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A NOTE ON A PROOF OF F. HAHN CONCERNING THE GROSS SUBSTITUTABILITY

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ABSTRACT. The aim of this note is to investigate the assumptions and the proofs of some results by F. Hahn, which have been stated in several other classics of Economic Theory. The results deal with the relations between the gross substitutability and the weak axiom of revealed preference, in a pure exchange economy, ruled by the Walras law.

1. Introduction

Relations between gross substitutability and the weak axiom of revealed preference, in a pure exchange economy, have been deeply investigated in Economics [1, 2, 3, 4, 5]. The problem proved to be very challenging also for a mathematical point of view . The original proof in [1] is a breakthrough for Mathematical Economics but it is quite elaborate. Several other proofs have been proposed in the past years, mainly with the task of making it simpler to read and to present. For this scope, assumptions such as the existence of partial derivatives of the excess supply functions have been introduced. However, the proof in [3], which seems to be accepted by the literature as the easier, still is not detailed from a mathematical point of view. In this note we present and comment the assumptions commonly accepted in the literature and we try to emend the mathematical lacks in the proof by Hahn. Section 2 is devoted to the statement of the problem and the main result about gross substitutability and the weak axiom of revealed preference. In Section 3, for the sake of completeness, we also consider the case when weak gross substitutability is assumed. As it is known the latter do not imply the weak axiom, but its consequences are of some interest for economic theory.

2. Gross substitutability and the weak axiom of revealed preference

According to the setting imposed by Hahn, we consider a Walrasian economy of pure exchange, where $(n+1)$ goods, labeled by $i = 0, 1, \ldots, n$, are traded. The (non normalized) price of the i-th good is denoted by p_i and we refer to the vector $\mathbf{p} \in \mathbb{R}^{n+1}$ as the vector of all prices. The following order relations are classical and refer to a Pareto order in the space $\mathbb{R}^n$:

i) $\mathbf{x} \geqq \mathbf{y}$ if and only if $x_i \geqq y_i, \forall i = 1, \ldots, n$;
ii) $\mathbf{x} \geq \mathbf{y}$ if and only if $\mathbf{x} \geqq \mathbf{y}, \mathbf{x} \neq \mathbf{y}$;

iii) $\mathbf{x} > \mathbf{y}$ if and only if $x_i < y_i, \forall i = 1, \ldots, n$.

When $\mathbf{y} = \mathbf{0}$, $\mathbf{x}$ is said, respectively, *nonnegative*, *semipositive*, *strictly positive*. The economy is characterized by a supply side and a demand side, which define two different kinds of agent. For each given price $\mathbf{p}$ one can define quantities x_i and y_i as the total demand and supply (respectively) of i-th good in the economy. Therefore we recall the following definitions:

Definition 1. *For $i = 0, \ldots, n$ the function $s_i(\mathbf{p}) : \mathbb{R}_+^{n+1} \to \mathbb{R}$ is the* excess supply *for the i-th good at the price $\mathbf{p}$, defined by:*

$$s_i(\mathbf{p}) = x_i(\mathbf{p}) - y_i(\mathbf{p}) - \bar{x}_i$$

where $x_i(\mathbf{p})$ and $y_i(\mathbf{p})$ are, respectively, the supply and the demand for the i-th good at the price level $\mathbf{p}$ and $\bar{x}_i$ is the total quantity of i owned by the demand side. The excess supply can be aggregated into a column vector $\mathbf{s}(\mathbf{p}) \in \mathbb{R}^{n+1}$, which is a vector valued function of $\mathbf{p}$.

Definition 2. *In a similar way it can be defined an* excess demand *function, which is, in the aggregate notation, the vector valued map:*

$$\mathbf{e}(\mathbf{p}) = -\mathbf{s}(\mathbf{p})$$

We make, according to [6] the following assumptions on $\mathbf{s}(\mathbf{p}) : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$:

$(H1)$ is defined and continuous over all $\mathbf{p} \geq \mathbf{0}$;

$(H2)$ is positively homogeneous of degree 0 (PH), i.e. $s_i(\lambda \mathbf{p}) = s_i(\mathbf{p})$ for all $\lambda > 0$ and $i = 0, \ldots, n$;

$(H3)$ *Walras Law* is in force, that is $\mathbf{p}^\top \mathbf{s}(\mathbf{p}) = 0$, $\forall \mathbf{p} \geq \mathbf{0}$.

Definition 3. *A feasible vector of prices $\mathbf{p}^*$ is said to be an equilibrium for the excess supply function when:*

$$\mathbf{s}(\mathbf{p}^*) \geqq 0, \quad \textit{with} \quad \mathbf{p}^* \geq \mathbf{0}$$

Remark 1. The definition of equilibrium price implies that:

$$\begin{aligned} s_i(\mathbf{p}^*) = 0 \quad &\text{iff} \quad p_i^* > 0 \\ s_i(\mathbf{p}^*) > 0 \quad &\text{iff} \quad p_i^* = 0 \end{aligned}$$

Otherwise it would be contradicted $(H3)$, having $p_i^* s_i(\mathbf{p}^*) > 0$. Therefore we have $\mathbf{s}(\mathbf{p}^*) = 0$ if and only if $\mathbf{p}^* > 0$.

The next assumption allows to simplify some of the proofs which follow.

$(H4)$ $s_i(\mathbf{p})$, $i = 0, \ldots, n$ has first order partial derivatives in its domain.

Under assumption $(H4)$ we can state the notion of *gross substitutability* in the "differentiable" form, which is stricter than the nondifferentiable one.

Definition 4. *We say that* gross substitutability *assumption is in force, or, equivalently that the $(n + 1)$ goods of the economy are* gross substitute, *for any feasible price $\mathbf{p}$, when:*

$$\frac{\partial s_i(\mathbf{p})}{\partial p_j} < 0 \quad i, j = 0, 1, \ldots, n;\ i \neq j \tag{1}$$

Remark 2.
- Usually Definition 4 is formulated via excess demand functions. If this is the case, it can be said that the Jacobian matrix of $\mathbf{e}(\mathbf{p})$ is a *Metzlerian matrix.*
- Condition (1) implies that the excess supply for the i-th good is decreasing with respect to all others goods ($j \neq i$)

Condition (1), presented in [7, 8, 9], has been widely used to study the existence of equilibrium prices and the (local and global) stability of the (dynamic) economic system (see e.g. [1, 6, 10]). Moreover, (1) can be used to obtain results of comparative statics analysis and the uniqueness of equilibrium price vector $\mathbf{p}^*$.
Some authors, see e.g. [11, 12, 13, 14], introduce, for the study of the above problems the following weaker assumption.

Definition 5. *We say that* weak gross substitutability *assumption is in force, or, equivalently that the* $(n+1)$ *goods of the economy are* weak gross substitute*, for any feasible price* $\mathbf{p}$*, when:*

$$\frac{\partial s_i(\mathbf{p})}{\partial p_j} \leqq 0 \quad i, j = 0, 1, \ldots, n;\ i \neq j \tag{2}$$

Remark 3. Similarly to Definition 4 we can recall that (2) implies that the excess supply of the i-th good is non-decreasing w.r.t. all other goods. Therefore some constant interval for the values of $s_i(\mathbf{p})$ can be allowed.

In [15] another basic concept of economic analysis is stated, the *weak axiom of revealed preference*. This axiom plays a crucial role in the proofs of stability properties of Walras's "tâtonnement" process. According to our notation, we state the following restricted definition of this axiom (see e.g. [3, 15]).

Definition 6. *The weak axiom of revealed preferences is in force at the equilibrium price* $\mathbf{p}^*$*, when:*

$$\mathbf{p}^{*\top}\mathbf{s}(\mathbf{p}) < \mathbf{p}^{*\top}\mathbf{s}(\mathbf{p}^*) = 0 \quad \forall \mathbf{p} \neq k\mathbf{p}^*,\ k > 0. \tag{3}$$

This note focuses on the relations between Definitions 4 and 5 and condition (3). We recall that in [1] the authors have first proved that (5) implies (3). Their proof does not need assumption $(H4)$, but is very elaborated one. Other authors have made it simpler, by assuming also $(H4)$ is in force. With regard to this case, we quote the proofs from [3, 4, 6, 16, 17]. All these proofs are, however, quite inaccurate from a mathematical point of view. Some interest is also on the graphic proof shown in [18], which however holds only for $n = 2$. We can list some facts which are common to the different proofs we have quoted.

I) Both $(H2)$ and (1) can be void if some $p_i = 0$ (see e.g. [10, 19, 20, 21, 22]). Hence one should require (1) to hold $\forall \mathbf{p} \in \operatorname{int} \mathbb{R}_+^{n+1}$, that is $\mathbf{p} > 0$.
Moreover, (see e.g. [1, 2, 10]) when (1) is in force, it follows that
$$p_i \to 0 \Longrightarrow s_i(\mathbf{p}) \to -\infty$$
which means the equilibrium price vector $\mathbf{p}^*$ must be strictly positive and therefore $\mathbf{s}(\mathbf{p}^*) = \mathbf{0}$.

II) Taking I) into account, the continuity of $\mathbf{s}(\mathbf{p})$ only over $\operatorname{int}\mathbb{R}_+^{n+1}$, does not ensure the existence of equilibrium prices, as shown in [23]. For this mathematical reason (and some economic arguments) some boundary conditions need to be assumed. The most classical assumptions (see e.g. [24, 25, 26]) are the following:
   i) the vector valued function $\mathbf{s}(\mathbf{p})$ is defined and continuous over $\operatorname{int}\mathbb{R}_+^{n+1}$;
   ii) $(H2)$ and $(H3)$ hold true;
   iii) $\mathbf{s}(\mathbf{p})$ is bounded from above (i.e. $\exists k < 0\,, s_i(\mathbf{p}) < k$ for all feasible $\mathbf{p}$);
   iv) if $\mathbf{p}^s \to \mathbf{p}$, where $\mathbf{p} \neq 0$, but $p_i = 0$ for some $i = 0, 1, \ldots, n$, then:

$$\text{Min}\left\{s_0(\mathbf{p}^s), \ldots, s_n(\mathbf{p}^s)\right\} \to -\infty$$

   These assumptions ensure the existence of a positive equilibrium price vector $\mathbf{p}^*$ *without* the gross substitutability assumption. If also (1) is in force, we have also the uniqueness of $\mathbf{p}^*$, as it is well known.

III) Due to $(H2)$, we shall assume that the domain of $\mathbf{s}(\cdot)$ is the unit simplex of $\mathbb{R}^{n+1}$:

$$S^{n+1} = \left\{\mathbf{p} \in \mathbb{R}^{n+1} \,|\, \mathbf{p} \geq \mathbf{0},\ \sum_i p_i = 1\right\}.$$

   The former set is closed, bounded and convex, however, under assumptions i)-iv) of II) $\mathbf{s}(\cdot)$ is continuous only on $\operatorname{int} S^{n+1}$, so the usual Weierstrass Theorem cannot be applied.

IV) Another consequence of (1), which can be found in [1], is that, from $p^* > 0$ and $\frac{p_r}{p_r^*} \leqq \frac{p_i}{p_i^*}$, for all i, it follows $s_r(\mathbf{p}) < s_r(\mathbf{p}^*) = 0$.

The following result which we quote from [26] states alternative conditions to guarantee the existence of strictly positive equilibrium price.

Theorem 1 ([26])**.** *Let $s_i(\mathbf{p})$, $i = 0, \ldots, n$ be functions $s_i : \mathbb{R}_+^{n+1} \to \mathbb{R}$ such that:*

(1) *$s_i(\cdot)$ is continuous over $\operatorname{int}\mathbb{R}_+^{n+1}$;*
(2) *$\mathbf{p}^\top \mathbf{s}(\mathbf{p}) = 0\ \forall \mathbf{p} \in \operatorname{int}\mathbb{R}_+^{n+1}$;*
(3) *If $\left\{\mathbf{p}^n\right\}_{n\geq 0} \in \operatorname{int}\mathbb{R}_+^{n+1}$ is a sequence of price vectors converging to $\bar{\mathbf{p}} \neq \mathbf{0}$, and $\bar{p}_k = 0$ for some good k, then for some other good k' with $\bar{p}_{k'} \neq 0$, the associated sequence of excess supply $\left\{s_{k'}(\mathbf{p}^n)\right\}_{n\geq 0}$ is unbounded from below.*

Then there exists a strictly positive vector of prices $\mathbf{p}^*$ such that $\mathbf{s}(\mathbf{p}^*) = 0$.

If the boundary condition (3) is violated, the equilibrium may have some zero components.

Example 1. *Consider an exchange economy with two goods. Suppose that the aggregate excess supply function is*

$$\mathbf{s}(p_1, p_2) = \begin{bmatrix} 1 \\ -\dfrac{p_1}{p_2} \end{bmatrix} \quad \text{for all } \mathbf{p} \in \operatorname{int}\mathbb{R}_+^2$$

Hence assumptions (1) and (2) of Theorem 1 are fulfilled, but not (3). It is easy now to prove that the (normalized) equilibrium is $\mathbf{p}^* = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, which is not strictly positive.

We wish to stress that some boundary conditions are necessary to have strict positive equilibria. The next example shows that it is not enough to require more continuity, on the whole simplex, as stated in Theorem 3.9 of [27].

Example 2. *Consider an exchange economy with two goods. Suppose that the aggregate excess supply function is*

$$\mathbf{s}(p_1, p_2) = \begin{bmatrix} p_2 e^{-p_1 p_2} \\ -p_1 e^{-p_1 p_2} \end{bmatrix} \quad \text{for all } \mathbf{p} \in \mathbb{R}^2_+$$

Hence assumptions (1) and (2) of Theorem 1 are fulfilled. Moreover the function is continuous over the whole set $\mathbb{R}^2$. *It is easy now to prove that the (normalized) equilibrium is* $\mathbf{p}^* = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, *which is not strictly positive.*

As already said a simple proof of the implication from (1) to (3) can be found in [3]. Although easy to read, the proof provided by Hahn contains some misleading mathematical arguments, which we try to amend.

Theorem 2. *Let* $\mathbf{s}(\cdot) : S^{n+1} \to \mathbb{R}^{n+1}$ *be differentiable over* $\operatorname{int} \mathbb{R}^{n+1}_+$ *and such that assumptions i)-iv) hold true. Assume also (1) is fulfilled and* $\mathbf{p}^*$ *is an equilibrium. Then it holds:*

$$\mathbf{p}^{*\top} \mathbf{s}(\mathbf{p}) < 0, \ \forall \mathbf{p} > \mathbf{0}, \ \mathbf{p} \neq k\mathbf{p}^*, \ k > 0$$

Proof: The thesis is clearly equivalent to prove that $\mathbf{p}^*$ is the unique maximizer for the real valued function $g(\mathbf{p}) = \mathbf{p}^{*\top}\mathbf{s}(\mathbf{p})$.
First we claim that $\mathbf{p}^*$ is a maximizer. By assumption iii) we have that there exists some $-\infty < k < +\infty$ such that:

$$g(\mathbf{p}) = \sum_{i=0}^{n} p_i^* s_i(\mathbf{p}) \leq \sum_{i=0}^{n} p_i^* k = K \qquad \forall \mathbf{p} \in S^{n+1} \quad ,$$

that is the range of g is bounded from above. Hence one can choose a sequence converging to the upper bound. Without loss of generality we can say there exists a sequence $\mathbf{p}^k \in S^{n+1}$ such that $g(\mathbf{p}^k) \to K$. Since S^{n+1} is a compact set, we can always think to $\mathbf{p}^k$ as a convergent sequence. Let then $\mathbf{p}^k \to \hat{\mathbf{p}} \in S^{n+1}$. We have two possibilities:

$$\hat{\mathbf{p}} \in \operatorname{int} S^{n+1} \quad \text{or} \quad \hat{\mathbf{p}} \in \operatorname{bd} S^{n+1} \quad ,$$

where $\operatorname{bd} S^{n+1}$ stands for the boundary of S^{n+1}.
Assume the latter holds true. By iv) we have $\operatorname{Min}\left\{s_0(\mathbf{p}^s), \ldots, s_n(\mathbf{p}^s)\right\} \to -\infty$, that is:

$$\forall M > 0 \quad \exists k, j : \quad \sum_{i=0}^{n} p_i^* s_i(\mathbf{p}^k) \leq \sum_{i \neq j} p_i^* s(\mathbf{p}^k) + p_j^*(-M) \quad ,$$

that is $g(\mathbf{p}^k) \to -\infty$ which is absurd. Therefore we have proved that $\hat{\mathbf{p}} \in \operatorname{int} S^{n+1}$.
Hence there exists a neighborhood $U(\hat{\mathbf{p}})$ such that ($\operatorname{cl} A$ means the closure of set A):

$$\left(\operatorname{cl} U(\hat{\mathbf{p}}) \cap S^{n+1}\right) \cap \operatorname{bd} S^{n+1} = \emptyset$$
$$\text{and } \mathbf{p}^* \in \left(\operatorname{cl} U(\hat{\mathbf{p}}) \cap S^{n+1}\right).$$

So we have defined a compact subset of S^{n+1}, $C = \big(\mathrm{cl}\, U(\hat{\mathbf{p}}) \cap S^{n+1}\big)$ where the maximizer of g must lay. Clearly Weierstrass theorem applies to g over C and any point which fulfill the necessary condition:

$$\langle \nabla g(\tilde{\mathbf{p}}), \mathbf{p} - \tilde{\mathbf{p}} \rangle \leq 0 \quad \forall \mathbf{p} \in C$$

must be that maximizer. Indeed, due to the previous remark I) and (1), it follows that the former condition is also sufficient. The same condition can be written as:

$$\sum_j \left[\left(\sum_i p_i^* \frac{\partial s_i}{\partial p_j}(\tilde{\mathbf{p}}) \right) (p_j - \tilde{p}_j) \right] \leq 0 \tag{4}$$

We now prove that $\mathbf{p}^*$ fulfils (4). Recalling that by $(H3)$ the function $h(\mathbf{p}) = \mathbf{p}^\top \mathbf{s}(\mathbf{p}) = 0$ is constant and differentiable, it follows that $\forall j = 0, \ldots, n$:

$$\frac{\partial h}{\partial p_j}(\mathbf{p}) = 0 = \sum_i p_i \frac{\partial s_i}{\partial p_j}(\mathbf{p}) + s_j(\mathbf{p})$$

$$\text{that is} \quad \sum_i p_i \frac{\partial s_i}{\partial p_j}(\mathbf{p}) = -s_j(\mathbf{p}).$$

Therefore we have

$$\sum_j \left[\left(\sum_i p_i^* \frac{\partial s_i}{\partial p_j}(\mathbf{p}^*) \right) (p_j - p_j^*) \right] = \sum_j \left[(-s_j(\mathbf{p}^*)) (p_j - p_j^*) \right] =$$

$$= -\left\{ \sum_j p_j s_j(\mathbf{p}^*) - \sum_j p_j^* s_j(\mathbf{p}^*) \right\}.$$

Because of $(H3)$ $\sum_j p_j^* s_j(\mathbf{p}^*) = 0$ and, since $\mathbf{p}^*$ is an equilibrium and $p_j > 0$ we have (4) is fulfilled by $\mathbf{p}^*$.

We need now to prove that $\mathbf{p}^*$ is the unique maximizer, up to positive multipliers. By contradiction assume there exists some $\mathbf{p}^0 \neq k\mathbf{p}^*$, which is a maximizer of g. Then it should be $g(\mathbf{p}^*) = g(\mathbf{p}^0)$, that is:

$$\sum_i p_i^* s_i(\mathbf{p}^0) = \sum_i p_i^* s_i(\mathbf{p}^*)$$

or, equivalently

$$\sum_i p_i^* \big(s_i(\mathbf{p}^0) - s_i(\mathbf{p}^*)\big) = 0 \tag{5}$$

Since $\mathbf{p}^0 \neq k\mathbf{p}^*$, there exists at least one index $r = 0, \ldots, n$, such that $\frac{p_r^0}{p_r^*} \leqq \frac{p_i^0}{p_i^*}$ for all $i = 0, \ldots, n$. As stated in IV), (1), implies that it must hold at least $s_r(\mathbf{p}^0) < s_r(\mathbf{p}^*)$. The latter, together with (5) implies the contradiction $p_r = 0$. $\square$

Remark 4. The previous proof suggest a slight change in the assumptions, in order to get rid of redundant requests. Indeed we need only to guarantee that a maximizer exists. To do that, it is far too much to ask for continuity and one may impose the upper semicontinuity of $s_i(\mathbf{p})$ for all $i = 0, \ldots, n$ on $\operatorname{int} S^{n+1}$. Once this assumption is in force, we may also move to consider extended real valued functions, that is $s_i : \mathbb{R}^{n+1} \to \mathbb{R} \cup \{-\infty\}$ so that we shall assume s_i defined on $\operatorname{bd} S^{n+1}$ where the value $-\infty$ is reached. Hence the function is upper semicontinuous on a compact set and there is no more reason to define the set $\left(\operatorname{cl} U(\hat{\mathbf{p}}) \cap S^{n+1}\right)$ we use in the proof.

3. Weak gross substitutability

In [3] the case when (2) is in force is considered also. Under such assumptions it is possible to prove the weaker consequence

$$\mathbf{p}^{*\top}\mathbf{s}(\mathbf{p}) < 0, \ \forall \mathbf{p} \notin E, \ \mathbf{p}^* \in E \tag{6}$$

where $E := \{\mathbf{p} \,|\, \mathbf{s}(\mathbf{p})) \leqq \mathbf{0}, \ \mathbf{p} \geq \mathbf{0}\}$ is the set of all equilibrium price. also (6) has some interest for the economic theory ([3, 6, 11, 13, 23, 28]). Condition (6) has been applied in [11] to the analysis of stability of Walrasian tatonnement process and in [13] to prove some results of comparative statics. The problem which arise now is that, basically, we allow equilibria with some zero components on the price vector.
The original proof of (6) given by Arrow and Hurwicz is long and complicate. An alternative proof has been proposed in [5, 29]. The latter is even longer and more complicate than the original one of Arrow and Hurwicz, but offers a more general setting and proves the result under weaker assumptions. Finally in [3] a shorter and more elementary proof is presented, but, as in the previous section, some remarks have to be done.
First when it is assumed that the Jacobian matrix

$$\left[\frac{\partial s_i(\mathbf{p})}{\partial p_j}\right], \qquad i, j = 0, \ldots, n$$

is indecomponible (in the usual sense of Linear Algebra), then the (normalized) equilibrium price vector is unique and strictly positive (see e.g. [2, 13]). Therefore the conclusions are those already established by Theorem 2, under the assumption of gross substitutability (see [2]).

Also for the case of weak gross substitutability, if we want to obtain positive equilibrium price vectors, we have to impose, besides usual conditions on $\mathbf{s}(\cdot)$, some suitable boundary condition. For example, in [23] the boundary condition (3) of Theorem 1, is replaced by

$(H5)$ for any fixed boundary point $\bar{\mathbf{p}}$ of $\operatorname{int}\mathbb{R}^{n+1}_+ \backslash \{\mathbf{0}\}$ ($\operatorname{int}\mathbb{R}^{n+1}_+$ is the interior of $\mathbb{R}^{n+1}_+$), let $K_{\bar{\mathbf{p}}} := \{k \,|\, \bar{\mathbf{p}}_k = 0\}$ be the set of commodities with zero price. Then $s_i(\mathbf{p})$ tends to minus infinity for every $i \in K_{\bar{\mathbf{p}}}$ as $\mathbf{p} \to \bar{\mathbf{p}}$ and $\liminf_{\mathbf{p}\to\bar{\mathbf{p}}} s_j(\mathbf{p}) > -\infty$ for any $j \notin K_{\bar{\mathbf{p}}}$.

Moreover, following the remarks from [29, 30], to assume continuity of the excess supply function on the nonnegative, nonzero price domain can be too strict under the weak gross substitutability hypothesis (the function in Example 2 do not satisfy Definition 5). Indeed, the function $\mathbf{s}(\mathbf{p})$ reduces to the trivially vanishing case under the former combined assumptions.
We seek, therefore, for some conditions which guarantee the existence of an equilibrium, but do not imply it is strictly positive. It is useful to assume $\mathbf{s}(\cdot)$ is differentiable only on the interior of the simplex S^{n+1} (and eventually on some points of the boundary), together

to suitable boundary conditions, which imply the existence of semipositive equilibria. Some results with very general assumptions are in the papers [5, 29], but a result with a simpler proof, a "classroom note" as the author states, can be found in [28].

Theorem 3. *Let $s_i(\mathbf{p})$, $i = 0, \ldots, n$ be functions $s_i : \mathbb{R}^{n+1}_+ \to \mathbb{R}$ such that:*

(1) *$(H2)$ is in force;*
(2) *$s_i(\cdot)$ is differentiable (hence continuous) in $\operatorname{int} S^{n+1} = \{\mathbf{p} \in \mathbb{R}^{n+1}_+ \mid \sum_{i=0}^n p_i = 1\}$, and bounded from above;*
(3) *$\sum_{i=0}^n s_i(\mathbf{p}) \to -\infty$ as $\mathbf{p} \to \bar{\mathbf{p}} \in \operatorname{bd} S^{n+1}$;*
(4) *$\sum_{i=0}^n p_i s_i(\mathbf{p}) \geqq 0$, $\forall \mathbf{p} \in S^{n+1}$ (the latter clearly implies $(H3)$ and it is somewhere referred to as* weak Walras Law *[29]);*
(5) *$\mathbf{s}(\cdot)$ admits partial derivatives on the points of $\operatorname{bd} S^{n+1}$ where $\mathbf{s}(\cdot)$ is defined. In all points of S^{n+1} where $\mathbf{s}(\cdot)$ is defined weak gross substitutability is in force.*

Then there exists an equilibrium price vector $\mathbf{p}^* \in S^{n+1}$ (that is some components are allowed to be null).

Remark 5. The function in Example 1 satisfies the assumptions in Theorem 3

We can now give a correct proof of the arguments followed by Hahn.

Theorem 4. *Let $\mathbf{s}(\cdot) : S^{n+1} \to \mathbb{R}^{n+1}$ be differentiable over $\operatorname{int} \mathbb{R}^{n+1}_+$ and such that assumptions i)-iv) hold true. Assume also that $\mathbf{s}(\cdot)$ admits partial derivatives on $\operatorname{bd} S^{n+1}$, (2) is in force and $\mathbf{p}^* \in S^{n+1}$ is an equilibrium. Then it holds:*

$$\mathbf{p}^{*\top}\mathbf{s}(\mathbf{p}) < 0, \ \forall \mathbf{p} \notin E, \ p \in S^{n+1}$$

where E is the set of all equilibria

Proof: The equilibrium price $\mathbf{p}^*$ now might have some zero components. Following [11, 12], we define the sets

$$R^+ = \{i\,, p_i^* > 0\} \qquad R^0 = \{i\,, p_i^* = 0\}.$$

Without loss of generality we may assume $\mathbf{p}^* = (\mathbf{p}^*_{R^+}, \mathbf{p}^*_{R^0})$ that is we have first the positive components. Note that $\displaystyle\sum_{i \in R^+} p_i^* = 1$ since $\mathbf{p}^* \in S^{n+1}$. Hence also the vector $\mathbf{p}_{R^+}$ belongs to a unit sphere.
Now, by 5 we have that for all $j = 0, \ldots, n$, $s_j(\cdot)$ are decreasing, that is

$$s_j(\mathbf{p}_{R^+}, \mathbf{p}_{R^0}) \leq s_j(\mathbf{p}_{R^+}, \mathbf{0}) \qquad \text{since } 0 \leq p_k, \ \forall k \in R^0$$

Therefore the thesis is equivalent to prove that $\mathbf{p}^*$ is a solution of the problem

$$\max_K g(\mathbf{p}) = \mathbf{p}^{*\top}\mathbf{s}(\mathbf{p})$$

where $K := \{\mathbf{p} \in \mathbb{R}^{n+1}_+\,, p_k = 0, \ \forall k \in R^0\}$, and that no maximizer of g can be found in the set $K \backslash E$.
In order to prove that $\mathbf{p}^*$ is a solution we can proceed as in Theorem 2, since again we are considering a closed simplex. We omit the proof since it would be a repetition of the arguments. To prove that no maxima occurs at $K \backslash E$, we can assume by contradiction that

$\tilde{\mathbf{p}} \in K \backslash E$ is a maximizer of g. For all $r \in R^+$ one has $k_r = \tilde{p}_r / p_r^*$ and, without loss of generality it might be $k_0 \geq k_1 \geq \ldots$, for $k_0 > 0$. Therefore it holds by (2) and (H2)

$$0 = s_0(\mathbf{p}^*) = s_0(k_0 \mathbf{p}^*) \leq s_0(\tilde{\mathbf{p}}) \tag{7}$$

If the latter inequality is strict then Walras law implies the contradiction

$$\begin{aligned} k_0 \sum_{i \in R^+} p_i^* \frac{\partial s_i}{\partial p_0}(\tilde{\mathbf{p}}) &\leq p_0 \frac{\partial s_0}{\partial p_0}(\tilde{\mathbf{p}}) + \sum_{i \neq 0} p_i \frac{\partial s_i}{\partial p_0}(\tilde{\mathbf{p}}) \\ &= -s_0(\tilde{\mathbf{p}}) < 0 \end{aligned}$$

Hence (7) is an equality which implies $\dfrac{\partial s_i}{\partial p_j}(\tilde{\mathbf{p}}) \tilde{p}_j = 0$ for all $j \in R^+$. If the latter is not true, weak gross substitutability and continuity would imply the strict inequality in (7). So we must find a different index $r \in R^+$ for which the optimality condition (4) is violated. Say there exists $\bar{r} \in R^+$ such that $s_{\bar{r}}(\tilde{\mathbf{p}}) > 0$. The previous reasoning implies there must be an index $h \in R^+$ such that $k_h = 0$.
Hence assume $s_i(\tilde{\mathbf{p}}) = 0$ for all $i < h$. Then it follows (for the details we refer to [3]) that $s_i(\tilde{\mathbf{p}}) \leq 0$, for all $i \geq h$, with strict inequality for some i. Hence ${\mathbf{p}^*}^\top \mathbf{s}(\tilde{\mathbf{p}}) < 0$ which contradict the assumption $\tilde{\mathbf{p}}$ is a maximizer. $\square$

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