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BERTRAND MATE OF TIMELIKE BIHARMONIC GENERAL HELICES IN THE LORENTZIAN $\mathbb{E}(1,1)$

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ABSTRACT. In this paper we study a Bertrand mate of timelike biharmonic general helices in the Lorentzian group of rigid motions $\mathbb{E}(1,1)$. Finally, we find out their explicit parametric equations.

1. Introduction

The classification and characterization of curves can be done by investigating the relationship between the Frenet vectors of the curves. For example, in 1845 Saint Venant proposed the question whether upon the surface generated by the principal normal of a curve, a second curve can exist which has for its principal normal of the given curve. This question was answered by Bertrand in 1850; he showed that a necessary and sufficient condition for the existence of such a second curve is that a linear relationship with constant coefficients exists between the first and second curvatures of the given original curve. The pairs of curves of this kind have been called conjugate Bertrand curves or more commonly Bertrand curves [1, 2].

A smooth map $\phi : N \longrightarrow M$ is said to be biharmonic if it is a critical point of the bienergy functional:

$$E_2(\phi) = \int_N \frac{1}{2} \left|\mathcal{T}(\phi)\right|^2 dv_h,$$

where $\mathcal{T}(\phi) := \mathrm{tr}\nabla^{\phi} d\phi$ is the tension field of ϕ.

The Euler–Lagrange equation of the bienergy [3–7] is given by $\mathcal{T}_2(\phi) = 0$. Here the section $\mathcal{T}_2(\phi)$ is defined by

$$\mathcal{T}_2(\phi) = -\Delta_\phi \mathcal{T}(\phi) + \mathrm{tr}R\left(\mathcal{T}(\phi), d\phi\right) d\phi, \tag{1.1}$$

and called the bitension field of ϕ. Non-harmonic biharmonic maps are called proper biharmonic maps.

In this paper, we study a Bertrand mate of timelike biharmonic general helices in the Lorentzian group [8–10] of rigid motions $\mathbb{E}(1,1)$. Finally, we find out their explicit parametric equations.

2. Preliminaries

Let $\mathbb{E}(1,1)$ be the group of rigid motions of Euclidean 2-space. This consists of all matrices of the form

$$\begin{pmatrix} \cosh x & \sinh x & y \\ \sinh x & \cosh x & z \\ 0 & 0 & 1 \end{pmatrix}.$$

Topologically, $\mathbb{E}(1,1)$ is diffeomorphic to $\mathbb{R}^3$ under the map

$$\mathbb{E}(1,1) \longrightarrow \mathbb{R}^3 : \begin{pmatrix} \cosh x & \sinh x & y \\ \sinh x & \cosh x & z \\ 0 & 0 & 1 \end{pmatrix} \longrightarrow (x,y,z).$$

Its Lie algebra has a basis consisting of

$$\mathbf{e}_1 = \frac{\partial}{\partial x}, \ \mathbf{e}_2 = \cosh x \frac{\partial}{\partial y} + \sinh x \frac{\partial}{\partial z}, \ \mathbf{e}_3 = \sinh x \frac{\partial}{\partial y} + \cosh x \frac{\partial}{\partial z},$$

for which

$$[\mathbf{e}_1, \mathbf{e}_2] = \mathbf{e}_3, \ [\mathbf{e}_2, \mathbf{e}_3] = 0, \ [\mathbf{e}_1, \mathbf{e}_3] = \mathbf{e}_2.$$

Put

$$x^1 = x, \ x^2 = \frac{1}{2}(y+z), \ x^3 = \frac{1}{2}(y-z).$$

Then, we get

$$\mathbf{e}_1 = \frac{\partial}{\partial x^1}, \ \mathbf{e}_2 = \frac{1}{2}\left(e^{x^1}\frac{\partial}{\partial x^2} + e^{-x^1}\frac{\partial}{\partial x^3}\right), \ \mathbf{e}_3 = \frac{1}{2}\left(e^{x^1}\frac{\partial}{\partial x^2} - e^{-x^1}\frac{\partial}{\partial x^3}\right). \tag{2.1}$$

The bracket relations are

$$[\mathbf{e}_1, \mathbf{e}_2] = \mathbf{e}_3, \ [\mathbf{e}_2, \mathbf{e}_3] = 0, \ [\mathbf{e}_1, \mathbf{e}_3] = \mathbf{e}_2. \tag{2.2}$$

We consider a left-invariant Lorentzian metric which has a pseudo-orthonormal basis $\{\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3\}$. We consider left-invariant Lorentzian metric [11], given by

$$g = -\left(dx^1\right)^2 + \left(e^{-x^1}dx^2 + e^{x^1}dx^3\right)^2 + \left(e^{-x^1}dx^2 - e^{x^1}dx^3\right)^2, \tag{2.3}$$

where

$$g(\mathbf{e}_1, \mathbf{e}_1) = -1, \ g(\mathbf{e}_2, \mathbf{e}_2) = g(\mathbf{e}_3, \mathbf{e}_3) = 1. \tag{2.4}$$

Let coframe of our frame be defined by

$$\theta^1 = dx^1, \ \theta^2 = e^{-x^1}dx^2 + e^{x^1}dx^3, \ \theta^3 = e^{-x^1}dx^2 - e^{x^1}dx^3.$$

Proposition 2.1. *For the covariant derivatives of the Levi-Civita connection of the left-invariant metric g defined above the following is true:*

$$\nabla = \begin{pmatrix} 0 & 0 & 0 \\ -\mathbf{e}_3 & 0 & -\mathbf{e}_1 \\ -\mathbf{e}_2 & -\mathbf{e}_1 & 0 \end{pmatrix}, \tag{2.5}$$

where the (i,j)-element in the table above equals $\nabla_{\mathbf{e}_i}\mathbf{e}_j$ for our basis

$$\{\mathbf{e}_k, k = 1,2,3\} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}.$$

3. Timelike Biharmonic General Helices in the Lorentzian Group of Rigid Motions $\mathbb{E}(1,1)$

Let $\gamma : I \longrightarrow \mathbb{E}(1,1)$ be a non geodesic timelike curve in the group of rigid motions $\mathbb{E}(1,1)$ parametrized by arc length. Let $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ be the Frenet frame fields tangent to the group of rigid motions $\mathbb{E}(1,1)$ along γ defined as follows:

$\mathbf{T}$ is the unit vector field γ' tangent to γ, $\mathbf{N}$ is the unit vector field in the direction of $\nabla_{\mathbf{T}}\mathbf{T}$ (normal to γ) and $\mathbf{B}$ is chosen so that $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ is a positively oriented orthonormal basis. Then, we have the following Frenet formulas:

$$\begin{aligned} \nabla_{\mathbf{T}}\mathbf{T} &= \kappa \mathbf{N}, \\ \nabla_{\mathbf{T}}\mathbf{N} &= \kappa \mathbf{T} + \tau \mathbf{B}, \\ \nabla_{\mathbf{T}}\mathbf{B} &= -\tau \mathbf{N}, \end{aligned} \tag{3.1}$$

where κ is the curvature of γ, τ is its torsion and

$$\begin{aligned} g(\mathbf{T}, \mathbf{T}) &= -1, \; g(\mathbf{N}, \mathbf{N}) = 1, \; g(\mathbf{B}, \mathbf{B}) = 1, \\ g(\mathbf{T}, \mathbf{N}) &= g(\mathbf{T}, \mathbf{B}) = g(\mathbf{N}, \mathbf{B}) = 0. \end{aligned} \tag{3.2}$$

With respect to the orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ we can write

$$\begin{aligned} \mathbf{T} &= T_1 \mathbf{e}_1 + T_2 \mathbf{e}_2 + T_3 \mathbf{e}_3, \\ \mathbf{N} &= N_1 \mathbf{e}_1 + N_2 \mathbf{e}_2 + N_3 \mathbf{e}_3, \\ \mathbf{B} &= \mathbf{T} \times \mathbf{N} = B_1 \mathbf{e}_1 + B_2 \mathbf{e}_2 + B_3 \mathbf{e}_3. \end{aligned} \tag{3.3}$$

Theorem 3.1. (see Ref. [12]) $\gamma : I \longrightarrow \mathbb{E}(1,1)$ *is a non geodesic timelike biharmonic curve in the Lorentzian group of rigid motions* $\mathbb{E}(1,1)$ *if and only if*

$$\begin{aligned} \kappa &= \text{constant} \neq 0, \\ \kappa^2 - \tau^2 &= 1 + 2B_1^2, \\ \tau' &= -2N_1 B_1. \end{aligned} \tag{3.4}$$

Theorem 3.2. (see Ref. [12]) *Let* $\gamma : I \longrightarrow \mathbb{E}(1,1)$ *is a non geodesic timelike biharmonic general helix in the Lorentzian group of rigid motions* $\mathbb{E}(1,1)$. *Then, the parametric equations of* γ *are*

$$\begin{aligned} x^1(s) &= \cosh \Game \kappa s + \wp_3, \\ x^2(s) &= \frac{\sinh \Game e^{\cosh \Game \kappa s + \wp_3}}{2\left(\wp_1^2 + \cosh^2 \Game\right)} \{(\cosh \Game - \wp_1) \cos(\wp_1 \kappa s + \wp_2) \\ &\quad + (\cosh \Game + \wp_1) \sin(\wp_1 \kappa s + \wp_2)\} + \wp_4, \\ x^3(s) &= \frac{\sinh \Game e^{-\cosh \Game \kappa s - \wp_3}}{2\left(\wp_1^2 + \sinh^2 \Game\right)} \{-(\cosh \Game - \wp_1) \cos(\wp_1 \kappa s + \wp_2) \\ &\quad + (\cosh \Game + \wp_1) \sin(\wp_1 \kappa s + \wp_2)\} + \wp_5, \end{aligned} \tag{3.5}$$

where $\wp_1, \wp_2, \wp_3, \wp_4, \wp_5$ *are constants of integration.*

Hence, we have the following theorem.

Theorem 3.3. *Let* $\gamma : I \longrightarrow \mathbb{E}(1,1)$ *is a non geodesic timelike biharmonic general helix in the Lorentzian group of rigid motions* $\mathbb{E}(1,1)$*. Then, the vector equation of* γ *is*

$$\begin{aligned}
\gamma(s) \;=\;& (\cosh \eth \kappa s + \wp_3)\mathbf{e}_1 \\
&+[\frac{\sinh \eth}{2\left(\wp_1^2 + \cosh^2 \eth\right)}\{(\cosh \eth - \wp_1)\cos(\wp_1 \kappa s + \wp_2) \\
&+ (\cosh \eth + \wp_1)\sin(\wp_1 \kappa s + \wp_2)\} + \wp_4 e^{-\cosh \eth \kappa s - \wp_3} \\
&+ \frac{\sinh \eth}{2\left(\wp_1^2 + \cosh^2 \eth\right)}\{-(\cosh \eth - \wp_1)\cos(\wp_1 \kappa s + \wp_2) \\
&+ (\cosh \eth + \wp_1)\sin(\wp_1 \kappa s + \wp_2)\} + \wp_5 e^{\cosh \eth \kappa s + \wp_3}]\mathbf{e}_2 \\
&+[\frac{\sinh \eth}{2\left(\wp_1^2 + \cosh^2 \eth\right)}\{(\cosh \eth - \wp_1)\cos(\wp_1 \kappa s + \wp_2) \\
&+ (\cosh \eth + \wp_1)\sin(\wp_1 \kappa s + \wp_2)\} + \wp_4 e^{-\cosh \eth \kappa s - \wp_3} \\
&- \frac{\sinh \eth}{2\left(\wp_1^2 + \cosh^2 \eth\right)}\{-(\cosh \eth - \wp_1)\cos(\wp_1 \kappa s + \wp_2) \\
&+ (\cosh \eth + \wp_1)\sin(\wp_1 \kappa s + \wp_2)\} - \wp_5 e^{\cosh \eth \kappa s + \wp_3}]\mathbf{e}_3
\end{aligned} \tag{3.6}$$

where $\wp_1, \wp_2, \wp_3, \wp_4, \wp_5$ *are constants of integration.*

Proof. Assume that γ is a non geodesic timelike biharmonic general helix in the Lorentzian $\mathbb{E}(1,1)$. Using (2.1) yields

$$\frac{\partial}{\partial x^1} = \mathbf{e}_1,\ \frac{\partial}{\partial x^2} = e^{-x^1}(\mathbf{e}_2 + \mathbf{e}_3),\ \frac{\partial}{\partial x^3} = e^{x^1}(\mathbf{e}_2 - \mathbf{e}_3). \tag{3.7}$$

Substituting (3.7) in (3.5), we have (3.6) as desired.

4. Bertrand Mate of Timelike Biharmonic General Helices in the Lorentzian Group of Rigid Motions $\mathbb{E}(1,1)$

Definition 4.1. *A curve* $\gamma : I \longrightarrow \mathbb{E}(1,1)$ *with* $\kappa \neq 0$ *is called a Bertrand curve if there exist a curve* $\mathfrak{B} : I \longrightarrow \mathbb{E}(1,1)$ *such that the principal normal lines of* γ *and* $\mathfrak{B}$ *at* $s \in I$ *are equal. In this case* $\mathfrak{B}$ *is called a Bertrand mate of* γ*.*

Let $\gamma : I \longrightarrow \mathbb{E}(1,1)$ be a Bertrand curve. A Bertrand mate of γ is as follows:

$$\mathfrak{B}(s) = \gamma(s) + \lambda \mathbf{N}(s), \quad \forall s \in I, \tag{4.1}$$

where λ is constant.

Theorem 4.2. *Let $\gamma : I \longrightarrow \mathbb{E}(1,1)$ be a unit speed timelike biharmonic general helix and $\mathfrak{B}$ its Bertrand mate on $\mathbb{E}(1,1)$. Then, the equation of $\mathfrak{B}$ is:*

$$
\begin{aligned}
\mathfrak{B}(s) \quad = \quad & (\cosh\eth\kappa s + \wp_3 - \frac{2\lambda}{\kappa}\left(\sinh^2\eth\cos\left(\wp_1\kappa s + \wp_2\right)\sin\left(\wp_1\kappa s + \wp_2\right)\right))\mathbf{e}_1 \\
& +[\frac{\sinh\eth}{2\left(\wp_1^2 + \cosh^2\eth\right)}\{(\cosh\eth - \wp_1)\cos\left(\wp_1\kappa s + \wp_2\right) \\
& + (\cosh\eth + \wp_1)\sin\left(\wp_1\kappa s + \wp_2\right)\} + \wp_4 e^{-\cosh\eth\kappa s - \wp_3} \\
& + \frac{\sinh\eth}{2\left(\wp_1^2 + \cosh^2\eth\right)}\{-(\cosh\eth - \wp_1)\cos\left(\wp_1\kappa s + \wp_2\right) \\
& + (\cosh\eth + \wp_1)\sin\left(\wp_1\kappa s + \wp_2\right)\} + \wp_5 e^{\cosh\eth\kappa s + \wp_3} \\
& - \frac{\lambda}{\kappa}\sinh\eth\sin\left(\wp_1\kappa s + \wp_2\right)\left(\frac{1}{\wp_1\kappa} + \cosh\eth\right)]\mathbf{e}_2 \\
& +[\frac{\sinh\eth}{2\left(\wp_1^2 + \cosh^2\eth\right)}\{(\cosh\eth - \wp_1)\cos\left(\wp_1\kappa s + \wp_2\right) \\
& + (\cosh\eth + \wp_1)\sin\left(\wp_1\kappa s + \wp_2\right)\} + \wp_4 e^{-\cosh\eth\kappa s - \wp_3} \\
& - \frac{\sinh\eth}{2\left(\wp_1^2 + \cosh^2\eth\right)}\{-(\cosh\eth - \wp_1)\cos\left(\wp_1\kappa s + \wp_2\right) \\
& + (\cosh\eth + \wp_1)\sin\left(\wp_1\kappa s + \wp_2\right)\} - \wp_5 e^{\cosh\eth\kappa s + \wp_3} \\
& + \frac{\lambda}{\kappa}\sinh\eth\cos\left(\wp_1\kappa s + \wp_2\right)\left(\frac{1}{\wp_1\kappa} - \cosh\eth\right)]\mathbf{e}_3,
\end{aligned}
\tag{4.2}
$$

where $\wp_1, \wp_2, \wp_3, \wp_4, \wp_5$ are constants of integration.

Proof. From Theorem 3.2, we have

$$
\mathbf{T} = \cosh\eth\mathbf{e}_1 + \sinh\eth\cos\left(\wp_1\kappa s + \wp_2\right)\mathbf{e}_2 + \sinh\eth\sin\left(\wp_1\kappa s + \wp_2\right)\mathbf{e}_3. \tag{4.3}
$$

Using the first equation of (3.3) and basis, we have

$$
\nabla_{\mathbf{T}}\mathbf{T} = \left(T_1' - 2T_2T_3\right)\mathbf{e}_1 + \left(T_2' - T_1T_3\right)\mathbf{e}_2 + \left(T_3' - T_1T_2\right)\mathbf{e}_3. \tag{4.4}
$$

Hence, we express

$$
\begin{aligned}
\mathbf{N} \quad = \quad & -\frac{2}{\kappa}\left(\sinh^2\eth\cos\left(\wp_1\kappa s + \wp_2\right)\sin\left(\wp_1\kappa s + \wp_2\right)\right)\mathbf{e}_1 \\
& -\frac{1}{\kappa}\sinh\eth\sin\left(\wp_1\kappa s + \wp_2\right)\left(\frac{1}{\wp_1\kappa} + \cosh\eth\right)\mathbf{e}_2 \\
& +\frac{1}{\kappa}\sinh\eth\cos\left(\wp_1\kappa s + \wp_2\right)\left(\frac{1}{\wp_1\kappa} - \cosh\eth\right)\mathbf{e}_3.
\end{aligned}
\tag{4.5}
$$

From (4.1) and (4.5), by direct calculation we have (4.2), which proves the theorem.

Using Theorem 4.3, we can give the parametric equations of this curve.

Theorem 4.3. *Let $\gamma : I \longrightarrow \mathbb{E}(1,1)$ be a unit speed timelike biharmonic general helix and $\mathfrak{B}$ its Bertrand mate on $\mathbb{E}(1,1)$. Then, the parametric equation of $\mathfrak{B}$ are:*

$$x^1_{\mathfrak{B}}(s) = \cosh \eth \kappa s + \wp_3 - \frac{2\lambda}{\kappa}\left(\sinh^2 \eth \cos(\wp_1 \kappa s + \wp_2) \sin(\wp_1 \kappa s + \wp_2)\right),$$

$$\begin{aligned}
x^2_{\mathfrak{B}}(s) &= \frac{1}{2}\exp(\cosh \eth \kappa s + \wp_3 - \frac{2\lambda}{\kappa}\left(\sinh^2 \eth \cos(\wp_1 \kappa s + \wp_2) \sin(\wp_1 \kappa s + \wp_2)\right)) \\
&\quad [[\frac{\sinh \eth}{2\left(\wp_1^2 + \cosh^2 \eth\right)}\{(\cosh \eth - \wp_1)\cos(\wp_1 \kappa s + \wp_2) \\
&\quad + (\cosh \eth + \wp_1)\sin(\wp_1 \kappa s + \wp_2)\} + \wp_4 e^{-\cosh \eth \kappa s - \wp_3} \\
&\quad + \frac{\sinh \eth}{2\left(\wp_1^2 + \cosh^2 \eth\right)}\{-(\cosh \eth - \wp_1)\cos(\wp_1 \kappa s + \wp_2) \\
&\quad + (\cosh \eth + \wp_1)\sin(\wp_1 \kappa s + \wp_2)\} + \wp_5 e^{\cosh \eth \kappa s + \wp_3} \\
&\quad - \frac{\lambda}{\kappa}\sinh \eth \sin(\wp_1 \kappa s + \wp_2)\left(\frac{1}{\wp_1 \kappa} + \cosh \eth\right)] \\
&\quad + [\frac{\sinh \eth}{2\left(\wp_1^2 + \cosh^2 \eth\right)}\{(\cosh \eth - \wp_1)\cos(\wp_1 \kappa s + \wp_2) \\
&\quad + (\cosh \eth + \wp_1)\sin(\wp_1 \kappa s + \wp_2)\} + \wp_4 e^{-\cosh \eth \kappa s - \wp_3} \\
&\quad - \frac{\sinh \eth}{2\left(\wp_1^2 + \cosh^2 \eth\right)}\{-(\cosh \eth - \wp_1)\cos(\wp_1 \kappa s + \wp_2) \\
&\quad + (\cosh \eth + \wp_1)\sin(\wp_1 \kappa s + \wp_2)\} - \wp_5 e^{\cosh \eth \kappa s + \wp_3} \\
&\quad + \frac{\lambda}{\kappa}\sinh \eth \cos(\wp_1 \kappa s + \wp_2)\left(\frac{1}{\wp_1 \kappa} - \cosh \eth\right)]],
\end{aligned}$$

$$\begin{aligned}
x^3_{\mathfrak{B}}(s) &= \frac{1}{2}\exp(-\cosh \eth \kappa s - \wp_3 + \frac{2\lambda}{\kappa}\left(\sinh^2 \eth \cos(\wp_1 \kappa s + \wp_2) \sin(\wp_1 \kappa s + \wp_2)\right)) \\
&\quad [[\frac{\sinh \eth}{2\left(\wp_1^2 + \cosh^2 \eth\right)}\{(\cosh \eth - \wp_1)\cos(\wp_1 \kappa s + \wp_2) \\
&\quad + (\cosh \eth + \wp_1)\sin(\wp_1 \kappa s + \wp_2)\} + \wp_4 e^{-\cosh \eth \kappa s - \wp_3} \\
&\quad + \frac{\sinh \eth}{2\left(\wp_1^2 + \cosh^2 \eth\right)}\{-(\cosh \eth - \wp_1)\cos(\wp_1 \kappa s + \wp_2) \\
&\quad + (\cosh \eth + \wp_1)\sin(\wp_1 \kappa s + \wp_2)\} + \wp_5 e^{\cosh \eth \kappa s + \wp_3} \\
&\quad - \frac{\lambda}{\kappa}\sinh \eth \sin(\wp_1 \kappa s + \wp_2)\left(\frac{1}{\wp_1 \kappa} + \cosh \eth\right)]\mathbf{e}_2 \\
&\quad - [\frac{\sinh \eth}{2\left(\wp_1^2 + \cosh^2 \eth\right)}\{(\cosh \eth - \wp_1)\cos(\wp_1 \kappa s + \wp_2) \\
&\quad + (\cosh \eth + \wp_1)\sin(\wp_1 \kappa s + \wp_2)\} + \wp_4 e^{-\cosh \eth \kappa s - \wp_3} \\
&\quad - \frac{\sinh \eth}{2\left(\wp_1^2 + \cosh^2 \eth\right)}\{-(\cosh \eth - \wp_1)\cos(\wp_1 \kappa s + \wp_2)
\end{aligned}$$

$$+(\cosh \eth+\wp_1)\sin(\wp_1\kappa s+\wp_2)\}-\wp_5 e^{\cosh\eth\kappa s+\wp_3}$$
$$+\frac{\lambda}{\kappa}\sinh\eth\cos(\wp_1\kappa s+\wp_2)\left(\frac{1}{\wp_1\kappa}-\cosh\eth\right)]],$$

where $\wp_1, \wp_2, \wp_3, \wp_4, \wp_5$ *are constants of integration.*

5. Conclusions

Curves of constant curvature or torsion constitute particular Bertrand curves. Bertrand curves are well studied classical curves and may be defined by their property that any Bertrand curve shares its principal normals with another Bertrand curve.

The helix is one of the most fascinating curves in Science and Nature. Scientists have long held a fascination, sometimes bordering on mystical obsession, for helical structures in nature. Therefore, we constructed a Bertrand mate of timelike biharmonic general helices in the Lorentzian group of rigid motions $\mathbb{E}(1,1)$. Moreover, we found out their explicit parametric equations.

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