STRUCTURE OF PHASE MATRICES OF LIGHT SCATTERING PARTICLES DERIVED FROM SYMMETRY CONSIDERATIONS

J. W. Hovenier[a]* and O. Muñoz[b]

(Invited paper)

ABSTRACT. Symmetry considerations are used to deduce equations from which the main structure of the phase matrix of one or more light scattering particles is derived for special directions of incident and scattered light. For this purpose we use symmetry relations with a wide range of validity, as well as rotational symmetry about a vertical axis in a three dimensional coordinate system.

1. Introduction

Suppose a particle or collection of independently scattering particles is located at the origin of a Cartesian coordinate system with axes x, y and z (see Fig. 1). The 4×4 matrix transforming the Stokes parameters of an incident beam into those of a scattered beam is called the phase matrix. Here the meridian planes of the beams are the planes of reference for their Stokes parameters [1]. The phase matrix depends in general on the angles θ and φ of the scattered beam and θ' and φ' of the incident beam. We assume that there is rotational symmetry about the z-axis. Thus the azimuth dependence of the phase matrix is reduced to the difference φ - φ'. Writing the Stokes parameters I, Q, U and V of a beam of light as elements of a column vector **I** we have

$$\mathbf{I}^s(u,\varphi) = c\mathbf{Z}(u,u',\varphi-\varphi')\mathbf{I}^i(u',\varphi').$$ (1)

Here the superscripts s and i refer to the scattered and incident beam, respectively, $u = -\cos\theta$ and $u' = -\cos\theta'$. Furthermore, c is a scalar and $\mathbf{Z}(u,u',\varphi-\varphi')$ is the phase matrix of the particle or collection of particles at the origin.

In earlier works [2]-[4] we have shown that in many cases the phase matrix obeys the following seven symmetry relations.

$$\mathbf{Z}(-u',-u,\varphi'-\varphi) = \mathbf{P}\tilde{\mathbf{Z}}(u,u',\varphi-\varphi')\mathbf{P},$$ (2)

$$\mathbf{Z}(u,u',\varphi'-\varphi) = \mathbf{PQZ}(u,u',\varphi-\varphi')\mathbf{QP},$$ (3)

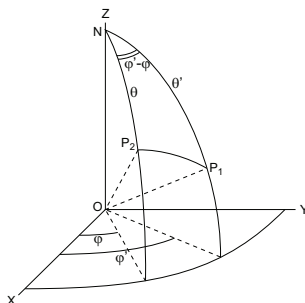$$\mathbf{Z}(-u,-u',\varphi'-\varphi) = \mathbf{Z}(u,u',\varphi-\varphi'),$$ (4)

Figure 1. Scattering by a particle or collection of particles located at O. Points N, P_1 and P_2 are situated on a unit sphere. The direction of the incident light is OP_1 and that of the scattered light is OP_2. The azimuthal angles φ and φ' are measured clockwise looking in the direction of the positive z-axis.

$$\mathbf{Z}(-u', -u, \varphi - \varphi') = \mathbf{Q}\tilde{\mathbf{Z}}(u, u', \varphi - \varphi')\mathbf{Q}, \tag{5}$$

$$\mathbf{Z}(u', u, \varphi - \varphi') = \mathbf{P}\tilde{\mathbf{Z}}(u, u', \varphi - \varphi')\mathbf{P}, \tag{6}$$

$$\mathbf{Z}(-u, -u', \varphi - \varphi') = \mathbf{PQZ}(u, u', \varphi - \varphi')\mathbf{QP}, \tag{7}$$

$$\mathbf{Z}(u', u, \varphi' - \varphi) = \mathbf{Q}\tilde{\mathbf{Z}}(u, u', \varphi - \varphi')\mathbf{Q}, \tag{8}$$

where the matrix $\mathbf{P} = \text{diag } (1,1,-1,1)$, the matrix $\mathbf{Q} = \text{diag}(1,1,1,-1)$, and a tilde above a matrix symbol stands for the transposed matrix. Relation (2) is due to reciprocity, relation (3) to mirror symmetry with respect to a meridian plane and relation (4) to turning the particle or collection of particles upside down. The other four relations are combinations of relations (2)-(4). It is important to note that relations (2)-(8) hold for all directions of the incident and scattered beams and thus are general as far as directions in three dimensional space are concerned. Some examples of cases when the seven symmetry relations hold for a single particle are a spherically symmetric particle and a rotationally symmetric particle (e.g. a cylinder, spheroid, bisphere composed of two identical spheres, Chebyshev particle with even degree n) with its rotational axis along the z-direction and with a plane of symmetry parallel to the x-y plane. The seven symmetry relations also hold for a collection of particles, each having a plane of symmetry, that are randomly or horizontally oriented or fluttering with respect to the x-y plane. In all cases we assume that the dielectric, permeability and conductivity tensors of the particles are symmetric and magnetic fields can be ignored, as is often the case. The main purpose of this work is to deduce the main structure of the phase matrix from symmetry relations

2. Structure equations

We first consider the special directions defined by $u = u'$ while $\varphi - \varphi'$ is arbitrary. Eq. (6) shows that

$$\mathbf{Z}(u, u, \varphi - \varphi') = \mathbf{P}\tilde{\mathbf{Z}}(u, u, \varphi - \varphi')\mathbf{P}. \tag{9}$$

This is an equation for $\mathbf{Z}(u, u, \varphi - \varphi')$ with an infinite number of matrix solutions. However, it is still very useful since it gives information about the structure of $\mathbf{Z}(u, u, \varphi - \varphi')$. Therefore, we shall call such an equation a structure equation. Writing out Eq. (9) shows that $\mathbf{Z}(u, u, \varphi - \varphi')$ is of the type

$$\begin{pmatrix} p_1 & q_1 & q_3 & q_5 \\ q_1 & p_2 & q_4 & q_6 \\ -q_3 & -q_4 & p_3 & q_2 \\ q_5 & q_6 & -q_2 & p_4 \end{pmatrix} \tag{10}$$

so that there are at most 10 different elements.

Let us now turn our attention to all directions of incidence and scattering in the special meridian plane given by $\varphi - \varphi' = 0$. We find from the mirror symmetry equation (3) the following structure equation

$$\mathbf{Z}(u, u', 0) = \mathbf{PQZ}(u, u', 0)\mathbf{QP}. \tag{11}$$

Writing out Eq. (11) we see that eight elements of the phase matrix $\mathbf{Z}(u, u', 0)$ must be identically equal to zero, since they differ in sign on the left hand and right hand side of Eq. (11), respectively, but have the same absolute value. Consequently, the structure of $\mathbf{Z}(u, u', 0)$ is block diagonal, i.e. of the type

$$\begin{pmatrix} p_1 & q_1 & 0 & 0 \\ r_1 & p_2 & 0 & 0 \\ 0 & 0 & p_3 & q_2 \\ 0 & 0 & r_2 & p_4 \end{pmatrix}. \tag{12}$$

The same structure is readily found for $\mathbf{Z}(u, u', \pi)$, taking into account that this matrix equals $\mathbf{Z}(u, u', -\pi)$. Hence the phase matrix in the principal plane is block diagonal and is composed of at most eight different functions of u and u' that are not identical to zero. Clearly this structure is easily understood as a result of mirror symmetry with respect to the principal plane.

We now consider the case that $u = u'$ and $\varphi - \varphi' = 0$. This corresponds to scattering in the exact forward direction. We then find the additional structure equation

$$\mathbf{Z}(u, u, 0) = \mathbf{P}\tilde{\mathbf{Z}}(u, u, 0)\mathbf{P}, \tag{13}$$

as follows from Eq. (6). Combining Eq. (12) and Eq. (13) we find that $\mathbf{Z}(u, u, 0)$ is not only block diagonal but also has $r_1 = q_1$ and $r_2 = -q_2$, so that it contains at most six different functions of u that are not identically equal to zero.

Another interesting case occurs when $\varphi - \varphi' = \pi$ and $u = -u'$, so for scattering in the exact backward direction. Then Eq. (5) yields the structure equation

$$\mathbf{Z}(u, -u, \pi) = \mathbf{Q}\tilde{\mathbf{Z}}(u, -u, \pi)\mathbf{Q}. \tag{14}$$

Combining this with Eq. (12) shows that $\mathbf{Z}(u, -u, \pi)$ is also block diagonal with $r_1 = q_1$ and $r_2 = -q_2$.

For light travelling in the positive or negative z-direction there is no unique plane of reference. We can handle this by first choosing a meridian plane and then letting u and/or u' tend to plus or minus one, as the case may be [5]. For example, if we let u and u' both tend to unity in the meridian plane given by $\varphi - \varphi' = 0$, the incident light is travelling perpendicularly downward and scattered in the exact forward direction. Writing the corresponding phase matrix as $\mathbf{Z}(1,1,0)$ the rotational symmetry about the z-axis of our scattering problems gives the structure equation

$$\mathbf{Z}(1,1,0) = \mathbf{L}(\varphi - \varphi')\mathbf{Z}(1,1,0)\mathbf{L}(\varphi' - \varphi). \tag{15}$$

Here the rotation matrix

$$\mathbf{L}(\alpha) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & C & S & 0 \\ 0 & -S & C & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{16}$$

where $C = \cos 2\alpha$ and $S = \sin 2\alpha$.

Eq. (15) can easily be understood because it is physically clear that the scattered light should have the same intensity and state of polarization if we had used the meridian planes with azimuthal angles φ' and φ as the plane of reference for the Stokes parameters of the incident beam and scattered beam, respectively. Writing out Eq. (15) shows that $\mathbf{Z}(1,1,0)$ must be a diagonal matrix of the type diag (p_1, p_2, p_2, p_4). Similarly, we find for incident light travelling perpendicularly downward and scattered in the exact backward direction the structure equation

$$\mathbf{Z}(-1,1,0) = \mathbf{L}(\varphi' - \varphi)\mathbf{Z}(-1,1,0)\mathbf{L}(\varphi' - \varphi), \tag{17}$$

which gives the simple structure diag $(p_1, p_2, -p_2, p_4)$ for $\mathbf{Z}(-1,1,0)$.

3. Conclusions

We have shown that symmetry arguments can be used in a simple and general way to deduce the main structure of a variety of phase matrices. In particular, we have determined the maximum number of different elements that are not identically equal to zero. Several examples have been given in this extended abstract. A more detailed treatment, with more examples, will be provided in a regular paper.

References

[1] J.W. Hovenier, C.V.M. van der Mee and H. Domke. *Transfer of polarized light in planetary atmospheres: basic concepts and practical methods*. Kluwer, Springer, Dordrecht. Berlin, 2004.

[2] J.W. Hovenier. Symmetry relationships for scattering of polarized light in a slab of randomly oriented particles. J. Atmos. Sci. 26, 488-499, 1969.

[3] J. W. Hovenier and O. Muñoz. Light scattering by horizontally oriented particles: Symmetry properties of the phase matrix. In: *Electromagnetic and Light Scattering XII. Conference Proceedings*, 2010, pp.78 -81.

[4] J.W. Hovenier and O. Muñoz. Symmetry relations for the phase matrix of horizontally oriented particles. JQSRT 2010;111:1449 -1453.

[5] J.W. Hovenier and J.F. de Haan. Polarized light in planetary atmospheres for perpendicular directions. Astron. Astrophys. 1985;146: 185 -191.

[a]   Astronomical Institute "Anton Pannekoek"
      University of Amsterdam
      P.O. Box 94249
      1090GE Amsterdam, The Netherlands

[b]   Instituto de Astrofísica de Andalucía, CSIC
      Glorieta de la Astronomía s/n
      18080 Granada, Spain

[*]   To whom correspondence should be addressed | Email: J.W.Hovenier@uva.nl