DOI: 10.1478/AAPP.91S1A17

AAPP | Atti della Accademia Peloritana dei Pericolanti
Classe di Scienze Fisiche, Matematiche e Naturali
ISSN 1825-1242

Vol. 91, Suppl. No. 1, A17 (2013)

ON THE STABILITY OF NONAUTONOMOUS BINARY DYNAMICAL SYSTEMS OF PARTIAL DIFFERENTIAL EQUATIONS

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ABSTRACT. Nonlinear nonautonomous binary reaction-diffusion dynamical systems of partial differential equations (PDE) are considered. Stability criteria - via a nonautonomous L^2-energy - are obtained. Applications to nonautonomous Lotka-volterra systems of PDEs and to "preys" struggle for the life, are furnished.

To Giuseppe Grioli for his hundredth birthday

1. Introduction

In [1]-[2] the nonlinear stability of the null solution of the nonautonomous binary dynamical systems

$$\begin{cases} \dot{x} = a(t)x + b(t)y + f(x,y,t), \\ \\ \dot{y} = c(t)x + d(t)y + g(x,y,t), \end{cases} \tag{1}$$

with f and g nonlinear functions such that $f(0,0,t) = g(0,0,t) = 0$, has been studied. Successively in [3] the asymptotic behaviour of solutions to the nonautonomous nonlinear third order PDE

$$u_{tt} + a(t)u_t = b(t)u_{xx} + c(t)u_{xxt} + F(u), \tag{2}$$

(with $F(u)$ nonlinear and such that $F(0) = 0$) modeling several phenomena, has been analyzed and, in particular, the stability of the null solution has been studied. It is to remark that, setting

$$u_t = v, \quad b(t) = \gamma_{12}, \quad c(t) = \gamma_{22}, \tag{3}$$

(2) is equivalent to the nonautonomous binary reaction-diffusion system with self and cross diffusion

$$\begin{cases} u_t = v, \\ \\ v_t = -av + \gamma_{12}u_{xx} + \gamma_{22}v_{xx} + F(u), \end{cases} \tag{4}$$

and that in [3], (2) has been studied by analyzing (4). In order to continuing a methodical study of stability of nonautonomous binary systems, the present paper is devoted to the stability of the null solution of the nonautonomous binary, reaction-diffusion, dynamical systems of PDEs given by

$$
\begin{cases} u_t = a_{11}(t)u + a_{12}(t)v + \gamma_1(t)\Delta u + F_1(t,u,v), \\ \\ v_t = a_{21}(t)u + a_{22}(t)v + \gamma_2(t)\Delta v + F_2(t,u,v), \end{cases} \tag{5}
$$

with $(\mathbf{x}, t) \in \Omega \times \mathbb{R}^+$, Ω being a smooth domain of $\mathbb{R}^3$, under the boundary conditions

$$
\begin{cases} \beta\varphi + (1-\beta)\nabla\varphi \cdot \mathbf{n} = 0, \\ \\ \varphi = u, v, \end{cases} \tag{6}
$$

with $\mathbf{n}$ the outward unit normal to $\partial\Omega$ and

$$
\beta : \mathbf{x} \in \partial\Omega \Rightarrow \beta(\mathbf{x}) \in (0,1), \quad \beta(\mathbf{x}) \not\equiv 0, \tag{7}
$$

smooth function and

$$
\begin{cases} a_{ij} : t \in \mathbb{R}^+ \Rightarrow a_{ij}(t) \in C^1(\mathbb{R}^+) \cap \mathbb{R}, \ \ i,j = 1,2,3, \\ \\ \gamma_i : t \in \mathbb{R}^+ \Rightarrow \gamma_i(t) \in \mathbb{R}^+, \\ \\ a_{ij}, \ \ \frac{da_{ij}}{dt}, \gamma_i \in L^\infty(\mathbb{R}^+), \ \ F_i(t,0,0) = 0, \end{cases} \tag{8}
$$

with F_i nonlinear functions of u and v.

Section 2 is devoted to some preliminaries. In particular the functional space in which is studied the system at stake is introduced together with the basic spectral problem. In the subsequent Section 3 the properties of a nonautonomous L^2-energy are analyzed while Sections 4 and 5 are respectively devoted to the linear and nonlinear stability. In Section 6 the stability of binary reaction-diffusion systems of PDEs of Lotka-Volterra type is considered. The "preys" struggle for the life and their survival strategy is put in evidence in the subsequent Section 7, while Section 8 is devoted to a final remark. The paper ends with an Appendix (Section 9) in which the main tools of the Direct Method for the stability of nonautonomous systems are recalled.

2. Preliminaries

We assume that $\Omega \subset \mathbb{R}^3$ is a smooth bounded domain having the interior cone property, and denote by

- $< \cdot, \cdot >$ the scalar product of $L^2(\Omega)$;
- $< \cdot, \cdot >_{\partial\Omega}$ the scalar product of $L^2(\partial\Omega)$;
- $\| \cdot \|$ the norm of $L^2(\Omega)$;
- $\| \cdot \|_{\partial\Omega}$ the norm of $L^2(\partial\Omega)$;
- $W^{1,2}(\Omega, \beta)$, the functional space such that

$$
W^{1,2}(\Omega,\beta) = \left\{\varphi \in W^{1,2}(\Omega) \cap W^{1,2}(\partial\Omega), \beta\varphi + (1-\beta)\nabla\varphi \cdot \mathbf{n} = 0, \text{ on } \partial\Omega\right\}.
$$

For $\beta > 0$, it follows that $\{$[4], pp.$92 - 98\}$,

$$
\left\| \sqrt{\frac{\beta}{1-\beta}}\varphi \right\|_{\partial\Omega}^2 + \|\nabla\varphi\|^2 \geq \alpha\|\varphi\|^2, \tag{9}
$$

where $\alpha = \alpha(\Omega, \beta)$ is the smallest eigenvalue of the spectral problem

$$\begin{cases} \Delta\varphi + \lambda\varphi = 0, \quad \text{in } \Omega, \\ \beta\varphi + (1-\beta)\nabla\varphi \cdot \mathbf{n} = 0, \quad \text{on } \partial\Omega. \end{cases} \tag{10}$$

Setting

$$\begin{cases} u = U_1, \quad v = U_2, \quad B_{11} = a_{11} - \alpha\gamma_1, \quad B_{22} = a_{22} - \alpha\gamma_2, \\ \varphi_1 = \gamma_1(\Delta U_1 + \alpha U_1), \\ \varphi_2 = \gamma_2(\Delta U_2 + \alpha U_2), \\ \psi_1 = f_1(t, U_1, U_2), \quad \psi_2 = f_2(t, U_1, U_2), \end{cases} \tag{11}$$

(5)-(6) are reduced respectively to

$$\begin{cases} \frac{\partial U_1}{\partial t} = B_{11}(t)U_1 + a_{12}(t)U_2 + \varphi_1 + \psi_1 \\ \frac{\partial U_2}{\partial t} = a_{21}(t)U_1 + B_{22}(t)U_2 + \varphi_2 + \psi_2, \end{cases} \quad \text{in } \Omega \times \mathbb{R}^+, \tag{12}$$

$$\beta U_1 + (1-\beta)\nabla U_i \cdot \mathbf{n} = 0, \qquad \text{on } \partial\Omega \times \mathbb{R}^+. \tag{13}$$

3. Properties of nonautonomous generalized energies

In the Appendix (Section 9), the essential tools of the Liapunov Direct Method for nonautonomous binary systems of ordinary differential equations (ODE) are recalled together with the properties requested to a function for being a Liapunov function in the nonautonomous case are recalled.
The analysis performed in the sequel is based on the behaviour of the functional

$$E = \frac{1}{2}\left[\mu_1(t)\|U_1\|^2 + \mu_2(t)\|U_2\|^2\right], \tag{14}$$

with μ_i, $(i = 1, 2)$, suitable positive derivable functions in $\mathbb{R}^+$ and bounded there together with the derivative $\dot{\mu}_i$.
Setting

$$F_* = \inf_{\mathbb{R}^+} F, \quad F^* = \sup_{\mathbb{R}^+} F, \tag{15}$$

for any function $F : \mathbb{R}^+ \to \mathbb{R}$, the following Lemma holds.

Lemma 1. *The functional E has the following properties:*

i) *if $(\mu_i)_* > 0$, $(i = 1, 2)$, then E is positive definite;*
ii) *the temporal derivative of E along the solutions of (12)-(13) is given by*

$$\begin{aligned} \dot{E} = &\tfrac{1}{2}\left[(\dot{\mu}_1 + 2B_{11}\mu_1)\|U_1\|^2 + (\dot{\mu}_2 + 2B_{22}\mu_2)\|U_2\|^2 + \right. \\ &\left. + 2(\mu_1 a_{12} + \mu_2 a_{21}) < U_1, U_2 > \right] + \Phi_1 + \Phi_2, \end{aligned} \tag{16}$$

with

$$\Phi_1 = \sum_{i=1}^{2} \gamma_i \mu_i < U_i, \varphi_i >, \ \ \Phi_2 = \sum_{i=1}^{2} \mu_i < U_i, f_i >; \tag{17}$$

iii) *admits an upper bound which is infinitely small at the origin;*
iv) *if one of the functions μ_i is negative in $\mathbb{R}^+$, then E is indefinite and, in any disk centered at the origin of the phase space, exists a domain in which E is positive.*

Proof Property $i)$ immediately follows since

$$E > \frac{1}{2} \min\left[(\mu_1)_*, (\mu_2)_*\right] (\|U_1\|^2 + \|U_2\|^2). \tag{18}$$

Analogously in view of

$$\dot{E} = \dot{\mu}_1 \|U_1\|^2 + \dot{\mu}_2 \|U_2\|^2 + \sum_{i=1}^{2} \mu_i < U_i, \dot{U}_i >, \tag{19}$$

and (12), (16) is easily obtained. As concerns $iii)$ it is enough to observe that

$$E < \frac{1}{2} \sup\left[(\mu_1)^*, (\mu_2)^*\right] (\|U_1\|^2 + \|U_2\|^2). \tag{20}$$

Finally $iv)$, in the case $\mu_2(t) < 0$, $\forall t \in \mathbb{R}^+$, is given by

$$E = \frac{1}{2}(\mu_1 \|U_1\|^2 - |\mu_2| \|U_2\|^2), \tag{21}$$

and hence on the domain $U_2 = 0$, one obtains

$$E > \frac{1}{2}(\mu_1)_* \|U_1\|^2. \tag{22}$$

4. Linear stability

In view of (10), it easily follows that

$$\Phi_1 \leq 0. \tag{23}$$

Therefore disregarding Φ_2, (16) reduces to

$$2\dot{E} \leq \begin{cases} (\dot{\mu}_1 + 2B_{11}\mu_1) \|U_1\|^2 + (\dot{\mu}_2 + 2B_{22}\mu_2) \|U_2\|^2 + \\ \\ +2(\mu_1 a_{12} + \mu_2 a_{21}) < U_1, U_2 > . \end{cases} \tag{24}$$

Theorem 1. *Let*

$$(a_{12}a_{21})^* < 0, \quad B_{11}^* \leq -h_1, \quad B_{22}^* < -h_2, \quad \forall t \in \mathbb{R}^+, \tag{25}$$

h_i, $(i = 1, 2)$, *being positive constant. Then*

$$|a_{21}| \leq |a_{21}(0)| e^{2h_1 t}, \quad |a_{12}| \leq |a_{12}(0)| e^{2h_2 t}, \ \ \forall t \in \mathbb{R}^+, \tag{26}$$

guarantee the stability of the null solution, while

$$|a_{12}| \leq |a_{12}(0)| e^{2(h_1 - \epsilon)t}, \quad |a_{21}| \leq |a_{21}(0)| e^{2(h_2 - \epsilon)t}, \tag{27}$$

with $0 < \epsilon =$*const.*$< \min(h_1, h_2)$, *guarantee the exponential asymptotic stability.*

Proof. We give the proof in the case $\{(a_{12})_* > 0,\ B_{21}^* < 0\}$. Choosing

$$\mu_1 = -a_{21}, \qquad \mu_2 = a_{12}, \tag{28}$$

one obtains

$$\mu_1 a_{12} + \mu_2 a_{21} = -a_{21}a_{12} + a_{12}a_{21} = 0, \tag{29}$$

and (23) reduces to

$$\dot{E} \leq \frac{1}{2}\left[(\dot{\mu}_1 + 2B_{11}\mu_1)\|U_1\|^2 + (\dot{\mu}_2 + 2B_{22}\mu_2)\|U_2\|^2\right]. \tag{30}$$

On the other hand (26) guarantee that

$$\begin{cases} \dot{\mu}_1 + 2B_{11}\mu_1 \leq 0, \\ \\ \dot{\mu}_2 + 2B_{22}\mu_2 \leq 0, \end{cases} \tag{31}$$

while (27) guarantee

$$\begin{cases} \dot{\mu}_1 + 2B_{11}\mu_1 < \epsilon B_{11}\mu_1 \leq -\epsilon h_1|a_{21}|_*, \\ \\ \dot{\mu}_2 + 2B_{22}\mu_2 < \epsilon B_{22}\mu_2 \leq -\epsilon h_2|a_{12}|_*. \end{cases} \tag{32}$$

Therefore $\dot{E} \leq 0$ in the case (26) and $\dot{E}$ is negative definite in the case (27). In view of (18) E is positive definite and exists a constant $m > 0$ such that

$$\dot{E} \leq -mE \Leftrightarrow E \leq E(0)e^{-mt}. \tag{33}$$

Theorem 2. *Let* $B_{11}B_{22} > a_{12}a_{21},\ \forall t \in \mathbb{R}^+$, *and*

$$\begin{cases} (a_{12}a_{21})_* > 0,\ \ (B_{11})^* \leq -h_1,\ \ (B_{22})^* \leq -h_2, \\ \\ \min(|a_{12}|_*, |a_{21}|_*) > 0, \end{cases} \tag{34}$$

h_1, $(i = 1, 2)$, *being positive constants. Then (26) guarantee the stability and (27) the asymptotic exponential stability.*

Proof. In view of (28) it turns out that

$$\begin{aligned} &2(\mu_1 a_{12} + \mu_2 a_{21}) < U_1, U_2 > \leq 2\sqrt{\mu_1\mu_2}\sqrt{B_{11}B_{22}} < |U_1|, |U_2| > \leq \\ &\leq \mu_1|B_{11}|\|U_1\|^2 + \mu_2|B_{22}|\|U_2\|^2, \end{aligned} \tag{35}$$

and (24) reduces to

$$\dot{E} \leq \frac{1}{2}\left[(\dot{\mu}_1 + 2B_{11}\mu_1)\|U_1\|^2 + (\dot{\mu}_2 + 2B_{22}\mu_2)\|U_2\|^2\right]. \tag{36}$$

Obviously, from now on, one has to follows, step by step, the proof of the previous theorem.

5. Nonlinear stability

Theorem 3. *Let*

$$\Phi_2 \leq kE^{1+\eta}, \tag{37}$$

with k and η positive constants and let the null solution be asymptotically linearly stable either according to theorem 1. Then the null solution is nonlinearly exponentially locally stable.

Proof. We give the proof in connection with theorem 1 (that one, in connection with theorem 2, can be obtained with an analogous procedure).
In view of $(33)_1$ and (37), it follows that

$$\dot{E} \leq -mE + kE^{1+\eta} \leq (kE^\eta - m)E. \tag{38}$$

Therefore for

$$E^\eta(0) < \frac{m}{k}, \tag{39}$$

one obtains (by an iterative procedure)

$$\dot{E} \leq -\delta E \Leftrightarrow E \leq E(0)e^{-\delta t}, \tag{40}$$

with

$$\delta = m - kE^\eta(0). \tag{41}$$

6. Stability of nonautonomous binary reaction-diffusion PDEs Lotka-Volterra models

We consider, in the case $\beta \equiv 1, \Omega = (0,1)$, the nonautonomous Lotka-Volterra models

$$\begin{cases} u_t = a(t)u - b(t)uv + \gamma_1(t)\Delta u, \\ \\ v_t = -c(t)v + d(t)uv + \gamma_2(t)\Delta v, \end{cases} \tag{42}$$

with a, b, c, d, positive functions of t. System (42) is a particular case of (5) and admits the positive steady solution ($\bar{u} = \frac{\bar{c}}{\bar{d}}$, $\bar{v} = \frac{\bar{a}}{\bar{b}}$), with $\bar{a}, \bar{b}, \bar{c}, \bar{d}$ positive constants if and only if

$$a = \bar{a}\varphi(t), \quad b = \bar{b}\varphi(t), \quad c = \bar{c}\varphi_1(t), \quad d = \bar{d}\varphi_2(t). \tag{43}$$

Assuming (43) for biological reasons, (42) reduces to

$$\begin{cases} u_t = \varphi(t)(\bar{a} - \bar{b}v)u + \gamma_1 \Delta u, \\ \\ v_t = \varphi_1(t)(-\bar{c} + \bar{d}u)v + \gamma_2 \Delta v. \end{cases} \tag{44}$$

We assume that φ and φ_1 are smooth functions defined in $\mathbb{R}^+$, bounded together with their first derivative $\dot{\varphi}$, $\dot{\varphi}_1$. Further - in order to guarantee that the "preys" density u grows in the absence of "predators" and diffusivity ($\gamma_1 = 0$) and vice versa the "predators" density v decreases in the absence of preys and absence of diffusivity ($\gamma_2 = 0$), we assume that φ

and φ_1 are positive functions of $t \in \mathbb{R}^+$. Following [2], we set

$$\begin{cases} U_1 = \frac{\bar{d}}{\bar{c}} u, \;\; U_2 = \frac{\bar{b}}{\bar{a}} v, \\ \tau = \bar{a} \int_0^t \varphi(z)\, dz, \quad r = \frac{\bar{c}}{\bar{a}}, \quad F(t) = \frac{\varphi_1}{\varphi}, \\ \bar{\gamma}_1 = \frac{\gamma_1}{\bar{a}\varphi}, \quad \bar{\gamma}_2 = \frac{\gamma_2}{\bar{a}\varphi}, \end{cases} \tag{45}$$

and (44) becomes

$$\begin{cases} \frac{\partial U_1}{\partial t} = (1 - U_2)U_1 + \bar{\gamma}_1 \Delta U_1, \\ \frac{\partial U_2}{\partial t} = rF(U_1 - 1)U_2 + \bar{\gamma}_2 \Delta U_2, \end{cases} \tag{46}$$

under the boundary conditions

$$U_1 = U_2 = 1, \qquad \text{for } x = 0, 1. \tag{47}$$

Remark 1. *We remark that:*

i) *(46)-(47) has $(1, 1)$ as equilibrium steady state where neither species is extinct;*

ii) *for*

$$\frac{\varphi_1}{\varphi} = \textit{const.}, \qquad \frac{\gamma_i}{\varphi} = \textit{const.}, \tag{48}$$

(46)-(47) reduces to the autonomous case;

iii) *the nonlinear stability of $(1, 1)$ guarantees the survival of both the species (and hence the ecological equilibrium).*

Theorem 4. *Let*

$$F \le F(0)e^{2(h_1 - \epsilon)t}, \quad h_1 - \epsilon > 0, \tag{49}$$

with h_1, ϵ positive constants. Then $(1, 1)$ is linearly, asymptotically, exponentially stable.

Proof. Setting

$$U_1 = 1 + X, \qquad U_2 = 1 + Y, \tag{50}$$

it turns out that

$$\begin{cases} \frac{\partial X}{\partial t} = -Y + \bar{\gamma}_1 \Delta X - XY, \\ \frac{\partial Y}{\partial t} = rFX + \bar{\gamma}_2 \Delta Y + rFXY, \end{cases} \tag{51}$$

under the boundary conditions

$$X = Y = 0, \qquad x = 0, 1. \tag{52}$$

Following (11) and linearizing one obtains

$$\begin{cases} \frac{\partial X}{\partial t} = -\alpha \bar{\gamma}_1 X - Y + \bar{\gamma}_1(\Delta X + \alpha X), \\ \frac{\partial Y}{\partial t} = rFX - \alpha \bar{\gamma}_2 Y + \bar{\gamma}_2(\Delta Y + \alpha Y), \end{cases} \tag{53}$$

with $\alpha = \pi^2$ [**?**], and it immediately follows that by virtue of (49) all the conditions requested by theorem 1 are verified.

Theorem 5. *Let*

$$0 < F \le m_1 = \textit{positive constant}. \tag{54}$$

Then $(1, 1)$ is locally nonlinearly asymptotically exponentially stable.

Proof. Let us begin by recalling that in $C_0(0,1)$ the following inequalities hold [10],

$$\begin{cases} \|\nabla\varphi\|^2 \geq \pi^2\|\varphi\|^2, \\ \int_0^1 \varphi^4\,dx \leq \frac{1}{\pi^2}\left[\int_0^1 (\nabla\varphi)^2\,dx\right]^2, \end{cases} \tag{55}$$

and hence it follows that

$$\begin{aligned} &\int_0^1 \varphi^2\varphi_1 dx \leq \left(\int_0^1 \varphi^4 dx \int_0^1 \varphi_1^2 dx\right)^{\frac{1}{2}} \leq \\ &\leq \frac{1}{\pi}\|\nabla\varphi\|^2 \cdot \|\varphi_1\| \leq \frac{1}{\pi}\|\nabla\varphi\|^2(\|\varphi\|^2 + \|\varphi_1\|^2)^{\frac{1}{2}}. \end{aligned} \tag{56}$$

Directly from (51) it follows that

$$\begin{aligned} &2\dot{E} = \dot{\mu}_1\|U_1\|^2 + \dot{\mu}_2\|U_2\|^2 + 2(-\mu_1 + \mu_2 rF) < X, Y > + \\ &+\mu_1\bar{\gamma}_1 < X, \Delta X > +\mu_2\bar{\gamma}_2 < Y, \Delta Y > . \end{aligned} \tag{57}$$

On the other hand in view of $(55)_1$, it follows that

$$< \varphi, \Delta\varphi >= -\|\nabla\varphi\|^2 \leq -(1-\eta)\|\nabla\varphi\|^2 - \eta\pi^2\|\varphi\|^2, \tag{58}$$

with $\eta < 1$ and hence (57) implies

$$\begin{aligned} &2\dot{E} \leq (\dot{\mu}_1 - \eta\pi^2\bar{\gamma}_1\mu_1)\|X\|^2 + (\dot{\mu}_2 - \eta\pi^2\bar{\gamma}_2\mu_2)\|Y\|^2 + \\ &+2(-\mu_1 + \mu_2 m_1) < |X|, |Y| > + \\ &+(1+2F)\mu_1\bar{\gamma}_1\|\nabla X\|^2\left[\tfrac{1}{\pi}\|X\| - (1-\eta)\right] + \\ &+(1+2F)\mu_2\bar{\gamma}_2\|\nabla Y\|^2\left[\tfrac{1}{\pi}\|Y\| - (1-\eta)\right]. \end{aligned} \tag{59}$$

Choosing

$$\mu_1 = m_1, \quad \mu_2 = 1, \tag{60}$$

one obtains

$$2\dot{E} \leq -\eta\pi^2 m_1\bar{\gamma}_1\|X\|^2 - \eta\pi^2\bar{\gamma}_2\|Y\|^2, \tag{61}$$

under the initial conditions

$$\|X(0)\| < (1-\eta)\pi, \quad \|Y(0)\| < (1-\eta)\pi, \tag{62}$$

and hence

$$\dot{E} \leq -\eta\pi^2\delta E \Leftrightarrow E \leq E_0 e^{-\eta\pi^2\delta t}, \tag{63}$$

with

$$\delta = \min[(\gamma_1)_*, (\gamma_2)_*]. \tag{64}$$

7. Struggle for the life

In the case $u = v = 0$, on $\partial\Omega$, (42) admits the null solution $(0,0)$ (which is the only steady state existing). The stability-instability of this solution is of primary interest since its asymptotic stability implies the extinction of the "preys" (the "predators" eat generally various species of preys). The "preys", in order to survive, have to adopt a strategy guaranteeing the instability of the null solution. In order to understand this strategy one has to consider the (linear) instability of the null solution of (42). Choosing as Liapunov function the energy norm $\|u\|^2 + \|v\|^2$, one immediately obtains

$$\frac{1}{2}\frac{d}{dt}(\|u\|^2 + \|v\|^2) = a(t)\|u\|^2 - c(t)\|v\|^2 + \bar{\Phi}, \tag{65}$$

with

$$\bar{\Phi} = \gamma_1(t) < u, \Delta u > +\gamma_2(t) < v, \Delta v >, \tag{66}$$

and, by virtue of $(55)_1$, one obtains

$$\frac{1}{2}\frac{d}{dt}(\|u\|^2 + \|v\|^2) < [a(t) - \alpha\gamma_1(t)]\|u\|^2 - [c(t) + \alpha\gamma_2(t)]\|v\|^2. \tag{67}$$

Therefore

$$a^* < \alpha(\gamma_1)_*, \tag{68}$$

implies asymptotic (linear) stability of the null solution. In view of (68) it follows that in order to survive the "preys" have to stay "as close as possible" in order to guarantee $\gamma_1(t) \simeq 0$. Since $\{\sin n\pi x\}$ is a complete system of eigenfunction associate to the sequence of the eigenvalues $\{n^2\pi^2\}$ of the spectral problem

$$\begin{cases} \Delta\varphi + \alpha\varphi = 0, & x \in]0,1[, \\ \varphi = 0, & x = 0,1, \end{cases} \tag{69}$$

then, in view of $u = \sum_{n=1}^{\infty} u_n$ with $u_n = X_n \sin n\pi x$, the (linear) instability is guaranteed by the instability of the null solution of

$$\frac{dX_1}{dt} = [a(t) - \pi^2\gamma_1(t)]X_1. \tag{70}$$

(70) implies

$$X_1(t) = X_1(0)\exp\int_0^t [a(\tau) - \pi^2\gamma_1(\tau)]dt, \tag{71}$$

therefore

$$X_1(0) > 0, \quad \int_0^t [a(\tau) - \pi^2\gamma_1(\tau)]dt, \tag{72}$$

guarantees the prey's survival in $[0,t]$.

8. A Final Remark

We remark that in [1]-[3] a Liapunov functional - different from (14) and directly linked to the eigenvalues of the problem at stake - has been introduced. In the case (12) this functional is

$$\tilde{V} = \frac{1}{2}\left[A(t)(\|U_1\|^2 + \|U_2\|^2) + \|B_{11}U_2 - a_{21}U_1\|^2 + \|a_{12}U_2 - B_{22}U_1\|^2\right], \quad (73)$$

with

$$A = B_{11}B_{22} - a_{12}a_{21}. \quad (74)$$

(73), in the autonomous case, allows to obtain conditions guaranteeing the nonexistence of subcritical instabilities [5] and hence appears to be ”a priori” the best candidate for the nonautonomous case. We here, for the sake of simplicity, have not taken it into account. The functional (73) has been used in [3].

9. Appendix

We recall here the basic conditions requested for the stability of nonautonomous systems.

i) *Stability*. The main theorem of the Direct Method for nonautonomous systems [9] guarantee that:
the existence of a positive definite function W i.e.:

$$W \geq m(\|U_1\|^2 + \|U_2\|^2), \quad m = \text{positive constant}, \quad (75)$$

implies

- stability if the temporal derivative along the solutions is negative semidefinite (i.e. $\dot{W} \leq 0$);
- asymptotic stability if admits an upper bound which is infinitely small at the origin (i.e. $W \leq m_1(\|U_1\|^2 + \|U_2\|^2),\ m_1$ =const>0) and its temporal derivative along the solutions is negative definite $\{$i.e. $\dot{W} < 0$ for $\|U_1\|^2 + \|U_2\|^2 \neq 0\,\}$.

ii) *Instability*. The null solution is unstable if exists a function W

- taking positive values in any disk centered at $U_1 = U_2 = 0$;
- $\dot{W}$ is positive definite for all $t \geq t_0 > 0$ in which W is bounded.

Acknowledgments

This paper has been performed under the auspices of the G.N.F.M. of I.N.D.A.M.

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Article contributed to the Festschrift volume in honour of Giuseppe Grioli on the occasion of his 100th birthday.

Communicated 13 April 2012; published online 29 January 2013