. If A and the closed fibre of u are Cohen-Macaulay, then B is also Cohen-Macaulay.

g) Let $u : A \to B$ be a flat morphism of noetherian local rings. If A and all the fibres of u have the property (S_n), then B has also the property (S_n).

h) Let A be a complete local ring and n a natural number. Then the sets $\mathrm{CM}(A) = \{ \mathfrak{p} \in \mathrm{Spec(A)} \mid \mathrm{A_{\mathfrak{p}}}$ is Cohen-Macaulay$\}$ and $S_n(A) = \{ \mathfrak{p} \in \mathrm{Spec(A)} \mid \mathrm{A_{\mathfrak{p}}}$ has the property $(S_n)\}$ are open in the Zariski topology of $\mathrm{Spec(A)}$.

i) Let $(A, \mathfrak{m}, k)$ be a local ring and $x \in \mathfrak{m}$ be a regular element. If A/xA is Cohen-Macaulay, then A is also Cohen-Macaulay.

j) Let $(A, \mathfrak{m}, k)$ be a local ring and $x \in \mathfrak{m}$ be a regular element. If A/xA has the property (S_n), then A has also the property (S_n).

Proof. The proofs of the assertions a) - i) can be found in [14]. As for j), it is also known, but we give a proof due to the lack of reference. Let $\mathfrak{q} \in \mathrm{Spec(A)}$, such that

$$\mathrm{depth(A_{\mathfrak{q}}) = n \leq k.}$$

If $x \in \mathfrak{q}$, then

$$\mathrm{depth(A/xA)_{\mathfrak{q}} = n - 1 < k,}$$

so by assumption we have $\mathrm{ht}(\mathfrak{q}/\mathrm{xA}) = \mathrm{n} - 1$ and it follows that $\mathrm{ht}(\mathfrak{q}) = \mathrm{n}$. If $x \notin \mathfrak{q}$, let us take $\mathfrak{p} \in \mathrm{Min}(\mathfrak{q} + \mathrm{xA})$. Then the ideal $(\mathfrak{q} + xA)A_{\mathfrak{q}}$ is $\mathfrak{q}A_{\mathfrak{q}}$-primary and

$$\mathrm{depth(A_{\mathfrak{p}}) \leq depth(A_{\mathfrak{q}}) + 1 = n + 1.}$$

Since $\mathrm{depth((A/xA)_{\mathfrak{q}}) \leq n}$, it follows that $\mathrm{ht}(\mathfrak{p}/\mathrm{xA}) \leq \mathrm{n}$ and we obtain $\mathrm{ht}(\mathfrak{p}) \leq \mathrm{n} + 1$. As x is a regular element, $\mathrm{ht}(\mathfrak{q}) \leq \mathrm{n}$. □

Remark **2.7**. It is also well-known that if $u : A \to B$ is a flat morphism of noetherian local rings and if A and the closed fibre of u have the property (S_n), then B does not necessarily have the property (S_n).

The following known lemma shows another important connection between the properties Cohen Macaulay and (S_n).

Lemma 2.8. *Let A be a local ring. Then A has the property (S_n) if and only if for any* $\mathfrak{p} \in \mathrm{Spec(A)}$ *such that* $\mathrm{depth(A_{\mathfrak{p}})} \leq \mathrm{n} - 1, \mathrm{A_{\mathfrak{p}}}$ *is Cohen-Macaulay.*

Proof. Suppose that A has the property (S_n) and let $\mathfrak{p} \in \mathrm{Spec(A)}$ such that $\mathrm{depth(A_{\mathfrak{p}})} \leq \mathrm{n} - 1$. Then obviously $\mathrm{depth(A_{\mathfrak{p}})} \geq \mathrm{ht}(\mathfrak{p})$, that is $A_{\mathfrak{p}}$ is Cohen Macaulay. Conversely, if $\mathrm{depth(A_{\mathfrak{p}})} \leq \mathrm{n} - 1$, by assumption $A_{\mathfrak{p}}$ is Cohen Macaulay and consequently

$$\mathrm{depth(A_{\mathfrak{p}})} = \mathrm{ht}(\mathfrak{p}) = \min\{\mathrm{n}, \mathrm{ht}(\mathfrak{p})\}.$$

If $\mathrm{depth(A_{\mathfrak{p}})} \geq \mathrm{n}$, then clearly $\mathrm{depth(A_{\mathfrak{p}})} \geq \min\{\mathrm{n}, \mathrm{ht}(\mathfrak{p})\}$. Finally it follows that A has the property (S_n). □

3. Cohen Macaulay and (S_n) fibres

We begin by noting the following lemma.

Lemma 3.1. [10] *Let k be a field, A a k-algebra and K a field, finite extension of k.*
a) *If A has the property (S_n), then $K \otimes_k A$ has the property (S_n).*
b) *If A is a Cohen-Macaulay ring, then $K \otimes_k A$ is a Cohen-Macaulay ring.*

Definition 3.2. Let n be a natural number and k a field. A k-algebra A is called geometrically (S_n), if for any field K, finite extension of k, the noetherian ring $K \otimes_k A$ has the property (S_n).

Definition 3.3. Let n be a natural number and k a field. A k-algebra A is called geometrically Cohen-Macaulay, if for any field K, finite extension of k, the noetherian ring $K \otimes_k A$ is Cohen-Macaulay.

Definition 3.4. Let n be a natural number. A morphism of noetherian rings $u : A \to B$ is called an (S_n) morphism if:
a) u is flat;
b) for any prime ideal $\mathfrak{p} \in \mathrm{Spec(A)}$, the ring $B \otimes_A k(\mathfrak{p})$ is a geometrically (S_n) $k(\mathfrak{p})-$ algebra.

Definition 3.5. A morphism of noetherian rings $u : A \to B$ is called a Cohen-Macaulay morphism if:
a) u is flat;
b) for any prime ideal $\mathfrak{p} \in \mathrm{Spec(A)}$, the ring $B \otimes_A k(\mathfrak{p})$ is a geometrically Cohen-Macaulay $k(\mathfrak{p})-$algebra.

Definition 3.6. Let n be a natural number. A noetherian ring A has geometrically (S_n) formal fibers if for any $\mathfrak{p} \in \mathrm{Spec(A)}$, the canonical completion morphism $A_{\mathfrak{p}} \to \widehat{A_{\mathfrak{p}}}$ is a (S_n) morphism.

Definition 3.7. A noetherian ring A has geometrically Cohen-Macaulay formal fibers if for any $\mathfrak{p} \in \mathrm{Spec(A)}$, the canonical completion morphism $A_{\mathfrak{p}} \to \widehat{A_{\mathfrak{p}}}$ is a Cohen-Macaulay morphism.

Remark **3.8**. From **3.1** it follows that the notions of *geometrically* (S_n) (resp. *geometrically Cohen-Macaulay*) and (S_n) (resp. *Cohen-Macaulay*) in the above definitions are equivalent. We used the term *geometrically* in order to keep the initial definitions of Grothendieck [8].

Example 3.9. a) Clearly any ring with Cohen-Macaulay formal fibres has also (S_n) formal fibres, for any natural number n.

b) Any complete local ring has Cohen-Macaulay formal fibres.

c) If A is a Cohen Macaulay ring and $\mathfrak{a}$ is an ideal of A, then $A/\mathfrak{a}$ has Cohen-Macaulay formal fibres [1].

In [8] Grothendieck posed two problems about rings with good formal fibres. In **3.10** and **3.11** let **P** be one of the properties Cohen Macaulay or (S_n). We shall consider these two problems only for the Cohen-Macaulay and (S_n) property, even if the original formulation of Grothendieck was for a general property **P** of noetherian local rings.

Problem 3.10. (The localization problem [8])

Consider $u : (A, \mathfrak{m}, k) \to (B, \mathfrak{n}, K)$ a local and flat morphism of noetherian local rings. Suppose that:
i) A is a **P**-ring;
ii) $u \otimes k : k \to B \otimes_A k$ is a **P**-morphism.
Does it follow that u is a **P**-morphism?

Problem 3.11. (The lifting problem for **P**-rings [8])

Let A be a noetherian ring, $\mathfrak{a}$ an ideal of A. Suppose that:
i) $A/\mathfrak{a}$ is a **P**-ring;
ii) A is complete in the $\mathfrak{a}-$adic topology.
Does it follow that A is a **P**-ring?

The problem **3.11** is connected with the following problem.

Problem 3.12. (The stability to completion of **P**-rings [8])

Let A be a noetherian ring, $\mathfrak{a}$ an ideal of A and let B be the completion of A in the $\mathfrak{a}-$adic topology. Suppose that A is a **P**-ring.
Does it follow that B is a **P**-ring?

Remark **3.13**. a) The original formulation of these problems by Grothendieck was for a general property **P** of noetherian local rings. We shall deal only with the properties Cohen-Macaulay and (S_n).

b) Consider the assumptions in problem **3.10**. The condition ii) means that the fiber at the closed point of u is a **P**-morphism. The question is if this imply that all the fibers of u are **P**-morphisms, in other words, does the property of being a geometrically **P** algebra, *localizes* from the fiber of a morphism at the closed point to all the fibers of the morphism?

c) It is clear that a positive answer to the problem **3.11** implies also a positive answer to the problem **3.12**.

d) It is obvious that Problem **3.12** is equivalent to the folowing: if A is a noetherian ring **P**-ring, does it follow that $A[[X]]$ is also a **P**-ring?

In the rest of the paper we shall deal with the status of these problems.

4. The localization problem for Cohen-Macaulay and (S_n) morphisms

The localization problem for Cohen-Macauly and (S_n) have a positive answer. We shall present a proof of this.

Theorem 4.1. *Let $u : (A, \mathfrak{m}, k) \to (B, \mathfrak{n}, K)$ be a local flat morphism of noetherian local rings. Suppose that:*
1) *A has geometrically Cohen-Macaulay formal fibers;*
2) *$B/\mathfrak{m}B$ is Cohen-Macaulay.*
Then u is a Cohen-Macaulay morphism.

Proof. Since $B/\mathfrak{m}B$ is Cohen-Macaulay, it follows that it has Cohen-Macaulay formal fibers. Hence, replacing the morphism $A \to B$ by the morphism $\widehat{A} \to \widehat{B}$, where $\widehat{A}$ is the completion of A in the $\mathfrak{m}-$adic topology and $\widehat{B}$ is the completion of B in the $\mathfrak{n}$-adic topology, we can suppose that A and B are complete local rings. Moreover, it is easily seen that we can assume that A is a domain and that we have only to prove that $B \otimes_A Q(A)$ is Cohen-Macaulay. Let $X := \mathrm{Spec(B)},\ \mathrm{Y := Spec(A)},\ \mathrm{f := Spec(u)}$. By Kawasaki's Macaulayficaton [13], it follows that there is a Cohen-Macaulay scheme Y' and a proper and birational morphism $g : Y' \to Y$. Let $X' := X \times_Y Y'$ and let $f' : X' \to Y'$ and $g' : X' \to X$ be the canonical morphisms. Denote by $a := \mathfrak{n} \in X$. It suffices to prove that X' is a Cohen-Macaulay scheme. Indeed, as g is proper and birational, there exists an open set $U \subseteq Y$, such that $g^{-1}(U) \cong U$ and then $f^{-1}(U) \cong f'^{-1}(g^{-1}(U))$. This proves that $f^{-1}(U)$ is a Cohen-Macaulay scheme and that the fibre of f in the maximal point of Y is contained in $f^{-1}(U)$. In order to prove that X' is Cohen-Macaulay, it is sufficient to prove that $g'^{-1}(a) \subseteq \mathrm{CM}(X')$. Indeed, since B is a complete local ring, $\mathrm{CM}(X')$ is open. Then there exists an open neighbourhood V of a, such that $g^{-1}(V) \subseteq \mathrm{CM}(X')$. But since B is local $V = X$. Let

$$x' \in g'^{-1}(a),\ z' := f'(x'),\ b := g(z').$$

Then $b = f(a)$. Let also

$$O := O_{Y',z'},\ O' := O_{X',x'}.$$

It follows that O' is a localization of $B \otimes_A O$, hence $O'/\mathfrak{m}_{z'}O'$ is a localization of $(B/\mathfrak{n}) \otimes_k \mathbf{k}(z')$. Since the field extension $k \subseteq \mathbf{k}(z')$ is of finite type, from the assumption 2) it results that $O'/\mathfrak{m}_{z'}O'$ is Cohen-Macaulay. Because the morphism $O \to O'$ is flat, we get that O' is Cohen-Macaulay. □

Remark **4.2**. The first complete proof of the theorem **4.1** was given by Avramov and Foxby [1], using the Cohen factorization of a morphism [3]. The above proof was given in [4], in the case $\dim(A) \leq 4$, because at the time [4] was written, the Macaulayfication [13] was not proved in the full generality, only Falting's partial result [9] was available. Another proof, in the more general case of morphisms of finite flat dimension, was given also by Avramov and Foxby[1]. The paper [2] contains also generalizations of **4.1**, as well as a very interesting treatment of the study of the fibers of a morphism, in a more general frame that the original one of Grothencieck.

From **4.1** we can obtain at once the localization property for (S_n).

Theorem 4.3. *Let* $u : (A, \mathfrak{m}, k) \to (B, \mathfrak{n}, K)$ *be a local flat morphism of noetherian local rings, n a natural number. Suppose that:*
1) *A has geometrically* (S_n) *formal fibers;*
2) $B/\mathfrak{m}B$ *is a geometrically* (S_n) *k-algebra.*
Then u is a (S_n) *morphism.*

Proof. As above we can assume that A is a complete domain, that all non-zero fibers of u are geometrically (S_n) and we have only to prove that $B \otimes_A Q(A)$ has the property (S_n). Let $\mathfrak{P} \in \mathrm{Spec(B)}$ such that $\mathfrak{P} \cap A = (0)$ and suppose that $\mathrm{depth(B_{\mathfrak{P}})} < \mathrm{n}$. Let also $x \in \mathfrak{m}, x \neq 0$, let $\mathfrak{Q} \in \mathrm{Min}(\mathfrak{P} + \mathrm{xB})$ and put $\mathfrak{q} := \mathfrak{Q} \cap A$. Then $\mathrm{depth(B_{\mathfrak{Q}})} \leq \mathrm{n} + 1$, hence

$$\mathrm{depth(B_{\mathfrak{Q}}/\mathfrak{q}B_{\mathfrak{Q}}) = depth(B_{\mathfrak{Q}}) - depth(A_{\mathfrak{q}}) < n.}$$

It follows that the morphism $k(\mathfrak{q}) \to B_{\mathfrak{Q}}/\mathfrak{q}B_{\mathfrak{Q}}$ is geometrically Cohen-Macaulay. By **4.1** it folows that $A_{\mathfrak{q}} \to B_{\mathfrak{Q}}$ is Cohen-Macaulay, hence the morphism $Q(A) \to B_{\mathfrak{P}}$ is Cohen-Macaulay. Consequently, $B_{\mathfrak{P}}$ is a Cohen-Macaulay ring, and we apply **2.8**. □

Remark **4.4**. The fact that **4.3** can be deduced from **4.1**, was also observed for the first time in [4]. In a more general frame, the same was observed in [7].

For $n = 1$ we can give a direct proof, that does not depend on **4.1**, patterned on the proof given in [12] for reduced morphisms.

Theorem 4.5. *Let* $u : (A, \mathfrak{m}, k) \to (B, \mathfrak{n}, K)$ *be a local flat morphism of noetherian local rings. Suppose that:*
1) *A has geometrically* (S_1) *formal fibers;*
2) $B/\mathfrak{m}B$ *is a geometrically* (S_1) *k-algebra.*
Then u is a (S_1) *morphism.*

Proof. Let B^* be the $\mathfrak{m}B-$adic completion of B. We have the canonical commutative diagram:

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "f"'] & B \arrow[d, "g"] \\ \widehat{A} \arrow[r, "v"] & B^* \end{tikzcd}$$

Since A has geometrically (S_1) formal fibers, it follows that f is an (S_1) morphism. If we prove that v is an (S_1) morphism, it will follow that u is an (S_1) morphism. Hence we can suppose that A is complete. By noetherian induction, we can suppose that A is a domain, that for any $\mathfrak{q} \in \mathrm{Spec(A)},\ \mathfrak{q} \neq 0,\ \mathrm{B \otimes_A k(\mathfrak{q})}$ is a geometrically (S_1) $k(\mathfrak{q})-$algebra and we have to show that the fiber of u in 0 is geometrically (S_1), or what is the same that it is (S_1). Replacing A with its normalization, we can also suppose that A is normal. Finally we can assume that A is a local normal domain having geometricaly regular formal fibers and we have only to show that $B \otimes_A Q(A)$ has the property (S_1). For this, take a prime ideal $\mathfrak{P} \in \mathrm{Spec(B)}$ such that $\mathfrak{P} \cap A = 0$ and $\mathrm{depth(B_{\mathfrak{P}})} = 0$. Let $x \in A$ be a non-zero and non-invertible element and take $\mathfrak{Q} \in \mathrm{Min}(\mathfrak{P} + \mathrm{xB})$. Then $\mathrm{depth(B_{\mathfrak{Q}})} = 1$, because by flatness x is also B-regular. Let $\mathfrak{q} := \mathfrak{Q} \cap A$. Then clearly we have $\mathrm{depth(A_{\mathfrak{q}})} = 1$. Consequently,

$$\mathrm{depth(B_{\mathfrak{Q}}/\mathfrak{q}B_{\mathfrak{Q}})} = \mathrm{depth(B_{\mathfrak{Q}})} - \mathrm{depth(A_{\mathfrak{q}})} = 0.$$

Since the fibre of u in $\mathfrak{Q}$ is (S_1), from **2.8** it follows that $B_{\mathfrak{Q}}/\mathfrak{q}B_{\mathfrak{Q}}$ is Cohen-Macaulay. But as $\mathrm{depth(A_{\mathfrak{q}})} = 1$ and A is normal, it follows that $A_{\mathfrak{q}}$ is regular. Hence $B_{\mathfrak{Q}}$ is Cohen-Macaulay and consequently also $B_{\mathfrak{P}}$ is Cohen-Macaulay, being a localization of $B_{\mathfrak{Q}}$. Now we apply again **2.8**. □

5. Lifting properties and stability to completion

In the case of a semilocal ring the lifting property got a satisfactory answer [4]:

Theorem 5.1. [4] *Let A be a semilocal Nagata ring and let $\mathfrak{a}$ be an ideal of A. Suppose that:*
1) *A is complete in the $\mathfrak{a}$-adic topology;*
2) *$A/\mathfrak{a}$ has Cohen-Macaulay formal fibres.*
Then A has Cohen-Macaulay formal fibres.

Theorem 5.2. [4] *Let A be a semilocal Nagata ring, n a natural number and $\mathfrak{a}$ be an ideal of A. Suppose that:*
1) *A is complete in the $\mathfrak{a}$-adic topology;*
2) *$A/\mathfrak{a}$ has (S_n) formal fibres.*
Then A has (S_n) formal fibres.

Remark **5.3**. The proof of **5.1** and **5.2** in [4] was given in a more general frame, which allowed to derive the same conclusions also for other properties of noetherian local rings. The proof was patterned on the proof of Rotthaus [16], on the completion of semilocal quasi-excellent rings.

Question 5.4. Are the Theorems **5.1** and **5.2** true without the assumption that A is a Nagata ring?

For studying the lifting property in the general case we need some preparations.

Notation 5.5. For a noetherian ring A and a natural number n, we shall use the following notations:
Reg(A):= $\{\mathfrak{p} \in \mathrm{Spec(A)} \mid \mathrm{A_{\mathfrak{p}}}$ is regular$\}$, the regular locus of A;
Nor(A):= $\{\mathfrak{p} \in \mathrm{Spec(A)} \mid \mathrm{A_{\mathfrak{p}}}$ is normal$\}$, the normal locus of A.

Definition 5.6. A noetherian ring A is called CM-2, if for any $A-$algebra of finite type B, CM(B) is open in the Zariski topology of $\mathrm{Spec(B)}$.

Definition 5.7. A noetherian ring A is called Reg-2, if for any A-algebra of finite type B, Reg(B) is open in the Zariski topology of $\mathrm{Spec(B)}$.

Definition 5.8. A noetherian ring A is called Nor-2, if for any A-algebra of finite type B, Nor(B) is open in the Zariski topology of $\mathrm{Spec(B)}$.

Definition 5.9. A noetherian ring A is called (S_n)-2, if for any A-algebra of finite type B, $S_n(B)$ is open in the Zariski topology of $\mathrm{Spec(B)}$.

With these notations we can state the following result about the lifting of rings with Cohen-Macaulay formal fibres.

Theorem 5.10. [4],[18] *Let A be a noetherian ring and $\mathfrak{a}$ be an ideal of A. Suppose that:*
1) *A is complete in the $\mathfrak{a}$-adic topology;*
2) *$A/\mathfrak{a}$ has Cohen-Macaulay formal fibres;*
3) *$A_{\mathfrak{m}}$ is* CM-2*, for any maximal ideal $\mathfrak{m}$ of A.*
Then A has Cohen-Macaulay formal fibres.

As a corollary one can immediately obtain the positive answer for the stability to completion of rings with Cohen-Macaulay formal fibres.

Corollary 5.11. [4] Let A be a noetherian ring and $\mathfrak{a}$ be an ideal of A. If A has Cohen-Macaulay formal fibres, then the $\mathfrak{a}$-adic completion of A has also Cohen-Macaulay formal fibres.

Corollary 5.12. Let A be a noetherian ring with Cohen-Macaulay formal fibres. Then $A[[X]]$ has also Cohen-Macaulay formal fibres.

For the case of the property (S_n) we can obtain only the following, somewhat weaker, result:

Theorem 5.13. [11] *Let A be a noetherian ring, n a natural number and $\mathfrak{a}$ be an ideal of A. Suppose that:*
1) *A is complete in the $\mathfrak{a}$-adic topology;*
2) *$A/\mathfrak{a}$ has (S_n) formal fibres;*
3) *$A_{\mathfrak{m}}$ is* CM-2*, for any maximal ideal $\mathfrak{m}$ of A.*
Then A has (S_n) formal fibres.

Question 5.14. Is **5.13** true if we suppose only that $A_{\mathfrak{m}}$ is (S_n)-2 and not necessarily CM-2, for any maximal ideal $\mathfrak{m}$ of A ?

Regarding the general case of the lifting problem, in the case of rings with geometrically regular or geometrically normal formal fibers there are more general results.

Theorem 5.15. [17],[15] *Let A be a noetherian ring and $\mathfrak{a}$ be an ideal of A. Suppose that:*
1) *A is complete in the $\mathfrak{a}$-adic topology;*
2) *$A/\mathfrak{a}$ has geometrically regular formal fibres and is* Reg-2*;*
3) *A contains a field of characteristic 0.*
Then A has geometrically regular formal fibres and is Reg-2.

Theorem 5.16. [5],[15] *Let A be a noetherian ring and $\mathfrak{a}$ be an ideal of A. Suppose that:*
1) *A is complete in the $\mathfrak{a}$-adic topology;*
2) *$A/\mathfrak{a}$ has geometrically normal formal fibres and is* Nor-2*;*
Then A has geometrically normal formal fibres and is Nor-2.

Remark **5.17**. The proof of **5.15** uses Hironaka's resolution of singularities, that's why the condition on the characteristic of the ring was needed. Instead of the resolution of singularities, in the proof of **5.16** the normalization of a domain was used, thus allowing the result in full generality.

Taking account of **5.17** and of the results of [13] on Macaulayfication, the following question might have a positive answer:

Question 5.18. Let A be a noetherian ring and $\mathfrak{a}$ be an ideal of A. Suppose that:
1) A is complete in the $\mathfrak{a}$-adic topology;
2) $A/\mathfrak{a}$ has Cohen-Macaulay formal fibres and is CM-2.
Does it follow that A has Cohen-Macaulay formal fibres and is CM-2?

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