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TOPOLOGICAL CHARACTERIZATIONS OF $\mathcal{S}$-LINEARITY

DAVID CARFÌ

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ABSTRACT. We give several characterizations of basic concepts of $\mathcal{S}$-linear algebra in terms of weak duality on topological vector spaces. On the way, some classic results of Functional Analysis are reinterpreted in terms of $\mathcal{S}$-linear algebra, by an application-oriented fashion. The results are required in the $\mathcal{S}$-linear algebra formulation of infinite dimensional Decision Theory and in the study of abstract evolution equations in economical and physical Theories.

1. Intoduction

The paper contains several and various results. In section 4 we show some relations among the $\mathcal{S}$-linear hull and the $\beta(\mathcal{S}'_n, \mathcal{S}_n)$ closed linear hull of an $\mathcal{S}$-family, ${}^{\mathcal{S}}span$ is characterized. In section 5 the fundamental concept of $\mathcal{S}$-linear independence is characterized; $\mathcal{S}$-linear independence is the main tool to prove the uniqueness of solution of abstract Cauchy problems of evolution. In section 6, $\mathcal{S}$-bases are characterized. In section 7 and 8, $\mathcal{S}$-linear operators are introduced and characterized: they are the core of the infinite dimensional formulation of Decisions Theory and, moreover, of the linear evolution of infinite dimensional economical and physical systems. In section 9, the new concept of $\mathcal{S}$-closed set are introduced and studied.

2. Preliminaries and notations on tempered distributions

In this paper we shall use some notations. The letters n, m, k are natural numbers, $\mathbb{N}(\leq k)$ is the set of positive integer lower than or equal to k; μ_n is the Lebesgue measure on $\mathbb{R}^n$; $\mathbb{I}_{(\mathbb{R},\mathbb{C})}$ is the immersion of $\mathbb{R}$ in $\mathbb{C}$; if X is a non-empty set, $\mathbb{I}_X$ is the identity map on X. If X and Y are two topological vector spaces on $\mathbb{K}$, $\mathrm{Hom}(\mathrm{X}, \mathrm{Y})$ is the set of all the linear operators from X to Y, $\mathcal{L}(X, Y)$ is the set of all the linear and continuous operators from X to Y, $X^* := \mathrm{Hom}\,(\mathrm{X}, \mathbb{K})$ is the algebraic dual of X and $X' = \mathcal{L}\,(X, \mathbb{K})$ is the topological dual of X. $\mathcal{S}_n := \mathcal{S}(\mathbb{R}^n, \mathbb{K})$ is the $(n, \mathbb{K})$-*Schwartz space*, that is to say, the set of all the smooth functions (i.e., of class C^∞) of $\mathbb{R}^n$ in $\mathbb{K}$ *rapidly decreasing at infinity with all their derivatives* (the functions and all its derivatives tend to 0 at $\mp\infty$ faster than the reciprocal of any polynomial); $\mathcal{S}_{(n)}$ is the *standard Schwartz topology* on $\mathcal{S}_n$, and $(\mathcal{S}_n)$ is the topological vector space on $\mathcal{S}_n$ with its standard topology; the topology $\mathcal{S}_{(n)}$ is generated by a metric, in fact ($\mathcal{S}_n$ is closed under differentiation and multiplication by polynomials) by the family of seminorms (p_k) on $\mathcal{S}_n$ defined by

$$p_k(f) = \sup_{x\in\mathbb{R}^n} \max_{\alpha,\beta\in\mathbb{N}_0^n} \left\{ \left| x^\beta D^\alpha f(x) \right| : 0 \le |\alpha|, |\beta| \le k \right\},$$

for every non-negative integer k; each p_k is a norm on $\mathcal{S}_n$, and $p_k(f) \le p_{k+1}(f)$ for all $f \in \mathcal{S}_n$, the pair $\left(\mathcal{S}_n, (p_k)_{k\in\mathbb{N}_0}\right)$ is a countably complete normed space and consequentely $(\mathcal{S}_n)$ is a Frèchet space (see also $[Ho]$ and $[Ba]$); $\mathcal{S}'_n := \mathcal{S}'(\mathbb{R}^n, \mathbb{K})$ is the space of tempered distributions from $\mathbb{R}^n$ to $\mathbb{K}$, that is, the topological dual of the topological vector space $(\mathcal{S}_n, \mathcal{S}_{(n)})$, i.e., $\mathcal{S}'_n$=$(\mathcal{S}_n, \mathcal{S}_{(n)})'$; if $x \in \mathbb{R}^n$, δ_x is the *distribution of Dirac on $\mathcal{S}_n$ centered at x*, i.e., the functional $\delta_x : \mathcal{S}_n \to \mathbb{K} : \phi \mapsto \phi(x)$; if $f \in \mathcal{O}_M(\mathbb{R}^n, \mathbb{K})$, where

$$\mathcal{O}_M(\mathbb{R}^n, \mathbb{K}) = \{g \in C^\infty(\mathbb{R}^n, \mathbb{K}) : \forall \phi \in \mathcal{S}_n(\mathbb{K}), \phi g \in \mathcal{S}_n(\mathbb{K})\},$$

then the functional

$$[f] = [f]_n : \mathcal{S}_n \to \mathbb{K} : \phi \mapsto \int_{\mathbb{R}^n} f\phi d\mu_n$$

is a tempered distribution, called the *regular distribution generated by f* (see [1] page 110, [2], [3], [4]).

Let $a, b \in \mathbb{R}^{\neq} = \mathbb{R}\backslash\{0\}$, $\mathcal{S}_{(a,b)}$ is the (a, b) *-Fourier-Schwartz transformation*, i.e., the operator $\mathcal{S}_{(a,b)} : \mathcal{S}_n \to \mathcal{S}_n$, defined, for all $f \in \mathcal{S}_n$ and $\xi \in \mathbb{R}^n$, by

$$\mathcal{S}_{(a,b)}(f)(\xi) = \left(\frac{1}{a}\right)^n \int_{\mathbb{R}^n} f e^{-ib(\cdot|\xi)} d\mu_n = \left[\left(\frac{1}{a}\right)^n e^{-ib(\cdot|\xi)}\right](f),$$

where $(\cdot \mid \cdot)$ is the standard scalar product on $\mathbb{R}^n$. Moreover, we recall that $\mathcal{S}_{(a,b)}$ is a homeomorphism with respect to the standard topology $\mathcal{S}_{(n)}$ and, concerning its inverse, for every $x \in \mathbb{R}^n$ and $g \in \mathcal{S}_n$,one has

$$\mathcal{S}^-_{(a,b)}(g)(x) = \left(\frac{|b|\, a}{2\pi}\right)^n \int_{\mathbb{R}^n} g e^{ib(x|\cdot)} d\mu_n = \mathcal{S}_{(2\pi/(|b|a),-b)}(g)(x).$$

Let $a, b \in \mathbb{R}^{\neq} = \mathbb{R}\backslash\{0\}$, $\mathcal{F}_{(a,b)}$ denotes the (a, b)*-Fourier transformation on the space of tempered distributions*, i.e., the operator $\mathcal{F}_{(a,b)} : \mathcal{S}'_n \to \mathcal{S}'_n$, defined, for all $u \in \mathcal{S}'_n$ and for every $\phi \in \mathcal{S}_n$, by

$$\mathcal{F}_{(a,b)}(u)(\phi) = u\left(\mathcal{S}_{(a,b)}(\phi)\right),$$

in other terms it is the transpose of $\mathcal{S}_{(a,b)}$:

$$\mathcal{F}_{(a,b)} = {}^t\left(\mathcal{S}_{(a,b)}\right).$$

Moreover, we recall that $\mathcal{F}_{(a,b)}$ is a homeomorphism in the weak* topology $\sigma\left(\mathcal{S}'_n, \mathcal{S}_n\right)$ (even more it is a topological isomorphism). Moreover, one has

$$\mathcal{F}^-_{(a,b)} =_{(2\pi/(|b|a),-b)} \mathcal{F}.$$

Two properties that we shall use are the following ones: for all $\alpha \in \mathbb{N}_0^n$,

$$\mathcal{F}_{(a,b)}(u^{(\alpha)}) = (bi)^\alpha \left(\mathbb{I}_{\mathbb{R}^n}\right)^\alpha \mathcal{F}_{(a,b)}(u);$$

and

$$\mathcal{F}_{(a,b)}((\mathbb{I}_{\mathbb{R}^n})^\alpha u) = \left(\frac{i}{b}\right)^\alpha \left(\mathcal{F}_{(a,b)}(u)\right)^{(\alpha)},$$

where, $\mathbb{I}_{\mathbb{R}^n}$ is (as we said) the identity operator on $\mathbb{R}^n$, and where $(\mathbb{I}_{\mathbb{R}^n})^\alpha$ is the α-th power of the identity in multi-indexed notation.

3. Preliminaries and notations on $\mathcal{S}$-linear algebra

Let I be a non-empty set. We denote by $s(I, \mathcal{S}'_n)$ the space of all the families in $\mathcal{S}'_n$ indexed by I, i.e., the set of all the surjective maps from I onto a subset of $\mathcal{S}'_n$. Moreover, as usual, if v is one of these families, for each $p \in I$, the distribution $v(p)$ is denoted by v_p, and the family v itself is also denoted by $(v_p)_{p\in I}$. The set $s(I, \mathcal{S}'_n)$ is a vector space with respect to the standard operations of addition $+ : s(I, \mathcal{S}'_n)^2 \to s(I, \mathcal{S}'_n)$ defined by $v+w := (v_p + w_p)_{p\in I}$, and multiplication by scalars $\cdot : \mathbb{K}\times s(I, \mathcal{S}'_n) \to s(I, \mathcal{S}'_n)$ defined by $\lambda v := (\lambda v_p)_{p\in I}$. In other words, the family $v + w$ is defined by $(v + w)_p = v_p + w_p$, for every p in I, and the family λv is defined by $(\lambda v)_p = \lambda v_p$, for every p in I. In the theory of superpositions on $\mathcal{S}'_n$ the class of the $\mathcal{S}$-families plays a basic role.

Let v be a family in $\mathcal{S}'_n$ indexed by $\mathbb{R}^m$. The family v is called ***family of class*** $\mathcal{S}$ *or* $\mathcal{S}$-***family*** *if, for each $\phi \in \mathcal{S}_n$, the function $v(\phi) : \mathbb{R}^m \to \mathbb{K}$, defined by $v(\phi)(p) := v_p(\phi)$, for each $p \in \mathbb{R}^m$, belongs to the space $\mathcal{S}_m$. We denote the set of all these families by $\mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$. Let $v \in \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$ be a family of class $\mathcal{S}$. We call* ***operator generated by the family*** *v (or* ***associated with*** *v) the operator $\widehat{v} : \mathcal{S}_n \to \mathcal{S}_m : \phi \mapsto v(\phi)$.*

In the following we shall denote by $\mathcal{L}(\mathcal{S}_n, \mathcal{S}_m)$ the set of all the linear and continuous operators among the two topological vector spaces $(\mathcal{S}_n)$ and $(\mathcal{S}_m)$. Moreover, consider a linear operator $A : \mathcal{S}_n \to \mathcal{S}_m$, we say that A is *transposable* if its algebraic adjoint ${}^*A : \mathcal{S}^*_m \to \mathcal{S}^*_n$ (X^* denote the algebraic adjoint of X), defined by ${}^*A(a) = a \circ A$, maps $\mathcal{S}'_m$ into $\mathcal{S}'_n$.

Let $v \in \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$ be a family of tempered distributions. Then, the following assertions hold and are equivalent:

i) for every $a \in \mathcal{S}'_m$ the composition $u = a \circ \widehat{v}$, i.e., the functional

$$u : \mathcal{S}_n \to \mathbb{K} : \phi \mapsto a(\widehat{v}(\phi)),$$

is a tempered distribution;

ii) $\widehat{v}$ is transposable;

iii) $\widehat{v}$ is $(\sigma(\mathcal{S}_n, \mathcal{S}'_n), \sigma(\mathcal{S}_m, \mathcal{S}'_m))$-continuous from $\mathcal{S}_n$ to $\mathcal{S}_m$;

iv) $\widehat{v}$ is strongly-continuous from $(\mathcal{S}_n)$ to $(\mathcal{S}_m)$.

The two vector spaces $\mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$ and $\mathcal{L}(\mathcal{S}_n, \mathcal{S}_m)$ are isomorphic, being the map

$$(\cdot)^\wedge : \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n) \to \mathcal{L}(\mathcal{S}_n, \mathcal{S}_m) : v \mapsto \widehat{v}$$

an isomorphism, moreover, its inverse is the map

$$(\cdot)^\vee : \mathcal{L}(\mathcal{S}_n, \mathcal{S}_m) \to \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n) : A \mapsto A^\vee := (\delta_x \circ A)_{x\in\mathbb{R}^m}.$$

Let $v \in \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$ and $a \in \mathcal{S}'_m$. The distribution $a \circ \widehat{v} = {}^t(\widehat{v})(a)$ is called ***the*** $\mathcal{S}$-***linear superposition of*** *v* ***with respect to (the system of coefficients)*** *a and we denote it by*

$$\int_{\mathbb{R}^m} av.$$

Moreover, if $u \in \mathcal{S}'_n$ *and there exists an* $a \in \mathcal{S}'_m$ *such that*

$$u = \int_{\mathbb{R}^m} av,$$

u *is said an* $\mathcal{S}$***-linear superposition of*** v.

As a particular case, we can consider the linear superposition of v with respect to the regular distribution generated by the $\mathbb{K}$ -constant functional on $\mathbb{R}^m$ of value 1, the distribution $\left[1_{(\mathbb{R}^m,\mathbb{K})}\right]$, we denote it simply by $\int_{\mathbb{R}^m} v$, and then

$$\int_{\mathbb{R}^m} v := \int_{\mathbb{R}^m} \left[1_{(\mathbb{R}^m,\mathbb{K})}\right] v.$$

An alternative definition of superposition can be obtained defining the superposition of a family of numbers (real or complex) with respect to a distributional system of coefficients.

We say that a family of real or complex number $x = (x_i)_{i\in\mathbb{R}^m}$ *is a family of class* $\mathcal{S}$ *if the function* $f_x : \mathbb{R}^m \to \mathbb{K}$*, defined by* $f_x(i) = x_i$*, for each* i *in* $\mathbb{R}^m$*, is a function of class* $\mathcal{S}$*. We call* f_x *the test function associated with the family* x*.*

In this conditions, we put

$$\int_{\mathbb{R}^m} a\, x := a(f_x),$$

for every tempered distribution $a \in \mathcal{S}'_m$ *, and we call the number* $\int_{\mathbb{R}^m} ax$ *superposition of the family* x *with respect to* a*.*

Introducing a notation, the relation between the two kind of superpositions is very natural.

Notation. Let $\langle \cdot, \cdot \rangle$ be the canonical bilinear form on $\mathcal{S}'_n \times \mathcal{S}_n$ and let v be an $\mathcal{S}$-family of tempered distributions in $\mathcal{S}'_n$ indexed by $\mathbb{R}^m$. For every test function $\phi \in \mathcal{S}_n$ by the symbol $\langle v, \phi\rangle$ we denote the family of numbers defined by

$$\langle v, \phi\rangle_i := \langle v_i, \phi\rangle ,$$

for every i in $\mathbb{R}^m$.

Let v *be an* $\mathcal{S}$*-family of tempered distributions in* $\mathcal{S}'_n$ *indexed by* $\mathbb{R}^m$*, let* a *be a tempered distribution in* $\mathcal{S}'_m$ *and let* $\langle \cdot, \cdot \rangle$ *be the canonical bilinear form on* $\mathcal{S}'_n \times \mathcal{S}_n$*. Then, for every* $\phi \in \mathcal{S}_n$*, we have*

$$\left\langle \int_{\mathbb{R}^m} av, \phi \right\rangle = \int_{\mathbb{R}^m} a \langle v, \phi\rangle .$$

We shall see that the preceding result can be restated saying that the canonical bilinear form on $\mathcal{S}'_n \times \mathcal{S}_n$ is $\mathcal{S}$-linear in the first argument.

The operators

$$\int_{\mathbb{R}^m} (\cdot, \cdot) : \mathcal{S}'_m \times \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n) \to \mathcal{S}'_n : (a, v) \mapsto \int_{\mathbb{R}^m} av,$$

and

$$\int_{\mathbb{R}^m} (\cdot, v) : \mathcal{S}'_m \to \mathcal{S}'_n : a \mapsto \int_{\mathbb{R}^m} av,$$

are called the superposition operator in $\mathcal{S}'_n$ with coefficients-systems in $\mathcal{S}'_m$ and the superposition operator associated to v.

4. $\mathcal{S}$-linear hull

Definition 4.1 (of $\mathcal{S}$-linear hull). *Let $v \in \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$. The $\mathcal{S}$-**linear hull** of v is the set*

$$^{\mathcal{S}}\mathrm{span}\,(v) := {}^t\widehat{v}\,(\mathcal{S}'_m) = \left\{ u \in \mathcal{S}'_n : \exists a \in \mathcal{S}'_m : u = \int_{\mathbb{R}^m} av \right\}. \ \square$$

Example 4.2 (on the Dirac and Fourier families). Let δ be the Dirac family, then $^{\mathcal{S}}\mathrm{span}\,(\delta) = \mathcal{S}'_{\mathrm{n}}$. In fact, for all $u \in \mathcal{S}'_n$, one has

$$u = u \circ \mathbb{I}_{\mathcal{S}_n} = u \circ \widehat{\delta} = \int_{\mathbb{R}^n} u\delta.$$

Let $\varphi = \left(\left[(1/a)^n e^{-ib(p|\cdot)}\right]\right)$ be the Fourier family, we have $^{\mathcal{S}}\mathrm{span}\,(\varphi) = \mathcal{S}'_{\mathrm{n}}$, as follows from the Fourier expansion theorem. $\triangle$

Definition 4.2 (system of $\mathcal{S}$-generators). *Let $v \in \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$. The family v is called system of $\mathcal{S}$-**generators** for a set $V \subseteq \mathcal{S}'_n$ if and only if $^{\mathcal{S}}\mathrm{span}(v) = V$.* $\square$

Example 4.2. The Dirac family and the Fourier families are systems of $\mathcal{S}$-generators for $\mathcal{S}'(\mathbb{R}^n, \mathbb{C})$. $\triangle$

Theorem 4.1 (on the structure of $^{\mathcal{S}}\mathrm{span}$). *Let $u \in \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$. Then, $^{\mathcal{S}}\mathrm{span}\,(u)$ is a subspace of $\mathcal{S}'_n$, it contains all the elements of u and consequently*

$$\mathrm{span}\,(u) \subseteq {}^{\mathcal{S}}\,\mathrm{span}\,(u)\,.$$

Proof. Let $\lambda \in \mathbb{K}$ and $v, w \in {}^{\mathcal{S}}\,\mathrm{span}\,(u)$, then, there exist $a, b \in \mathcal{S}'_m$ such that

$$v = \int_{\mathbb{R}^m} au, \ w = \int_{\mathbb{R}^m} bu.$$

Now, one has

$$\lambda v + w = \lambda \int_{\mathbb{R}^m} au + \int_{\mathbb{R}^m} bu = \int_{\mathbb{R}^m} (\lambda a + b)\, u,$$

and then $\lambda v + w \in {}^{\mathcal{S}}\mathrm{span}\,(u)$. Moreover, let δ be the Dirac basis of $\mathcal{S}'_m$, we have

$$\int_{\mathbb{R}^m} \delta_p u = u_p$$

and then $u_p \in {}^{\mathcal{S}}\mathrm{span}\,(u)$. ∎

Let us see the relation among the $\mathcal{S}$-linear hull of an $\mathcal{S}$-family and its $\beta(\mathcal{S}'_n, \mathcal{S}_n)$-closed linear hull. Note that, since $\mathcal{S}_n$ is reflexive, it is semireflexive and then the linear subspaces of $\mathcal{S}'_n$ are $\beta(\mathcal{S}'_n, \mathcal{S}_n)$-closed if and only if they are $\sigma(\mathcal{S}'_n, \mathcal{S}_n)$-closed, so the $\beta(\mathcal{S}'_n, \mathcal{S}_n)$-closed linear hull of a subset coincides with the $\sigma(\mathcal{S}'_n, \mathcal{S}_n)$-closed hull of the same set.

Theorem 4.2. *Let $v \in \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$ be a system of $\mathcal{S}$ -generators for $\mathcal{S}'_n$. Then,*

$$\overline{\mathrm{span}}_{\beta(\mathcal{S}'_n, \mathcal{S}_n)}\,(v) = \mathcal{S}'_n.$$

Proof. To prove that v is $\beta(\mathcal{S}'_n, \mathcal{S}_n)$-dense in $\mathcal{S}'_n$, we shall prove that every linear $\beta(\mathcal{S}'_n, \mathcal{S}_n)$-continuous form on $\mathcal{S}'_n$, that is zero on the family v, is zero on the whole $\mathcal{S}'_n$. In fact, let f be such a form, since $\mathcal{S}'_n$ is reflexive there is a test function g in $\mathcal{S}_n$ such that $f(u) = u(g)$, for every distribution u in $\mathcal{S}'_n$. Since f is zero on v, we have

$$0 = f(v_i) = v_i(g) = v(g)(i) = \widehat{v}(g)(i),$$

and hence $\widehat{v}(g)$ is the origin of $\mathcal{S}_m$. Now, v $\mathcal{S}$-generates $\mathcal{S}'_n$ if and only if ${}^t\widehat{v}$ is surjective and thus, by the Schwartz-Dieudonné theorem on Frechét spaces (see [5], theorem 7, pg. 92), the operator $\widehat{v}$ is injective, so g is the origin of $\mathcal{S}_n$, and then f is the origin of $\mathcal{S}''_n$. ■

This result can be generalized.

Theorem 4.3. *Let $v \in \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$ be an $\mathcal{S}$-family in $\mathcal{S}'_n$. Then,*

$$^{\mathcal{S}}\mathrm{span}\,(v) \subseteq \overline{\mathrm{span}}_{\beta(\mathcal{S}'_n,\mathcal{S}_n)}\,(v) = \overline{\mathrm{span}}_{\sigma(\mathcal{S}'_n,\mathcal{S}_n)}\,(v)\,.$$

Proof. We shall prove that every superposition of v is the $\beta(\mathcal{S}'_n, \mathcal{S}_n)$-limit of a sequence of finite linear combinations of the family v. Let a be in $\mathcal{S}_m$, then a is the $\beta(\mathcal{S}'_m, \mathcal{S}_m)$-limit of a sequence d of finite combinations of the Dirac family of $\mathcal{S}'_m$, since the Dirac family is $\beta(\mathcal{S}'_m, \mathcal{S}_m)$-dense in $\mathcal{S}'_m$. We have

$$\int_{\mathbb{R}^m} av = \int_{\mathbb{R}^m} \left({}^{\beta(\mathcal{S}'_m,\mathcal{S}_m)} \lim_{k\to\infty} d_k \right) v = {}^{\beta(\mathcal{S}'_n,\mathcal{S}_n)} \lim_{k\to\infty} \int_{\mathbb{R}^m} d_k v,$$

by the $(\beta(\mathcal{S}'_m, \mathcal{S}_m), \beta(\mathcal{S}'_n, \mathcal{S}_n))$-continuity of ${}^t\widehat{v}$. Moreover, by the selection property of the Dirac's distributions, the superposition $\int_{\mathbb{R}^m} d_n v$ is a finite combination of the family v, and this concludes the proof. ■

The following theorem shows when $^{\mathcal{S}}\mathrm{span}\,(v)$ is $\sigma(\mathcal{S}'_n, \mathcal{S}_n)$-closed.

Theorem 4.4. *Let $v \in \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$ be an $\mathcal{S}$-family in $\mathcal{S}'_n$. Then, the following conditions are equivalent:*
1) $^{\mathcal{S}}\mathrm{span}\,(v)$ is $\sigma(\mathcal{S}'_n, \mathcal{S}_n)$-closed in $\mathcal{S}'_n$, i.e., $\beta(\mathcal{S}'_n, \mathcal{S}_n)$-closed;
2) $^{\mathcal{S}}\mathrm{span}\,(v) = \overline{\mathrm{span}}_{\sigma(\mathcal{S}'_n,\mathcal{S}_n)}\,(v)$;
3) $\int_{\mathbb{R}^m}(\cdot, v)$ is a topological homomorphism for $\sigma(\mathcal{S}'_m, \mathcal{S}_m)$ and $\sigma(\mathcal{S}'_n, \mathcal{S}_n)$;
4) $\widehat{v}\,(\mathcal{S}_n)$ is closed in the space $(\mathcal{S}_m)$;
5) $\widehat{v}$ is a topological homomorphism for the pair of topologies $(\sigma(\mathcal{S}_n, \mathcal{S}'_n), \sigma(\mathcal{S}_m, \mathcal{S}'_m))$;
6) $\widehat{v}$ is a topological homomorphism from $(\mathcal{S}_n)$ to $(\mathcal{S}_m)$.

Proof. It is the Dieudonné-Schwartz theorem (see [5] , theorem 7, pg. 92) reread in our context ($(\mathcal{S}_n)$ and $(\mathcal{S}_m)$ are two Frèchet spaces), taking into account the preceding theorem. ■

Remark. If $^{\mathcal{S}}\mathrm{span}\,(v)$ is $\sigma(\mathcal{S}'_n, \mathcal{S}_n)$-closed, then $\int_{\mathbb{R}^m}(\cdot, v)$ is a topological homomorphism for $\beta(\mathcal{S}'_m, \mathcal{S}_m)$ and $\beta(\mathcal{S}'_n, \mathcal{S}_n)$ (by proposition 18, page 309, of $[Ho]$).

Theorem 4.5. *Let $v \in \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$. Then the following assertions are equivalent*
1) v is a system of $\mathcal{S}$-generators for $\mathcal{S}'_n$;
2) $\int_{\mathbb{R}^m}(\cdot, v)$ is a surjective topological homomorphism for $\sigma(\mathcal{S}'_m, \mathcal{S}_m)$ and $\sigma(\mathcal{S}'_n, \mathcal{S}_n)$;

3) $\int_{\mathbb{R}^m}(\cdot, v)$ *is a surjective topological homomorphism for* $\beta(\mathcal{S}'_m, \mathcal{S}_m)$ *and* $\beta(\mathcal{S}'_n, \mathcal{S}_n)$*;*
4) $\widehat{v}$ *is an injective topological homomorphism for* $\sigma(\mathcal{S}_n, \mathcal{S}'_n)$ *and* $\sigma(\mathcal{S}_m, \mathcal{S}'_m)$*;*
5) $\widehat{v}$ *is an injective topological homomorphism from* $(\mathcal{S}_n)$ *to* $(\mathcal{S}_m)$*.*

Now we see an infinite-dimensional version of a basic theorem of linear algebra, more precisely the following classic result:

Theorem. *Let* $v = (v_i)_{i=1}^n$ *be a family of linear forms on a vector space* X *and let* w *be a linear form vanishing on the kernel of every form* v_i*. Then* w *is a linear combination of the family* v*.*

Note, first of all, that the theorem can be restated as follows.

We say kernel of a family $v = (v_i)_{i\in I}$ of linear forms on a vector space X the intersection of all the kernels of the forms forming v:

$$\ker v := \bigcap_{i\in I} \ker v_i.$$

Moreover, if Y is a subspace of X, by $Y^\perp$ we denote the orthogonal of Y, i.e., the set of all the linear forms on X which vanish on every vector of Y.

With this notation we can restate the preceding theorem.

Theorem. *Let* $v = (v_i)_{i=1}^n$ *be a family of linear forms on a vector space* X *and let* w *be another linear form. Then,* w *vanishes on the kernel of the family* v *if and only if* w *is a linear combination of the family* v*, in other words*

$$(\ker v)^\perp = \operatorname{span}(v).$$

Finally, we state and prove the $\mathcal{S}$-linear version of the above result.

Theorem 4.6. *Let* $v = (v_p)_{p\in\mathbb{R}^m}$ *be an* $\mathcal{S}$*-family in* $\mathcal{S}'_n$*. Then*

$$(\ker v)^\perp = \overline{\operatorname{span}}_{\sigma(\mathcal{S}'_n, \mathcal{S}_n)}(v).$$

In particular, if v *is exhaustive (i.e., if* ${}^{\mathcal{S}}\operatorname{span}(v)$ *is* $\sigma(\mathcal{S}'_n, \mathcal{S}_n)$*-closed) we have*

$$(\ker v)^\perp = {}^{\mathcal{S}}\operatorname{span}(v).$$

Proof. A classic theorem on duality (see [5] theorem 11, pg. 119) affirms that

$$(\ker A)^\perp = \overline{(\operatorname{Im}({}^tA))}_{\sigma(E',E)},$$

for every weakly continuous operator $A : E \to F$. Now applying this theorem to the operator $\widehat{v}$ we have

$$(\ker \widehat{v})^\perp = \overline{(\operatorname{Im}({}^t\widehat{v}))}_{\sigma(\mathcal{S}'_n,\, \mathcal{S}_n)} = \overline{({}^{\mathcal{S}}\operatorname{span}(v))}_{\sigma(\mathcal{S}'_n, \mathcal{S}_n)} = \overline{\operatorname{span}}_{\sigma(\mathcal{S}'_n, \mathcal{S}_n)}(v).$$

On the other hand, ϕ belongs to $\ker \widehat{v}$ if and only if $v(\phi)(p) = 0$, for every m-tuple p, and this means that ϕ belongs to the kernel of each v_p, concluding $\ker \widehat{v} = \ker v$. ■

5. $\mathcal{S}$-linear independence

Definition 5.1 (of $\mathcal{S}$-linear independence). *Let $v \in \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$. The family v is said $\mathcal{S}$-linearly independent, if $a \in \mathcal{S}'_m$ and $\int_{\mathbb{R}^m} av = 0_{\mathcal{S}'_n}$ implies $a = 0_{\mathcal{S}'_m}$.* $\square$

Example 5.1. *The Dirac family in $\mathcal{S}'_n$ is $\mathcal{S}$-linearly independent.* In fact, one has

$$\int_{\mathbb{R}^n} u\delta = u,$$

for all $u \in \mathcal{S}'_n$, and then $\int_{\mathbb{R}^n} u\delta = 0_{\mathcal{S}'_n}$ implies $u = 0_{\mathcal{S}'_n}$. $\triangle$

Example 5.2 (the Fourier families). *The Fourier families are $\mathcal{S}$-linearly independent.* In fact, let φ be the (a,b)-Fourier family, and let $\int_{\mathbb{R}^n} u\varphi = 0_{\mathcal{S}'_n(\mathbb{C})}$. For every $\phi \in \mathcal{S}_n(\mathbb{C})$, one has

$$0 = \left(\int_{\mathbb{R}^n} u\varphi \right)(\phi) = u\left(\widehat{\varphi}\left(\phi\right)\right) = u\left(\mathcal{S}_{(a,b)}\left(\phi\right)\right) = \mathcal{F}_{(a,b)}\left(u\right)\left(\phi\right),$$

i.e., $\mathcal{F}_{(a,b)}\left(u\right) = 0_{\mathcal{S}'_n(\mathbb{C})}$, and thus $u = 0_{\mathcal{S}'_n(\mathbb{C})}$, being $\mathcal{F}_{(a,b)}$ injective. $\triangle$

Theorem 5.1. *Let $v \in \mathcal{S}\left(\mathbb{R}^m, \mathcal{S}'_n\right)$ be a family $\mathcal{S}$ -linearly independent. Then, v is linearly independent. Consequently, ${}^{\mathcal{S}}\mathrm{span}\left(v\right)$ is an infinite-dimensional subspace of $\mathcal{S}'_n$.*

Proof. Let $k \in \mathbb{N}$, $\alpha \in \left(\mathbb{R}^m\right)^k$, and $v_\alpha = \left(v_{\alpha_i}\right)_{i=1}^k$. By contradiction, let v_α be a linearly dependent system of $\mathcal{S}'_n$, then there exists a non-zero k-tuple $\lambda \in \mathbb{C}^k$ such that

$$\sum_{i=1}^k \lambda_i v_{\alpha_i} = 0_{\mathcal{S}'_n}.$$

Put $\Lambda = \sum_{i=1}^k \lambda_i \delta_{\alpha_i}$, we have

$$\int_{\mathbb{R}^m} \Lambda v = \int_{\mathbb{R}^m} \sum_{i=1}^k \lambda_i \delta_{\alpha_i} v = \sum_{i=1}^k \lambda_i \int_{\mathbb{R}^m} \delta_{\alpha_i} v = \sum_{i=1}^k \lambda_i v_{\alpha_i} = 0_{\mathcal{S}'_n}.$$

Now, since $\Lambda \neq 0_{\mathcal{S}'_m}$, the preceding equality contradicts the $\mathcal{S}$-linear independence of v, against the assumptions. $\blacksquare$

Remark. The above theorem shows that, for the $\mathcal{S}$ -families, the $\mathcal{S}$-linear independence implies the usual linear independence. Actually, the $\mathcal{S}$-linear independence is more restrictive than the linear independence, as we shall see later by a simple example. On the contrary it is less restrictive than the $\beta(\mathcal{S}'_n, \mathcal{S}_n)$-topological independence, as it is shown below.

We recall that a system $v = (v_i)_{i \in I}$ is $\beta(\mathcal{S}'_n, \mathcal{S}_n)$-topologically free (resp. $\sigma(\mathcal{S}'_n, \mathcal{S}_n)$-topologically free) if and only if there exists a family $(f_i)_{i \in I}$ of $\beta(\mathcal{S}'_n, \mathcal{S}_n)$-continuous (resp. $\sigma(\mathcal{S}'_n, \mathcal{S}_n)$-continuous) linear forms on $\mathcal{S}'_n$ such that $f_i(v_k) = \delta_{ik}$, where $(\delta_{ik})_{(i,k) \in I^2}$ is the Kronecker delta on $I \times I$. If v is not topologically free it is said topologically bound.

Theorem 5.2. *Every $\mathcal{S}$-family is $\beta(\mathcal{S}'_n, \mathcal{S}_n)$-topologically bound and, thus, $\sigma(\mathcal{S}'_n, \mathcal{S}_n)$ -topologically bound.*

Proof. Let v be an $\mathcal{S}$-family in $\mathcal{S}'_n$ indexed by $\mathbb{R}^m$. And let f be an arbitrary family in $\mathcal{S}''_n$ indexed by the same index set. Being the space $(\mathcal{S}_n)$ reflexive, for every i, there is a g_i in $\mathcal{S}_n$ such that $f_i(u) = u(g)$, for every u in $\mathcal{S}'_n$. If there is an index i such that $f_i(v_i) = 1$, then we have

$$1 = f_i(v_i) = v_i(g_i) = v(g_i)\,(i)\,,$$

being v an $\mathcal{S}$-family, the function $v(g_i)$ is continuous, then there is a neighborhood U of i in which the function $v(g_i)$ is strictly positive. Then, for every k in U, we have

$$f_i(v_k) = v_k(g_i) = v(g_i)\,(k) > 0,$$

and then f cannot verify the condition of topological independence for v. ■

With the same proof, it is possible to prove that *every C^0-family is strongly topologically bound*. Consequently every smooth family is also strongly topologically bound.

It's simple to prove that a family v in $\mathcal{S}'_n$ indexed by $\mathbb{R}^m$ is $\mathcal{S}$-linearly dependent if and only if, for every y in $\mathbb{R}^m$, there is a tempered distribution a different from δ_y such that $v_p = \int_{\mathbb{R}^m} av$.

By the Dieudonné-Schwartz theorem (see [5], theorem 7, pg. 92) we immediately deduce two characterizations.

Theorem 5.3. *Let $v \in \mathcal{S}\left(\mathbb{R}^m, \mathcal{S}'_n\right)$ such that ${}^{\mathcal{S}}\mathrm{span}\,(v)$ is $\sigma(\mathcal{S}'_n, \mathcal{S}_n)$-closed. Then the following assertions are equivalent*
1) v is $\mathcal{S}$-linearly independent;
2) $\int_{\mathbb{R}^m}(\cdot, v)$ is an injective topological homomorphism for $\sigma(\mathcal{S}'_m, \mathcal{S}_m)$ and $\sigma(\mathcal{S}'_n, \mathcal{S}_n)$;
3) $\int_{\mathbb{R}^m}(\cdot, v)$ is an injective topological homomorphism for $\beta(\mathcal{S}'_m, \mathcal{S}_m)$ and $\beta(\mathcal{S}'_n, \mathcal{S}_n)$;
4) $\widehat{v}$ is a surjective topological homomorphism for $\sigma(\mathcal{S}_n, \mathcal{S}'_n)$ and $\sigma(\mathcal{S}_m, \mathcal{S}'_m)$;
5) $\widehat{v}$ is an surjective topological homomorphism from $(\mathcal{S}_n)$ to $(\mathcal{S}_m)$.

Theorem 5.4. *Let $v \in \mathcal{S}\left(\mathbb{R}^m, \mathcal{S}'_n\right)$. Then the following assertions are equivalent*
1) v is $\mathcal{S}$-linearly independent and ${}^{\mathcal{S}}\mathrm{span}\,(v)$ is $\sigma(\mathcal{S}'_n, \mathcal{S}_n)$-closed;
2) $\int_{\mathbb{R}^m}(\cdot, v)$ is an injective topological homomorphism for $\sigma(\mathcal{S}'_m, \mathcal{S}_m)$ and $\sigma(\mathcal{S}'_n, \mathcal{S}_n)$;
3) $\widehat{v}$ is a surjective topological homomorphism for $\sigma(\mathcal{S}_n, \mathcal{S}'_n)$ and $\sigma(\mathcal{S}_m, \mathcal{S}'_m)$;
4) $\widehat{v}$ is an surjective topological homomorphism from $(\mathcal{S}_n)$ to $(\mathcal{S}_m)$.

Remark. If v is $\mathcal{S}$-linearly independent, we can consider the algebraic isomorphism from $\mathcal{S}'_m$ to ${}^{\mathcal{S}}\mathrm{span}\,(v)$ that sends every $a \in \mathcal{S}'_m$ to the superposition $\int_{\mathbb{R}^m} av$, that is the restriction of the injection $\int_{\mathbb{R}^m}(\cdot, v)$ to the pair of sets $(\mathcal{S}'_m, {}^{\mathcal{S}}\,\mathrm{span}\,(v))$. We shall denote the inverse of this isomorphism by $[\cdot \mid v]$, it is a consequence of the preceding theorem that this inverse operator is a topological isomorphism, with respect to the topology induced by $\sigma(\mathcal{S}'_n, \mathcal{S}_n)$ on ${}^{\mathcal{S}}\mathrm{span}\,(v)$ and to $\sigma(\mathcal{S}_m, \mathcal{S}'_m)$, in and only if ${}^{\mathcal{S}}\mathrm{span}\,(v)$ is $\sigma(\mathcal{S}'_n, \mathcal{S}_n)$-closed.

6. $\mathcal{S}$-bases

Definition 6.1 (of $\mathcal{S}$-basis). *Let $v \in \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$ and let $V \subseteq \mathcal{S}'_n$. v is an $\mathcal{S}$-**basis** of V if it is $\mathcal{S}$-linearly independent, and ${}^{\mathcal{S}}\mathrm{span}(v) = V$.* $\square$

The Dirac family δ in $\mathcal{S}'_n$ is an $\mathcal{S}$-basis of $\mathcal{S}'_n$. We call δ *the canonical $\mathcal{S}$-basis of $\mathcal{S}'_n$* or *the Dirac basis of $\mathcal{S}'_n$*.

Moreover, the following complete version of the Fourier expansion-theorem, allow us to call the Fourier families of $\mathcal{S}'(\mathbb{R}^n, \mathbb{C})$ with the name *Fourier bases of $\mathcal{S}'(\mathbb{R}^n, \mathbb{C})$*.

Theorem 6.1 (Fourier expansion theorem in geometric form). *The Fourier families in $\mathcal{S}'(\mathbb{R}^n, \mathbb{C})$ are $\mathcal{S}$-bases of $\mathcal{S}'(\mathbb{R}^n, \mathbb{C})$.*

The following is a meaningful generalization of the Fourier expansion theorem.

Theorem 6.2 (characterization of an $\mathcal{S}$-basis). *Let $v \in \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$. Then,*
i) v is $\mathcal{S}$-generates $\mathcal{S}'_n$ if and only if ${}^t(\widehat{v})$ is surjective.
ii) v is $\mathcal{S}$-linearly independent if and only if ${}^t(\widehat{v})$ is injective.
iii) v is an $\mathcal{S}$-basis of $\mathcal{S}'_n$ if and only if ${}^t(\widehat{v})$ is bijective.

Proof. First of all ${}^t(\widehat{v})$ is well defined because v is an $\mathcal{S}$-family. Moreover, it is obvious that v $\mathcal{S}$-generates $\mathcal{S}'_n$ if and only if ${}^t(\widehat{v})$ is surjective, and that v is $\mathcal{S}$ -linearly independent if and only if ${}^t(\widehat{v})$ is injective. ■

Example 6.1 (a system of linearly independent $\mathcal{S}$ -generators that is not an $\mathcal{S}$-basis). Let $v = (\delta'_x)_{x\in\mathbb{R}}$ be the family in $\mathcal{S}'_1$ of the first derivatives of the Dirac distributions. The family v is of class $\mathcal{S}$, in fact

$$v(\phi)(x) = v_x(\phi) = \delta'_x(\phi) = -\phi'(x),$$

and $-\phi'$ is an $\mathcal{S}$-function. Consequently, the operator associated with v is the derivation in $\mathcal{S}_n$ up to the sign and, then, ${}^t(\widehat{v})$ is the derivation in $\mathcal{S}'_n$. This last operator is a surjective operator (every tempered distribution has a primitive) but it is not injective (every tempered distribution has many primitives), then v is a system of $\mathcal{S}$-generators for $\mathcal{S}'_1$, but it is not $\mathcal{S}$-linearly independent. Moreover, note that v is linearly independent. In fact, let P be a finite subset of the real line $\mathbb{R}$, and let, for every p_0 in P, f_{p_0} be a function in the space $\mathcal{S}_1$ whose derivative verifies the relation $f'_{p_0}(p) = \delta_{p_0 p}$, for every index p in P (here $\delta_{p_0 p}$ is the Kronecker symbol on the index-set P calculate in the pair (p_0, p), not the Dirac distribution). If $a = (a_p)_{p\in P}$ is a finite family of scalars such that $\sum_{p\in P} a_p v_p = 0_{\mathcal{S}'_1}$, then

$$0 = \left(\sum_{p\in P} a_p v_p\right)(f_{p_0}) = \sum_{p\in P} a_p \delta_{p_0 p} = a_{p_0},$$

for every p_0 in P.

By the Dieudonné-Schwartz theorem (see [5], theorem 7, pg. 92) we immediately have a characterization.

Theorem 6.3. *Let $v \in \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$. Then the following assertions are equivalent*
1) v is an $\mathcal{S}$-basis of $\mathcal{S}'_n$;

2) $\int_{\mathbb{R}^m}(\cdot, v)$ is topological isomorphism for $\sigma(\mathcal{S}'_m, \mathcal{S}_m)$ and $\sigma(\mathcal{S}'_n, \mathcal{S}_n)$;
3) $\int_{\mathbb{R}^m}(\cdot, v)$ is a topological isomorphism for $\beta(\mathcal{S}'_m, \mathcal{S}_m)$ and $\beta(\mathcal{S}'_n, \mathcal{S}_n)$;
4) $\widehat{v}$ is a topological isomorphism for $\sigma(\mathcal{S}_n, \mathcal{S}'_n)$ and $\sigma(\mathcal{S}_m, \mathcal{S}'_m)$;
5) $\widehat{v}$ is a topological isomorphism from $(\ \mathcal{S}_n)$ to $(\mathcal{S}_m)$.

7. $\mathcal{S}$-operators and the definition of $\mathcal{S}$-linear operators

Definition 7.1 (image of a family of distributions). *Let $W \subseteq \mathcal{S}'_n$, $A : W \to \mathcal{S}'_m$ be an operator and $v = (v_p)_{p\in\ \mathbb{R}^k}$ be a family of tempered distributions in W, i.e., such that the set $\{v_p\}_{p\in\mathbb{R}^k}$ is contained in W. The **image of** v **under** A is by definition the family in $\mathcal{S}'_m$*

$$A(v) = (A(v_p))_{p\in\mathbb{R}^k} \, ,$$

i.e., the family such that, for all $p \in \mathbb{R}^k$, one has $A(v)_p = A(v_p)$. $\square$

We can read the above definition saying that “the image of a family of vectors is the family of the images of vectors”.

Definition 7.2 (operator of class $\mathcal{S}$). *Let $W \subseteq \mathcal{S}'_n$ and $L : W \to \mathcal{S}'_m$ be an operator. L is an $\mathcal{S}$-**operator** or **operator of class** $\mathcal{S}$ if, for each natural k and for each $v \in \mathcal{S}(\mathbb{R}^k, \mathcal{S}'_n)$, such that $\{v_p\}_{p\in\mathbb{R}^k} \subseteq W$, one has $L(v) \in \mathcal{S}(\mathbb{R}^k, \mathcal{S}'_m)$.* $\square$

We can read the above definition as follows: “L is of class $\mathcal{S}$ if the image of an $\mathcal{S}$-family is an $\mathcal{S}$-family”.

Example 7.1 (the transpose). Let $A : \mathcal{S}_n \to \mathcal{S}_m$ be a $(\sigma(\mathcal{S}_n, \mathcal{S}'_n), \sigma(\mathcal{S}_m, \mathcal{S}'_m))$-continuous operator. A is transposable (i.e., for every $a \in \mathcal{S}'_m$, $a \circ A$ is in $\mathcal{S}'_n$) and its transpose is

$${}^tA : \mathcal{S}'_m \to \mathcal{S}'_n : a \mapsto a \circ A.$$

Let $v \in \mathcal{S}\left(\mathbb{R}^k, \mathcal{S}'_n\right)$, one has, by definition,

$${}^tA\,(v)_p = {}^tA\,(v_p)\,,$$

and hence one deduces

$${}^tA\,(v)\,(\phi)\,(p) = {}^tA\,(v)_p\,(\phi) = {}^tA\,(v_p)\,(\phi) = v_p\,(A\,(\phi)) = v\,(A\,(\phi))\,(p),$$

so, taking into account that v is an $\mathcal{S}$-family, one has ${}^tA\,(v)\,(\phi) = \widehat{v}\,(A\,(\phi)) \in \mathcal{S}_k$. Concluding one has ${}^tA\,(v) \in \mathcal{S}\left(\mathbb{R}^k, \mathcal{S}'_n\right)$, and thus the operator ${}^tA$, sending $\mathcal{S}$-family in $\mathcal{S}$-family, is an $\mathcal{S}$-operator. $\triangle$

Application 7.1. Let $L : \mathcal{S}'_n \to \mathcal{S}'_n$ be a differential operator with constant coefficients and v be an $\mathcal{S}$-family in $\mathcal{S}'_n$. Then $L(v)$ is an $\mathcal{S}$-family, in fact L is the transpose of a certain differential opeator on $\mathcal{S}_n$. For instance, the family $(\delta_x)_{x\in\mathbb{R}^n}$ is obviously an $\mathcal{S}$-family, and so the families of derivatives $\left(\delta_x^{(i)}\right)_{x\in\mathbb{R}^n}$ are $\mathcal{S}$ -families for every multi-index i. ♠

Definition 7.3 ($\mathcal{S}$-linear operator). *Let $L : \mathcal{S}'_n \to \mathcal{S}'_m$ be an $\mathcal{S}$-operator. L is called **$\mathcal{S}$-linear operator** if, for each natural k, for each $v \in \mathcal{S}(\mathbb{R}^k, \mathcal{S}'_n)$ and for every $a \in \mathcal{S}'_k$, one has*

$$L\left(\int_{\mathbb{R}^k} av\right) = \int_{\mathbb{R}^k} aL(v).$$

The set of all the $\mathcal{S}$-linear operators from $\mathcal{S}'_n$ to $\mathcal{S}'_m$ is denoted by ${}^{\mathcal{S}}\mathrm{Hom}(\mathcal{S}'_n, \mathcal{S}'_m)$. $\square$

8. Characterization of the $\mathcal{S}$-linear operators

Now, we can show the nature of the $\mathcal{S}$-linear operators defined on $\mathcal{S}'_n$.

Recall that if $v \in s\left(\mathbb{R}^k, \mathcal{S}'_m\right)$ and $w \in \mathcal{S}\left(\mathbb{R}^m, \mathcal{S}'_n\right)$, the family in $\mathcal{S}'_n$

$$\int_{\mathbb{R}^m} vw := \left(\int_{\mathbb{R}^m} v_p w\right)_{p\in\mathbb{R}^k},$$

is called the superposition of w with respect to v. We have already proved that, if $v \in \mathcal{S}\left(\mathbb{R}^k, \mathcal{S}'_m\right)$ then $\int_{\mathbb{R}^m} vw \in \mathcal{S}\left(\mathbb{R}^k, \mathcal{S}'_n\right)$ and

$$\left(\int_{\mathbb{R}^m} vw\right)^{\wedge} = \widehat{v} \circ \widehat{w}.$$

In this case, $\int_{\mathbb{R}^m} vw$ is also denoted by vw and it is called the $\mathcal{S}$-product of v by w.

Lemma 8.1 (the image under a transpose operator). *Let $B \in \mathcal{L}\left(\mathcal{S}_n, \mathcal{S}_m\right)$ and $v \in \mathcal{S}(\mathbb{R}^k, \mathcal{S}'_m)$. Then,*

$${}^tB(v) = \int_{\mathbb{R}^k} vB^{\vee},$$

so in particular, ${}^tB$ is an $\mathcal{S}$-operator.

Proof. For each $p \in \mathbb{R}^k$, one has

$$\left(\int_{\mathbb{R}^k} vB^{\vee}\right)_p = \int_{\mathbb{R}} v_p B^{\vee} = v_p \circ (B^{\vee})^{\wedge} = v_p \circ B = {}^tB\left(v_p\right) = {}^tB(v)(p),$$

and hence

$$\int_{\mathbb{R}^k} vB^{\vee} = {}^tB(v). \blacksquare$$

Theorem 8.1 ($\mathcal{S}$-linearity of a transpose operator). *Let $B \in \mathcal{L}(\mathcal{S}_n, \mathcal{S}_m)$ and $v \in \mathcal{S}(\mathbb{R}^k, \mathcal{S}'_m)$. Then, for each $a \in \mathcal{S}'_k$ one has*

$${}^tB\left(\int_{\mathbb{R}^k} av\right) = \int_{\mathbb{R}^k} a\, {}^tB(v).$$

Proof. One has

$$\begin{aligned} {}^tB\left(\int_{\mathbb{R}^k} av\right) &= \left(\int_{\mathbb{R}^k} av\right) \circ B = (a \circ \widehat{v}) \circ B = a \circ (\widehat{v} \circ B) = \\ &= \int_{\mathbb{R}^k} a(\widehat{v} \circ B)^{\vee} = \int_{\mathbb{R}^k} a\left(\int_{\mathbb{R}^k} vB^{\vee}\right) = \int_{\mathbb{R}^k} a\, {}^tB(v). \blacksquare \end{aligned}$$

Application 8.1. As a simple application, we prove the formula: $u' = \int_{\mathbb{R}} u\delta'$, where δ' is the $\mathcal{S}$-family in $\mathcal{S}_1'$ defined by $\delta' = \left(\delta_p'\right)_{p\in\mathbb{R}}$. Let δ be the Dirac family of $\mathcal{S}_1'$, then for each $u \in \mathcal{S}_1'$, one has $u = \int_{\mathbb{R}} u\delta$, and thus

$$u' = D\left(\int_{\mathbb{R}} u\delta\right) = \int_{\mathbb{R}} uD(\delta) = \int_{\mathbb{R}} u\delta'. \ \spadesuit$$

Theorem 8.2 (characterization of $\mathcal{S}$- linearity). *Let $L : \mathcal{S}_n' \to \mathcal{S}_m'$. Then, L is $\mathcal{S}$-linear if and only if there exists a $B \in \mathcal{L}(\mathcal{S}_m,\ \mathcal{S}_n)$ such that $L = {}^t(B)$.*

Proof. Sufficiency. Follows from the above theorem.
Necessity. Let δ be the Dirac's family in $\mathcal{S}_n'$, one has

$$L(u) = L\left(\int_{\mathbb{R}^n} u\delta\right) = \int_{\mathbb{R}^n} uL(\delta) = {}^t\left(L(\delta)^\wedge\right)(u),$$

so

$$L = {}^t\left(L(\delta)^\wedge\right).$$

$Q.\,E.\,D.$ ■

Recalling that a linear operator $L : \mathcal{S}_n' \to \mathcal{S}_m'$ is said to be transposable with respect to the canonical bilinear form $\langle \cdot, \cdot \rangle$ if and only if there exists a $B \in \mathcal{L}(\mathcal{S}_m, \mathcal{S}_n)$ such that $L = {}^t(B)$, and recalling that L is weakly continuous if and only if it is strongly continuous and if and only if it is transposable, we derive the following definitive characterization.

Theorem 8.3 (characterization of $\mathcal{S}$-linearity). *Let $L : \mathcal{S}_n' \to \mathcal{S}_m'$ be a operator. Then, the following assertions are equivalent*
1) L is $\mathcal{S}$-linear
2) there exists a $B \in \mathcal{L}(\mathcal{S}_m, \mathcal{S}_n)$ such that $L = {}^t(B)$;
3) L is linear and weakly continuous;
4) L is linear and strongly continuous;
5) L is linear and transposable.

9. $\mathcal{S}$-closed subsets

A natural kind of stability arises with $\mathcal{S}$-linear combinations.

Definition 9.1 (of $\mathcal{S}$-closedness in $\mathcal{S}_n'$). *Let W be a subset of $\mathcal{S}_n'$. W is called $\mathcal{S}$- **closed** or $\mathcal{S}$-**stable in** $\mathcal{S}_n'$ if it contains all the superpositions of its $\mathcal{S}$-families. In other words, W is $\mathcal{S}$-closed if, for each $k \in \mathbb{N}$, for each $v \in \mathcal{S}(\mathbb{R}^m, \mathcal{S}_n')$ in W and for each $a \in \mathcal{S}_m'$, $\int_{\mathbb{R}^m} av \in W$.* □

Example 9.1. The empty set and $\mathcal{S}_n'$ are $\mathcal{S}$-closed. △

Theorem 9.1. *Let F be a family of $\mathcal{S}$-closed subset of $\mathcal{S}_n'$. Then $\bigcap F$ is $\mathcal{S}$-closed.*

Proof. Let $F = (F_i)_{i \in I}$ and let v be an $\mathcal{S}$-family in $\bigcap F$. Then, v is an $\mathcal{S}$-family in F_i for every $i \in I$ (in fact, v is a family in $\bigcap F$ if $v_p \in F_i$, for every $p \in \mathbb{R}^m$ and $i \in I$). Since F_i is $\mathcal{S}$-closed

$$\int_{\mathbb{R}^m} av \in F_i,$$

for every $a \in \mathcal{S}'_n$ and every $i \in I$, therefore $\int_{\mathbb{R}^m} av \in \bigcap F$. ∎

Remark (on the union of two $\mathcal{S}$-closed sets). The union of two $\mathcal{S}$-closed sets is not necessarily $\mathcal{S}$ -closed; neither in the case in which the two $\mathcal{S}$-closed subsets are subspaces. In fact, consider in $\mathcal{S}'_2$ the $\mathcal{S}$-linear hulls $F_1 = {}^{\mathcal{S}}\mathrm{span}\left(\delta_{(x,0)}\right)$ and $F_2 = {}^{\mathcal{S}}\mathrm{span}\left(\delta_{(0,y)}\right)$, and their union $F = F_1 \cup F_2$. F is obviously not a subspace, but it is star-shaped in the origin; in fact, if u is in F, then u lies in F_1 or in F_2, and then the segment joining u and the origin is containing in F. Now, a star-shaped $\mathcal{S}$-closed set is necessarily a subspace (see later), then F cannot be $\mathcal{S}$ -closed.

Open problem. If two $\mathcal{S}$-closed sets are disjoints, then is their union $\mathcal{S}$-closed ? And, if v is an $\mathcal{S}$-family in the union of two disjoint $\mathcal{S}$-closed sets, must v be contained in one and only one of the two component sets ? The question arises naturally, since the image of v is a path-connected subset of $\mathcal{S}'_n$ in the topology $\sigma(\mathcal{S}'_n, \mathcal{S}_n)$.

Theorem 9.2. *Let v be an $\mathcal{S}$ -family in $\mathcal{S}'_n$ and let F be the collection of all the $\mathcal{S}$-closed subsets of $\mathcal{S}'_n$ containing v. Then ${}^{\mathcal{S}}\mathrm{span}(v) \subseteq \bigcap F$. Consequently, ${}^{\mathcal{S}}\mathrm{span}(v)$ is $\mathcal{S}$-closed if and only if*

$${}^{\mathcal{S}}\mathrm{span}(v) = \bigcap F.$$

Proof. Since F_i is $\mathcal{S}$-closed and contains v, we have ${}^{\mathcal{S}}\mathrm{span}(v) \subseteq F_i$, for every $i \in I$, and consequently ${}^{\mathcal{S}}\mathrm{span}(v) \subseteq \bigcap F$. If ${}^{\mathcal{S}}\mathrm{span}(v)$ is $\mathcal{S}$-closed, since it contains v , we have ${}^{\mathcal{S}}\mathrm{span}(v) \in F$, and hence

$$\bigcap F \subseteq {}^{\mathcal{S}}\mathrm{span}(v). \blacksquare$$

Remark. Note that, in the conditions of the preceding theorem, in general it is not true that $\bigcap F$ is a subspace, however, if ${}^{\mathcal{S}}\mathrm{span}(v)$ is $\mathcal{S}$-closed then $\bigcap F$ is necessarily a subspace.

Theorem 9.3. *Let F be a $\beta(\mathcal{S}'_n, \mathcal{S}_n)$-closed subspace of $\mathcal{S}'_n$. Then F is $\mathcal{S}$-closed.*

Proof. Let δ be the Dirac family in $\mathcal{S}'_m$ and let v be an $\mathcal{S}$-family in F, then

$$\int_{\mathbb{R}^m} \delta_p v = v_p \in F.$$

Now, let $a \in \mathcal{S}'_m$, since $\mathcal{S}_m$ is reflexive, we know that

$$\overline{\mathrm{span}}_{\beta(\mathcal{S}'_m, \mathcal{S}_m)}(\delta) = \mathcal{S}'_m,$$

therefore there exists a sequence $\Delta = (\Delta_k)_{k \in \mathbb{N}}$ in $\mathrm{span}\,(\delta)$ converging to a in the topology $\beta(\mathcal{S}'_m, \mathcal{S}_m)$.

We have, by the selection property of the Dirac family,

$$\int_{\mathbb{R}^m} \Delta_k v \in \mathrm{span}\,(v) \subseteq F,$$

the set-inclusion holding since F is a subspace of $\mathcal{S}'_n$. Moreover, being the superposition-operator $\int_{\mathbb{R}^m}(\cdot, v)$ continuous with respect to the topologies $\beta(\mathcal{S}'_m, \mathcal{S}_m)$ and $\beta(\mathcal{S}'_n, \mathcal{S}_n)$, it follows that, for every a in $\mathcal{S}'_m$,

$$\int_{\mathbb{R}^m} av = \int_{\mathbb{R}^m} \left({}^{\beta(\mathcal{S}'_m, \mathcal{S}_m)} \lim_{k \to +\infty} \Delta_k \right) v = {}^{\beta(\mathcal{S}'_n, \mathcal{S}_n)} \lim_{k \to +\infty} \int_{\mathbb{R}^m} \Delta_k v.$$

The last limit belongs to F for $\beta(\mathcal{S}'_n,\ \mathcal{S}_n)$-closedness. And consequently, F is $\mathcal{S}$-closed. ■

Remark. Since $\mathcal{S}_n$ is semireflexive F is a $\sigma(\mathcal{S}'_n, \mathcal{S}_n)$-closed if and only if it is $\beta(\mathcal{S}'_n, \mathcal{S}_n)$-closed.

Definition 9.2 (of $\mathcal{S}$-homomorphism). *We say that an operator $A : \mathcal{S}'_m \to \mathcal{S}'_n$ is an $\mathcal{S}$-homomorphism if, for every positive integer k and for every family $a = (a_i)_{i \in \mathbb{R}^k}$ in $\mathcal{S}'_m$, the image of a under A, i.e., the family $A(a) := (A(a_i))_{i \in \mathbb{R}^k}$, is an $\mathcal{S}$-family if and only if a is an $\mathcal{S}$-family.*

Definition 9.3 (of $\mathcal{S}$-stable family). *We say that an $\mathcal{S}$-family v in $\mathcal{S}'_n$ indexed by $\mathbb{R}^m$ is $\mathcal{S}$-stable if, for every family $a = (a_i)_{i \in \mathbb{R}^k}$ in $\mathcal{S}'_m$, the superposition $\int_{\mathbb{R}^m} av$ (that is necessarily a family in the $\mathcal{S}$-linear hull of v and indexed by $\mathbb{R}^k$) is an $\mathcal{S}$ -family if and only if a is an $\mathcal{S}$ -family. In other terms v is $\mathcal{S}$-stable if and only if the superposition operator $\int_{\mathbb{R}^m}(\cdot, v)$ is an $\mathcal{S}$-homomorphism.*

Theorem 9.4. *Let v be an $\mathcal{S}$ -stable family in $\mathcal{S}'_n$. Then the ${}^{\mathcal{S}}\mathrm{span}(v)$ is $\mathcal{S}$-closed.*

Proof. Let v be indexed by $\mathbb{R}^m$ and let w be an $\mathcal{S}$-family in the $\mathcal{S}$-linear hull of v indexed by $\mathbb{R}^k$, then there is a family a such that $\int_{\mathbb{R}^m} av = w$. Since w is an $\mathcal{S}$-family and since v is $\mathcal{S}$-stable, we deduce that a is an $\mathcal{S}$-family. Concluding, applying the $\mathcal{S}$-linearity, for every b in $\mathcal{S}'_k$, we have

$$\int_{\mathbb{R}^k} bw = \int_{\mathbb{R}^k} b \left(\int_{\mathbb{R}^m} av \right) = \int_{\mathbb{R}^m} \left(\int_{\mathbb{R}^k} ba \right) v,$$

and hence the superpositions of w belong to the $\mathcal{S}$-linear hull of v. ■

Corollary. *Let v be an $\mathcal{S}$-stable family in $\mathcal{S}'_n$. Then the ${}^{\mathcal{S}}\mathrm{span}(v)$ is the intersection of all the $\mathcal{S}$ -closed subset of $\mathcal{S}'_n$ containing v.*

Note. For the basic facts about weak duality see [6]. For the basic formulation of Quantum Mechanics see [7]. For the $\mathcal{S}$-linear algebra formulation of infinite dimensional Decision Theory and for the study of abstract evolution equations in economical and physical Theories see [8], [9], [10], [11], [12], [13], [14], [15], [16].

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David Carfì
Università degli Studi di Messina
Facoltà di Economia
Via dei Verdi, 75
98122 Messina, Italy
E-mail: davidcarfi71@yahoo.it

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