

A FEW SPLITTING THEOREMS FOR RANK 2 VECTOR BUNDLES ON \mathbf{P}^4

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ABSTRACT. The paper investigates vanishing conditions on the intermediate cohomology of a normalized rank 2 vector bundle \mathcal{F} on \mathbf{P}^4 which force \mathcal{F} to split or, at least, to be a non-stable bundle (with few possible exceptions). The results are applied to see when subcanonical surfaces in \mathbf{P}^4 are forced to be complete intersections of two hypersurfaces, since subcanonical surfaces are zero loci of non-zero sections of rank 2 vector bundles.

1. Introduction

In [8], Theorem 15, it is proved that a normalized rank 2 vector bundle \mathcal{F} on \mathbf{P}^4 splits whenever one of the following conditions holds:

- a. $h^1\mathcal{F}(-1) = 0$ and $h^2\mathcal{F}(-2) \leq h^2\mathcal{F}(-1)$,
- b. $h^1\mathcal{F} = 0$ and $h^2\mathcal{F}(-1) \leq h^2\mathcal{F}$,
- c. $h^1\mathcal{F}(1) = 0$, $h^2\mathcal{F} \leq h^2\mathcal{F}(1)$ and $c_1 = -1$.

In the present paper we go into further investigation and examine new vanishing conditions on the intermediate cohomology of a normalized rank 2 vector bundle \mathcal{F} on \mathbf{P}^4 which force \mathcal{F} to split or, at least, to be a non-stable bundle (with few possible exceptions).

In particular we consider the following three cases:

- d. $h^2\mathcal{F}(-2) = 0$ (here we can prove that \mathcal{F} is forced to be non-stable),
- e. $h^1\mathcal{F}(0) = h^2\mathcal{F}(0) = 0$ (\mathcal{F} is forced to split with two exceptions),
- f. $h^1\mathcal{F}(-1) = h^2\mathcal{F}(-1) = 0$ (\mathcal{F} is forced to split with two exceptions).

These cases are new with respect to [8] and extend to \mathbf{P}^4 results of [2], [1] and [7]. The proofs can be obtained using three main tools: the properties of the Euler characteristic function of a vector bundle, the technique of passing to the general hyperplane restriction, the good properties of the spectrum of a rank 2 vector bundle on \mathbf{P}^3 .

We observe that, if a surface X in \mathbf{P}^4 is a -subcanonical, i.e. it is the zero locus of a non-zero section of a rank 2 vector bundle, then the above results can be applied to X ; therefore we can see that the vanishing of some strategic intermediate cohomology forces a surface to be a complete intersection. We want to emphasize that techniques concerning only vector bundles are useful to obtain results on surfaces.

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2. Notations and definitions

Unless otherwise explicitly stated:

1) k is an algebraically closed field of characteristic 0 and \mathbf{P}^n is the n -dimensional projective space over k . For every coherent sheaf \mathcal{G} on \mathbf{P}^n , $h^i \mathcal{G}(t)$ is the dimension of the k -vector space $H^i(\mathcal{G} \otimes \mathcal{O}_{\mathbf{P}^n}(t))$.

2) \mathcal{F} is a rank 2 vector bundle on \mathbf{P}^n and \mathcal{F}_H is its restriction to a general hyperplane $H = \mathbf{P}^{n-1}$: the two sheaves are linked by the standard exact sequence:

$$(1) \quad 0 \rightarrow \mathcal{F}(-1) \rightarrow \mathcal{F} \rightarrow \mathcal{F}_H \rightarrow 0$$

3) c_1 and c_2 are the first and second Chern class of \mathcal{F} , which we always suppose normalized so that c_1 is either 0 or -1 (Chern classes will always be treated as whole numbers); for every integer t the Euler characteristic function of $\mathcal{F}(t)$ is given by the following formula:

$$(2) \quad \chi(\mathcal{F}(t)) = 2 \binom{t+4}{4} - c_1 \binom{t+3}{3} - c_2 \binom{t+3}{2} - \frac{c_1 c_2 (t+2)}{2} + \frac{c_2 (c_2 + 1 - c_1)}{12}$$

where $\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!}$ for every $k \geq 1$ and every $n \in \mathbf{Z}$; \mathcal{F}^\vee denotes the dual of \mathcal{F} and $\mathcal{F}^\vee \cong \mathcal{F}(-c_1)$ for any normalized rank 2 bundle on \mathbf{P}^4 so that $h^i \mathcal{F}(t) = h^{4-i} \mathcal{F}(-t - c_1 - 5)$;

4) $\alpha = \alpha(\mathcal{F})$ and $\beta = \beta(\mathcal{F})$ ($\alpha \leq \beta$) are the smallest degrees of two independent generators of $H_*^0 \mathcal{F} = \bigoplus_n H^0 \mathcal{F}(n)$; \mathcal{F} is stable if $\alpha \geq 1$, semistable if $\alpha \geq -c_1$, strictly semistable if $c_1 = \alpha = 0$ and non-stable if $\alpha \leq -c_1 - 1$ (see [3] Lemma 3.1);

5) $r = r(\mathcal{F})$ is the integer $-\alpha - c_1$ if \mathcal{F} is non-stable and 0 if \mathcal{F} is semistable; the number $\delta = c_2 + c_1 r + r^2$ is strictly positive for every non-split bundle;

6) the spectrum of a normalized rank 2 bundle \mathcal{E} on \mathbf{P}^3 is the unique set of δ integers $\{k_i\}_{i=1, \dots, \delta}$ with the following property (see [3] Theorem 7.1 and [9] Theorem 3.1):

$$h^1(\mathcal{E}(l)) = \bigoplus_{i=1}^{\delta} h^0(\mathcal{O}_{\mathbf{P}^1}(k_i + l + 1))$$

for every $l \leq r - 1$;

7) X is a locally Cohen-Macaulay (possibly reducible or non-reduced) codimension 2 subvariety of \mathbf{P}^4 ; we will use the standard exact sequence:

$$(3) \quad 0 \rightarrow \mathcal{I}_X \rightarrow \mathcal{O}_{\mathbf{P}^4} \rightarrow \mathcal{O}_X \rightarrow 0$$

8) X is a -subcanonical if the dualizing sheaf ω_X is isomorphic to $\mathcal{O}_X(a)$ for some integer a . For an a -subcanonical surface X in \mathbf{P}^4 , we denote by t and v the only integers such that either $a = 2t - 5$ or $a = 2t - 6$ and either $a = 2v + 1$ or $a = 2v + 2$;

9) $s = \min\{n / h^0 \mathcal{I}_X(n) \neq 0\}$ is the least degree of a hypersurface containing X ;

We will use the following well-known facts:

10) the spectrum of a rank 2, normalized vector bundle \mathcal{E} on \mathbf{P}^3 is symmetric with respect to $-\frac{c_1}{2}$; if \mathcal{E} is stable or semistable, then it is connected, except possibly for a gap at 0 if \mathcal{E} is strictly semistable; if \mathcal{E} is non-stable and non-split, then $-r - 1$ belongs to the spectrum

and the subset $\{k_i \leq -r - 1\}$ is connected. (see [3], Theorem 7.1 and Proposition 7.2 and [9], Theorem 3.1 and Proposition 3.5).

11) If X is an a -subcanonical surface in \mathbf{P}^4 and t is the integer above defined, then there are a normalized rank 2 vector bundle \mathcal{F} and a global section of $\mathcal{F}(t)$ whose zero locus is X and the following sequence is exact:

$$(4) \quad 0 \rightarrow \mathcal{O}_{\mathbf{P}^4} \rightarrow \mathcal{F}(t) \rightarrow \mathcal{I}_X(2t + c_1) \rightarrow 0.$$

12) If X is the zero locus of a global section of $\mathcal{F}(t)$, then either $t = \alpha$ or $t \geq \beta$ (no codimension 2 subvariety corresponds to sections of $\mathcal{F}(t)$ if either $t < \alpha$ or $\alpha < t < \beta$) (see [6], Lemma 1 and Remark 2); moreover, if $t = \alpha < 0$, then X is non-reduced; the minimal degree of a hypersurface containing X is:

$$\begin{aligned} s &= \alpha + \beta + c_1 \text{ if } t = \alpha \\ s &= t + \alpha + c_1 \text{ if } t \geq \beta. \end{aligned}$$

13) If X is an integral degree d surface in \mathbf{P}^4 , and Y is its general hyperplane section, then Y is an integral curve (by the second theorem of Bertini) of degree d in \mathbf{P}^3 and the following sequence is exact:

$$(5) \quad 0 \rightarrow \mathcal{I}_X(-1) \rightarrow \mathcal{I}_X \rightarrow \mathcal{I}_Y \rightarrow 0;$$

if, moreover, X is a -subcanonical, then Y is $(a + 1)$ -subcanonical.

14) If $h^1\mathcal{F}(t_0) = 0$, for some $t_0 \leq 0$, then $h^1\mathcal{F}(t) = 0$, for every $t < t_0$ (this fact can be seen using the cohomology sequence of the exact sequence (1) and it holds more generally for reflexive sheaves on \mathbf{P}^n ; see also [8] Proposition 11).

3. Vanishing and Splitting Theorems for Rank 2 Vector Bundles

In the present section we state and prove three theorems, connecting the vanishing, in a strategic position, of the first and/or second cohomology module of the rank 2 vector bundle \mathcal{F} with the property of being split or not semistable.

Theorem 3.1. *Let \mathcal{F} be a non-split, normalized rank 2 vector bundle on \mathbf{P}^4 such that $h^2\mathcal{F}(-2) = 0$. Then \mathcal{F} is non-stable.*

More precisely, $\alpha \leq -3 - c_1$.

Proof: We consider separately the case $c_1 = 0$ and the case $c_1 = -1$.

Case a. Assume that $c_1 = 0$. Using duality and the hypothesis $H^2\mathcal{F}(-2) = 0$, we have: $\chi(\mathcal{F}(-2)) = h^0\mathcal{F}(-2) - h^1\mathcal{F}(-2) - h^1\mathcal{F}(-3) + h^0\mathcal{F}(-3) = \frac{(c_2^2 + c_2)}{12}$.

Therefore, \mathcal{F} cannot be stable, because in that case $c_2 \geq 1$ while $\chi(\mathcal{F}(-2)) \leq 0$

If $\alpha \leq -1$, using the spectrum of \mathcal{F}_H (H being a general hyperplane), we obtain $h^1\mathcal{F}(-2) \geq h^1\mathcal{F}_H(-2) \geq -\alpha + 1$; thus if $\alpha = -1$ or $\alpha = -2$ we get $\chi(\mathcal{F}(-2)) < 0$ while $\frac{(c_2^2 + c_2)}{12} \geq 0$ for every integer c_2 .

Case b. Assume that $c_1 = -1$. Using duality and the hypothesis $h^2\mathcal{F}(-2) = 0$, we have: $\chi(\mathcal{F}(-2)) = 2h^0\mathcal{F}(-2) - 2h^1\mathcal{F}(-2) = \frac{(c_2^2 + 2c_2)}{12}$.

As in the previous case, α must be strictly negative, because if either \mathcal{F} is stable or $\alpha = 0$, then c_2 is at least 1, while $\chi(\mathcal{F}(-2)) = -2h^1\mathcal{F}(-2) \leq 0$

Finally, let $\alpha = -1$; then $h^1\mathcal{F}(-2) = 0$ and $c_2 = 0$ ($c_2 = -2$ is not allowed because $\delta = c_2 + 2 > 0$). The minimal section of \mathcal{F} corresponds to a degree 2 subcanonical surface X . If H is a general hyperplane, $Y = X \cap H$ is a degree $c_2 + c_1\alpha + \alpha^2 = 2$ curve and then, by [5] Proposition 1.4, it is a double structure on a line L whose ideal sheaf is given by:

$$(6) \quad 0 \rightarrow \mathcal{I}_Y \rightarrow \mathcal{I}_L \rightarrow \mathcal{O}_L(5) \rightarrow 0.$$

By the cohomology exact sequences (3), (5) and (6) we get:

$$h^2\mathcal{F}(-1) = h^2\mathcal{I}_X(-3) \leq h^2\mathcal{I}_X(-4) + h^2\mathcal{I}_Y(-3) = h^2\mathcal{F}(-2) + h^1\mathcal{O}_L(2) = 0.$$

Thus, $\chi(\mathcal{F}(-1)) \leq 1 - h^1\mathcal{F}(-1) = 1 - h^1\mathcal{I}_X(-3) = 1 - h^1\mathcal{I}_Y(-3) = 1 - h^0\mathcal{O}_L(2) = -2$ and, on the other hand, using Chern classes, we get $\chi(\mathcal{F}(-1)) = 0$, which is a contradiction. \diamond

Theorem 3.2. *Let \mathcal{F} be a normalized rank 2 vector bundle on \mathbf{P}^4 such that $h^1\mathcal{F} = h^2\mathcal{F} = 0$. Then either \mathcal{F} splits or one of the following conditions holds:*

- i) $c_1 = 0$, $\alpha = -1$ and $c_2 = 36$,
- ii) $c_1 = -1$, $\alpha = 0$ and $c_2 = 22$.

Proof: We consider separately the case $c_1 = 0$ and the case $c_1 = -1$.

Case a. Assume that $c_1 = 0$. Then (see Notations, **14**) we have: $\chi\mathcal{F} = h^0\mathcal{F} + h^0\mathcal{F}(-5) = 2 + \frac{(c_2^2 - 35c_2)}{12}$.

Obviously $\alpha > 0$ is not allowed because the equation $2 + \frac{(c_2^2 - 35c_2)}{12} = 0$ has no whole number as a solution; $\alpha = 0$ is not allowed either because the equation $1 + \frac{(c_2^2 - 35c_2)}{12} = 0$ has no whole number as a solution.

Therefore we must have: $\alpha \leq -1$ and:

$$\chi\mathcal{F} = h^0\mathcal{F} + h^0\mathcal{F}(-5) = h^0\mathcal{O}_{\mathbf{P}^4}(-\alpha) + h^0\mathcal{O}_{\mathbf{P}^4}(-\alpha - 5) = \frac{(\alpha^4 + 35\alpha^2 + 24)}{12}.$$

On the other hand, using Chern classes:

$$\chi\mathcal{F} = 2 + \frac{(c_2^2 - 35c_2)}{12};$$

whence $\alpha^2 + 35 = c_2$. Moreover $h^1\mathcal{F}(-4) = h^2\mathcal{F}(-5) = 0$ implies that $h^1\mathcal{F}_H(-4) = 0$, i.e., using the spectrum of \mathcal{F}_H (see [9] and Notations **10**), $\sum_{k_i \geq 3} (k_i - 2) = 0$ and so no $k_i \geq 3$ can exist in the spectrum. But, if $\alpha \leq -2$, there is at least one $k_i = 3$ in the spectrum (see [9] and Notations **10**), since $r + 1 \geq 3$ belongs to the spectrum of \mathcal{F}_H .

Therefore $\alpha = -1$ and $c_2 = 36$.

Case b. Assume that $c_1 = -1$. Then (see Notations, **14**) we have:

$$\chi\mathcal{F} = h^0\mathcal{F} + h^0\mathcal{F}(-4) = 1 + \frac{(c_2^2 - 22c_2)}{12}.$$

Obviously $\alpha > 0$ is not allowed because the equation $1 + \frac{c_2^2 - 22c_2}{12} = 0$ has no whole number as a solution.

If $\alpha = 0$, then $1 + \frac{c_2^2 - 22c_2}{12} = 1$ and therefore $c_2 = 22$ (0 being excluded).

If $\alpha < 0$, then $h^1\mathcal{F}(0) = h^2\mathcal{F}(0) = 0$, hence $h^1\mathcal{F}(-3) = h^2\mathcal{F}(-4) = 0$ and so $h^1\mathcal{F}_H(-3) = 0$, which implies that no $k_i \geq 2$ can belong to the spectrum (see [9] and Notations **10**). But $r = 1 - \alpha$ hence $r = 1 - \alpha \geq 2$ belongs to the spectrum, which is a contradiction. Therefore $\alpha < 0$ is not allowed. \diamond

Theorem 3.3. *Let \mathcal{F} be a normalized rank 2 vector bundle on \mathbf{P}^4 such that $h^1\mathcal{F}(-1) = h^2\mathcal{F}(-1) = 0$. Then either \mathcal{F} splits or one of the following conditions holds:*

- i) $c_1 = 0$, $\alpha \geq 0$ and $c_2 = 11$,
- ii) $c_1 = -1$, $\alpha \geq 0$ and $c_2 = 4$.

Proof: We consider separately the case $c_1 = 0$ and the case $c_1 = -1$.

Case a. Assume that $c_1 = 0$ and $\alpha < 0$. Then (see Notations, **14**) $h^1\mathcal{F}(-3) = h^2\mathcal{F}(-4) = 0$ implies that $h^1\mathcal{F}_H(-3) = 0$, hence the spectrum of \mathcal{F}_H cannot contain any $k_i \geq 2$, which is absurd, because $r \geq 1$ (see [9] and Notations **10**).

If $\alpha \geq 0$, then $\chi(\mathcal{F}(-1)) = h^0\mathcal{F}(-1) + h^0\mathcal{F}(-4) = \frac{c_2^2 - 11c_2}{12} = 0$. But this implies $c_2 = 11$ (0 being excluded).

Case b. Assume that $c_1 = -1$ and $\alpha < 0$. Then (see Notations, **14**) $h^1\mathcal{F}(-3) = h^2\mathcal{F}(-3) = 0$ implies that $h^1\mathcal{F}_H(-2) = 0$, hence the spectrum of \mathcal{F}_H cannot contain any $k_i \geq 1$, which is absurd, because $r \geq 2$ (see [9] and Notations **10**).

If $\alpha \geq 0$, then $\chi\mathcal{F}(-1) = h^0\mathcal{F}(-1) + h^0\mathcal{F}(-3) = \frac{c_2^2 - 4c_2}{12} = 0$. But this implies $c_2 = 4$ (0 being excluded). \diamond

Remark 3.4. *We do not know whether non-split vector bundles as those described in the three theorems above really exist, But we recall that in [4] splitting criteria are given for a vector bundle on \mathbf{P}^n (so also \mathbf{P}^4), in terms of the vanishing of certain cohomology modules. In particular the paper (Theorem 2) proves that a non-split rank two vector bundle on \mathbf{P}^4 is a Horrocks-Mumford bundle whenever its second cohomology module is Buchsbaum, so restricting the possible range of non-split rank two vector bundles on \mathbf{P}^4 .*

4. Vanishing Properties of the Intermediate Cohomology and Complete Intersection Surfaces.

Let X be an a -subcanonical surface in \mathbf{P}^4 and consider the corresponding \mathcal{F} , t and v as in Notations **8**. It is well known that X is a complete intersection of two hypersurfaces if and only if \mathcal{F} splits.

Theorem 4.1. *Let X be an a -subcanonical surface in \mathbf{P}^4 and let s be the smallest degree of a hypersurface containing X .*

If $H^1\mathcal{O}_X(v+1) = 0$ (or equivalently $H^2\mathcal{I}_X(v+1) = 0$), then either X is a complete intersection or one of the following conditions holds:

- i) $a \leq -10$ and X is non-reduced
- ii) $a \geq 4$ and $s \leq v+1$ ($s \leq v$ when a is even).

Proof: Let \mathcal{F} be the rank 2 vector bundle such that X is the zero locus of a section of $\mathcal{F}(t)$. If \mathcal{F} is split, then X is a complete intersection. Otherwise $a = 2t + c_1 - 5 = 2v + 1 - c_1$

and, using (3) and (4), we obtain $0 = h^1\mathcal{O}_X(v+1) = h^2\mathcal{I}_X(v+1) = h^2\mathcal{I}_X(t+c_1-2) = h^2\mathcal{F}(-2)$. Thanks to Theorem 3.1 we know that \mathcal{F} is non-stable with $\alpha \leq -3 - c_1$.

If $t = \alpha$, then X is non-reduced and $a = 2\alpha + c_1 - 5 \leq 2(-3 - c_1) + c_1 - 5 \leq -10$.

If $t \geq \beta$, then $a = 2t + c_1 - 5 \geq 2(-\alpha - c_1 + 2) + c_1 - 5 \geq 2(3 + 2) + c_1 - 5 \geq 4$.

Moreover, the minimal degree of a hypersurface containing X is $t + \alpha + c_1 \leq v + 3 - c_1 - 3 - c_1 + c_1 = v - c_1$. \diamond

Theorem 4.2. *Let X be an a -subcanonical surface in \mathbf{P}^4 of degree d and let s be the smallest degree of a hypersurface containing X .*

Assume that $h^1\mathcal{I}_X(v+3) = h^2\mathcal{I}_X(v+3) = 0$. Then one of the following conditions holds:

- (i) X is non-reduced, $a = -7$ and $d = 37$;
- (ii) X is non-reduced, $a = -6$ and $d = 22$;
- (iii) $a \geq 1$ is odd, $d = v^2 + 6v + 45$ and $s = v + 2$;
- (iv) $a \geq 0$ is even, $d = v^2 + 7v + 34$ and $s = v + 3$.

Proof: It is a consequence of Theorem 3.2; in any event the proof can follow the lines of Theorem 4.1 \diamond

Theorem 4.3. *Let X be an a -subcanonical surface in \mathbf{P}^4 of degree d and let s be the smallest degree of a hypersurface containing X .*

Assume that $h^1\mathcal{I}_X(v+2) = h^2\mathcal{I}_X(v+2) = 0$. Then one of the following conditions holds:

- (i) $a \geq -5$ is odd, $d = v^2 + 6v + 20$ and $s \geq v + 3$;
- (ii) $a \geq -6$ is even, $d = v^2 + 7v + 16$ and $s \geq v + 3$.

Proof: It is a consequence of Theorem 3.3; in any event the proof can follow the lines of Theorem 4.1 \diamond

Remark 4.4. *We do not know whether non complete intersection surfaces as those described in the three theorems above really exist.*

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