

## ON PEDIGREE POLYTOPE AND ITS PROPERTIES

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**ABSTRACT.** The fact that linear optimization over a polytope can be done in polynomial time in the input size of the instance, has created renewed interest in studying  $0-1$  polytopes corresponding to combinatorial optimization problems. Studying their polyhedral structure has resulted in new algorithms to solve very large instances of some difficult problems like the symmetric traveling salesman problem. The multistage insertion formulation (*MI*) given by the author, in 1982, for the symmetric traveling salesman problem (*STSP*), gives rise to a combinatorial object called the pedigree. The pedigrees are in one-to-one correspondence with Hamiltonian cycles. Given  $n$ , the convex hull of all the pedigrees is called the corresponding pedigree polytope. In this article we bring together the research done a little over a decade by the author and his doctoral students, on the pedigree polytope, its structure, membership problem and properties of the *MI* formulation for the *STSP*. In addition we summarise some of the computational and other peripheral results relating to pedigree approach to solve the *STSP*. The pedigree polytope possesses properties not shared by the *STSP* (tour) polytope, which makes it interesting to study the pedigrees, both from theoretical and algorithmic perspectives.

### 1. Introduction

Several operations research, economic and computational biological problems can be formulated as programming problems with linear objectives seeking  $0-1$  integer solutions (Dell'Amico, Maffioli, and Martello 1997). These problems call for the study of polytopes whose vertices are  $0-1$  vectors, called  $0-1$  polytopes. Problems involving graphs formulated as combinatorial optimization problems (*COP*) also lead to the study of such polytopes (Chvátal 1975) (Sec. 2 contains formal definition of a *COP* and some typical problems). State-of-the-art concepts, theoretical results and algorithms for *COP* are presented in a recent book by Korte and Vygen (2012).

Finding whether a graph has a Hamiltonian cycle or not plays an important role directly or indirectly in solving problems in science, technology and business applications (Dell'Amico, Maffioli, and Martello 1997). The Symmetric Traveling Salesman Problem (*STSP*) is closely related to this problem and is about finding a minimum cost *tour* that starts from the home city and visits every city once and returns back to the home city, and the cost of traveling from city  $i$  to city  $j$  is the same as that of traveling from city  $j$  to city  $i$ . *STSP* is a typical, difficult combinatorial optimization problem, and has been extensively studied (Gutin and Punnen 2002; Jünger, Reinelt, and Rinaldi 1997; Lawler *et al.* 1985).

The standard formulation due to Dantzig, Fulkerson, and Johnson (1954) renders the asymmetric traveling salesman problem (ATSP), where the cost of traveling from city  $i$  to city  $j$  can be different from that of city  $j$  to city  $i$ , as a 0-1 integer program. Analogously the *STSP* can be formulated as an integer programming problem.

Dantzig, Fulkerson, and Johnson (1954) also solved a large size *TSP* to optimality by sequentially adding cutting planes (inequality constraints) to separate the current non-integer solution from the feasible region of the initial linear programming problem. This work became seminal for the polyhedral approaches that followed to solve the *STSP* (Applegate *et al.* 2006).

Studying adjacency structure of 0 – 1 polytopes has interested researchers both from theoretical and practical algorithmic perspectives. Success of Dantzig's simplex method, for solving linear programming problems, created interest in adjacent vertex improvement methods. This is one impetus for studying adjacency structure of 0 – 1 polytopes (Ikura and Nemhauser 1985). It is well known that testing whether a given pair of vertices are adjacent can be done in polynomial time in some classical *COP* polytopes. The description of the classical polytopes mentioned in this paper are available in Korte and Vygen (2012). Matching polytopes (Balinsky and Russakoff 1974; Chvátal 1975), set partitioning polytopes (Balas and Padberg 1972, 1975, 1976), vertex packing polytopes (Chvátal 1975), set packing polytopes (Ikura and Nemhauser 1985) permit polynomial time adjacency testing. On the other hand Papadimitriou (1978) has shown that the problem of checking nonadjacency on the traveling salesman polytope is *NP*-complete. Matsui shows that knapsack polytopes, set covering polytopes, among others, also share a similar fate (Matsui 1994).

Naddef and Pulleyblank define a sub-class of 0 – 1 polytopes called *combinatorial polytopes* that have an interesting property (combinatorial property) with respect to nonadjacency. If the midpoint of the line joining any pair of nonadjacent vertices is the midpoint of the line joining another pair of vertices, the polytope has combinatorial property. They show that perfect matching polytopes, stable set polytopes, permutation polytopes and matroid basis polytopes, node packing polytopes share this property (Naddef and Pulleyblank 1981). Naddef and Pulleyblank show that the graph of combinatorial polytopes are either hypercubes or else a hamiltonian path exists between every pair of nodes. And this result provides alternative proofs for the hamiltonicity of the graphs of the above mentioned combinatorial polytopes.

Matsui and Tamura (1995) observe two properties ( properties *A* and *B* given in Sec. 8) shared by 0 – 1 polytopes that are the convex hull of the set  $\{\mathbf{x} \in \{0, 1\}^n \mid \mathbf{Ax} = \mathbf{b}\}$ , where *A* is a  $m \times n$  matrix and *b* is an *m* dimensional vector. They show that every 0 – 1 polytope has property *A*. If a 0 – 1 polytope satisfies property *B* then it is a combinatorial polytope. Ikebe, Matsui, and Tamura (1993) say that a polytope *P* satisfies the *strong adjacency property* if every best valued vertex of *P* is adjacent to some second best valued vertex of *P* for each cost function. They also define a property (*P1*) that generalizes the combinatorial property of Naddef and Pulleyblank. They show that *P1* polytopes have strong adjacency property. As a corollary combinatorial polytopes have strong adjacency property (Ikebe, Matsui, and Tamura 1993).

Solving an optimization problem like the *STSP* efficiently is a theoretical challenge (Garey and Johnson 1979). Polyhedral combinatorics deals with the study of polytopes with the corner points (vertices) corresponding to objects of interest like the tours (Korte and

Vygen 2012). A polyhedral approach, like branch and cut, was used on the largest *STSP* problem solved to date (*Concorde TSP Solver*).

The research on the pedigree polytope departs from the standard formulation (Dantzig, Fulkerson, and Johnson 1954) and other formulations of the *STSP* (Bellman 1962; Fox, Gavish, and Graves 1980; Lawler *et al.* 1985; Miller, Tucker, and Zemlin 1960), and is based on the multistage insertion (*MI*) formulation of the *STSP* (Arthanari 1983; Arthanari and Usha 2000). *Insertion* is a local search heuristics commonly employed to generate a tour involving  $k + 1$  cities from a tour that involves  $k$  cities, where  $k$  varies from 3 to  $n - 1$  (Dell'Amico, Maffioli, and Martello 1997; Lawler *et al.* 1985). The sequence of insertion decisions made to insert city  $k + 1$  in an edge available in the  $k$ -tour resulting from the earlier insertion decisions starting with the unique 3-city tour,  $(1, 2, 3, 1)$  was formulated by Arthanari (1983) as an integer programming problem (*MI*-formulation), solving which yields an optimal tour.

Arthanari and Usha (2000) study some properties of the *MI*-formulation, which has polynomially many constraints and variables. Arthanari (2006) considers integer feasible solutions of the *MI* formulation and defines a combinatorial object called *pedigree* and study the properties of the corresponding polytope. A polynomial time nonadjacency testing algorithm for pedigree polytopes is also developed there. Arthanari (2005) observes that the pedigree polytope is a combinatorial polytope. As a consequence shows that the pedigree polytopes have the strong adjacency property. Also it is shown that the pedigree polytopes do not satisfy property *B* of Matsui and Tamura.

The paper is organised as follows: Sec. 2 gives notations and preliminaries and Subsec. 2.1 defines the main object of interest, namely, the pedigree. In Sec. 3 dimension of the pedigree polytope is established. Sec. 4 deals with the connections between the complexities of optimization, separation and membership problems for a polytope. In Sec. 5 some valid inequalities for pedigree polytope are given and it is shown that the integer solutions satisfying these inequalities and some equalities are precisely the pedigrees. Next in Sec. 6 we present the *MI* formulation for *STSP* given by Arthanari (1983), bringing out the connections between the pedigrees and the tours through the slack variables of the *MI* relaxation. Theorems characterising membership in pedigree polytope are given in Sec. 7.

Some properties of interest of  $0 - 1$  polytopes are defined and discussed in Sec. 8. Pedigree polytope is a combinatorial polytope is established in Sec. 9. Sec. 9.3 very briefly summarises other related works. Finally conclusions and future research planned are given in Sec. 10.

## 2. Notations and preliminaries

Let  $R$  denote the set of reals. Similarly  $Q$ ,  $Z$ ,  $N$  denote the rationals, integers and natural numbers respectively, and  $B$  stands for the binary set of  $\{0, 1\}$ . Let  $R_+$  denote the set of non negative reals. Similarly the subscript  $_+$  is understood with rationals. Let  $R^d$  denote the set of  $d$ -tuples of reals. Similarly the superscript  $^d$  is understood with rationals, etc.

Let  $K_n = (V_n, E_n)$  be the complete graph of  $n \geq 4$  vertices, where  $V_n = \{1, \dots, n\}$  is the set of vertices labelled in some order, and  $E_n = \{e = (i, j) \mid i, j \in V_n, i < j\}$  is the set of edges. We denote the elements of  $E_n$  by  $e$  where  $e = (i, j)$ . Let  $p_k$  denote  $|E_k| = k(k - 1)/2$ . Let the elements of  $E_n$  be labelled as follows:  $(i, j) \in E_n$ , has the label,  $l_{ij} = p_{j-1} + i$ .

This means, edges  $(1,2), (1,3), (2,3) \in E_3$  are labelled, 1, 2, and 3 respectively. Once the elements in  $E_{n-1}$  are labelled then the elements of  $E_n \setminus E_{n-1}$  are labelled in increasing order of the first coordinate, namely  $i$ . Let  $\tau_n = \sum_{k=4}^n p_{k-1}$ .

Assume that the edges in  $E_n$  are ordered in increasing order of the edge labels.

For a subset  $F \subset E_n$  we write the *characteristic* vector of  $F$  by  $x_F \in R^{p_n}$  where

$$x_F(e) = \begin{cases} 1 & \text{if } e \in F, \\ 0 & \text{otherwise.} \end{cases}$$

For a subset  $S \subset V_n$  we write

$$E(S) = \{(i, j) | (i, j) \in E, i, j \in S\}.$$

Given  $u \in R^{p_n}$ ,  $F \subset E_n$ , we define,

$$u(F) = \sum_{e \in F} u(e).$$

For any subset  $S$  of vertices of  $V_n$ , let  $\delta(S)$  denote the set of edges in  $E_n$  with one end in  $S$  and the other in  $S^c = V_n \setminus S$ . For  $S = \{i\}$ , we write  $\delta(\{i\}) = \delta(i)$ .

A subset  $H$  of  $E_n$  is called a *Hamiltonian cycle* in  $K_n$  if it is the edge set of a simple cycle in  $K_n$ , of length  $n$ . We also call such a Hamiltonian cycle a  $n$ -tour in  $K_n$ . At times we represent  $H$  by the vector  $(i_2 \dots i_n 1)$  where  $(i_2 \dots i_n)$  is a permutation of  $(2 \dots n)$ , corresponding to  $H$ . Let  $\mathcal{H}_n$  denotes the set of all *Hamiltonian cycles* (or  $n$ -tours) in  $K_n$ .

Let  $\mathbf{E}$  be a finite set, called the ground set. Let  $\mathcal{F}$  denote a collection of subsets of  $\mathbf{E}$ . Let  $c : \mathcal{F} \rightarrow \mathbf{R}$  denote a cost function. In an abstract way, a combinatorial optimization problem (*COP*) can be posed as: Find a  $X \in \mathcal{F}$  that minimizes  $c(X)$ .

Let  $\{0, 1\}^{|\mathbf{E}|}$  denote the set of all 0-1 vectors indexed by  $\mathbf{E}$ . Since any subset of  $\mathbf{E}$  can be given by a 0-1 vector, called the incidence vector, the collection  $\mathcal{F}$  can be equivalently given by a subset  $F$  of  $\{0, 1\}^{|\mathbf{E}|}$ . And the convex hull of  $F$ , denoted by  $\text{conv}(F)$ , is a 0-1 polytope. And the set of vertices of the polytope can be seen as  $F$ . We specify a combinatorial optimization problem by giving  $(\mathbf{E}, F, c)$ . For example, finding a Hamiltonian cycle that minimizes a linear objective function over the set of all Hamiltonian cycles (or  $n$ -tours) in  $K_n$  is a *COP*. This problem is also known as the *symmetric travelling salesman problem (STSP)*.

Here the ground set  $\mathbf{E}$  is the set of edges in a complete graph on  $n$  vertices,  $E_n$ .  $F$  is the set of incidence vectors of  $H \in \mathcal{H}_n$ . And we are given  $c \in R^{p_n}$ . Let  $Q_n$  denote the polytope  $\text{conv}(F)$ . *STSP* is a typical *COP* which is known to be *NP-hard*. In polyhedral combinatorics,  $Q_n$  is studied while solving the *STSP* (see Lawler *et al.* 1985; Jünger, Reinelt, and Rinaldi 1997).

**2.1. Defining pedigree.** Let  $e = (i, j) \in E_{k-1}$ . Inserting  $k$  in  $e$  is equivalent to replacing  $e$  by  $\{(i, k), (j, k)\}$ .

First we define a generator of an edge.

**Definition 2.1.** Edge Generators: Given  $e = (i, j) \in E_n$ ,  $G(e)$  is called the set of generators of  $e$ , and it is defined as follows:

$$G(e) = \begin{cases} \delta(i) \cap E_{j-1}, & \text{if } j \geq 4, \\ E_3 \setminus \{e\}, & \text{otherwise.} \end{cases}$$

Since an edge  $e = (i, j), j > 3$  is generated by inserting  $j$  in any  $e'$  in the set  $G(e)$ , the name *generator* is used to denote any such edge.

**Example 2.1.** Consider  $n = 5, e = (1, 5)$ . Here  $j \geq 4$ , so, we have  $G(e) = \delta(i) \cap E_{j-1}$ . Since  $i = 1, \delta(1) = \{(1, 2), (1, 3), (1, 4), (1, 5)\}$ , and  $E_{j-1} = E_4 = \{(1, 2), (1, 3), (2, 3), (1, 4), (2, 4), (3, 4)\}$ . Therefore,  $G(e) = \{(1, 2), (1, 3), (1, 4)\}$ .

Consider  $e = (2, 3)$ . Since  $j \leq 3$ , we have  $G(e) = E_3 \setminus \{e\} = \{(1, 2), (1, 3)\}$ .

The following is one of the possible ways to define our main object of interest, namely, a pedigree. The other equivalent definitions are given in later Sections.

**Definition 2.2.** Given  $n$ , consider  $W = (e_4, \dots, e_n)$ , where  $e_k = (i_k, j_k)$  for  $1 \leq i_k < j_k \leq k - 1, 4 \leq k \leq n$ .  $W$  is called a pedigree if and only if

- (1)  $e_k, 4 \leq k \leq n$ , are all distinct,
- (2)  $e_k \in E_{k-1}, 4 \leq k \leq n$ , and
- (3) for every  $k, 5 \leq k \leq n$ , there exists a  $e' \in G(e_k)$  such that,  $e_q = e'$ , where  $q = \max\{4, j_k\}$ .

Let  $\mathcal{P}_n$  denote the set of all pedigrees for a given  $n > 3$ . For any  $4 \leq k \leq n$ , given an edge  $e \in E_{k-1}$ , with edge label  $l$ , we can associate a 0 – 1 vector,  $\mathbf{x}(e) \in B^{p^{k-1}}$ , such that,  $\mathbf{x}(e)$  has a 1 in the  $l$ th coordinate, and zeros elsewhere. That is,  $\mathbf{x}(e)$  is the indicator of  $e$ .

Let  $\mathbf{E} = E_3 \times E_4 \dots \times E_{n-1}$  be the ground set. Let  $B^{\tau_n}$  denote the set of all binary vectors with  $\tau_n$  coordinates. That is, here  $\{0, 1\}^{|\mathbf{E}|} = B^{\tau_n}$ . Then, we can associate an  $X = (\mathbf{x}_4, \dots, \mathbf{x}_n) \in B^{\tau_n}$ , the characteristic vector of the pedigree  $W$ , where  $(W)_k = e_k$ , the  $(k - 3)$ rd component of  $W, 4 \leq k \leq n$  and  $\mathbf{x}_k$  is the indicator of  $e_k$ .

Let  $P_n = \{X \in B^{\tau_n} : X \text{ is the characteristic vector of } W, \text{ a pedigree}\}$ .

Consider the convex hull of  $P_n$ . We call this the *pedigree polytope*, denoted by  $\text{conv}(P_n)$ .

Given a cost vector  $C \in R^{\tau_n}$ , we wish to find a pedigree  $X^*$  in  $P_n$  that minimises  $CX^*$ . We have a combinatorial optimization problem, called the pedigree optimization problem (*POP*). Subsequently we shall show that the symmetric traveling salesman problem can be posed as a *POP*.

**Definition 2.3.** Let  $y(e)$  be the indicator of  $e \in E_k$ . Given a pedigree,  $W = (e_4, \dots, e_k)$  (with the characteristic vector,  $X \in P_k$ ) and an edge  $e \in E_k$ , we call  $(W, e) = (e_4, \dots, e_k, e)$  an *extension* of  $W$  in case  $(X, y(e)) \in P_{k+1}$ .

A pedigree  $W = (e_4, \dots, e_n) \in \mathcal{P}_n$  is such that  $(e_4, \dots, e_k)$  is a pedigree in  $\mathcal{P}_k$ , for  $4 \leq k \leq n$ . We state this interesting property as a fact below:

Given  $n > 3, X = (\mathbf{x}_4, \dots, \mathbf{x}_n) \in P_n$ , let  $X$  restricted to the first  $k - 3$  stage(s), be written as

$$X/k = (\mathbf{x}_4, \dots, \mathbf{x}_k)$$

**Fact:** Given  $X \in P_n$  and any  $k$  such that  $4 \leq k \leq n, X/k$  is in  $P_k$ .

Similarly,  $X/(k - 1)$  and  $X/(k + 1)$  are interpreted as restrictions of  $X$ .

**Example 2.2.** Consider  $W$  given by

$$W = (e_4 = (1, 3), e_5 = (2, 3), e_6 = (3, 4), e_7 = (2, 5)).$$

$W$  is a pedigree because 1] all the edges are distinct 2]  $e_k \in E_{k-1}, k = 4, \dots, 7$  and 3]  $e_4 = (1, 3)$  is a generator of  $e_5$  and  $e_6 = (3, 4)$ . Also  $e_5 = (2, 3)$  is a generator of  $e_7 = (2, 5)$ .

Let  $e = (3, 5) \in E_7$ . Here  $q = 5$ . Then  $(W, e)$  is a pedigree as 1]  $e$  does not appear in  $W$ , 2]  $G(e) = \{(1, 3), (2, 3), (3, 4)\}$  and  $e_5 = (2, 3)$  is in  $G(e)$ . However, if  $e$  were  $(2, 6)$ , we have  $q = 6$ , and  $G(e) = \{(1, 2), (2, 3), (2, 4), (2, 5)\}$ . But  $e_6 = (3, 4)$  is not in  $G(e)$ . So  $(W, e)$  is not an extension of  $W$ .

### 3. Dimension of the pedigree polytope

Grötschel and Padberg (1985) have shown that the dimension of the *STSP* polytope  $Q_n$  is  $d_n = \frac{n(n-3)}{2}$ . Here we show that the dimension of  $conv(P_n)$  is  $\tau_n - (n - 3)$ , recall that  $\tau_n$  is the number of coordinates of a  $X \in P_n$ .

First we notice that all pedigrees satisfy the Eq. 1.

**Lemma 3.1.**  $X \in P_n$  implies  $X \geq \mathbf{0}$  and

$$x_k(E_{k-1}) = 1, k \in V_n \setminus V_3. \tag{1}$$

**Proof:** As  $X \in B^{\tau_n}$ ,  $X$  trivially satisfies the non negativity restrictions. Since  $W$  is a pedigree,  $e_k, 4 \leq k \leq n$  are all distinct and  $e_k \in E_{k-1}$ . As  $\mathbf{x}_k$  is the indicator of  $e_k$ ,  $\sum_{e \in E_{k-1}} \mathbf{x}_k(e) = \mathbf{x}_k(E_{k-1}) = 1, k \in V_n \setminus V_3$ . Hence the lemma.  $\square$

Let  $\mathcal{D} = \{X \in R^{\tau_n} | X \text{ satisfies the Equations 1}\}$ . And these  $n - 3$  equations are non redundant. So  $dim(\mathcal{D}) = \tau_n - (n - 3)$ . As every pedigree satisfies Eq. (1),  $conv(P_n) \subset \mathcal{D}$ , therefore

$$dim(conv(P_n)) \leq \tau_n - (n - 3) \stackrel{def}{=} \tau'_n.$$

**Definition 3.1.** [Projection  $M$  ] Given  $X \in P_n$ , consider the transformation  $Y = MX$ , where  $M$  deletes the  $p_{k-1}^h$  component of  $\mathbf{x}_k$  in  $X$ , giving a vector  $Y \in B^{\tau'_n}$ .

The projection  $M$  is given by the matrix

$$M = \begin{bmatrix} I_{p_{3-1}} & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & I_{p_{4-1}} & \vdots & & \vdots \\ \vdots & \vdots & & \dots & 0 & 0 \\ 0 & 0 & \dots & & I_{p_{n-1-1}} & 0 \end{bmatrix}.$$

We next direct our search for a full dimensional polytope with dimension  $\tau'_n$ . And it is obtained as a projection of the pedigree polytope.

Let  $e^k = (k - 2, k - 1)$ , and  $E'_{k-1} = E_{k-1} \setminus \{e^k\}$ , for  $k \in V_n \setminus V_3$ .

**Definition 3.2** ( $A_n$ ). Let  $A_n = \{Y \in B^{\tau'_n} | Y = MX, X \in P_n\}$ .

**Lemma 3.2.** There is a 1 – 1 correspondence between  $A_n$  and  $P_n$ .

*Proof.* Given any  $X \in P_n$ ,  $Y \in A_n$ , given by the transformation  $M$  of  $X$  is unique. On the other hand, given a  $Y \in A_n$ , we can uniquely define,

$$x_k(e) = \begin{cases} y_k(e) & \text{if } e \in E'_{k-1} \\ 1 - y_k(E'_{k-1}) & \text{for } e = e^k. \end{cases}$$

Hence the lemma.  $\square$

We use the following lemma in proving the dimension of  $conv(A_n)$ .

**Lemma 3.3.** Given  $k \in V_{n-3} \setminus V_3$ , and  $e = (i, j) \in E'_k$ , we can select a pedigree in  $P_n$  such that  $x_{k+1}(e) = 1$ ,  $x_{k+2}((i, k+1)) = 1$  and  $x_l(e^l) = 1, k+3 \leq l \leq n$ .

**Proof:** Given  $k$  and  $e$  select a  $k$ -tour containing  $e$  as an edge and insert  $k+1$  in  $e$  obtaining a  $k+1$ -tour. Choose  $(i, k+1)$  for insertion of  $k+2$  obtaining a  $k+2$ -tour, which has  $e^{k+3} = (k+1, k+2)$  as an edge. Now extend this  $k+2$ -tour to a  $n$ -tour by sequentially inserting  $l$  in  $e^l$  for  $l$  ranging from  $k+3$  through  $n$ . Consider the corresponding pedigree  $X \in P_n$ . This has the required property.  $\square$

**Theorem 3.1.** Let  $A_n = \{Y \in R^{\tau'_n} \mid Y = MX, X \in P_n\}$ . The dimension of the polytope  $\text{conv}(A_n)$  is  $\tau'_n$ .

**Proof:** Since  $\text{conv}(A_n) \subset R^{\tau'_n}$ ,  $\dim(\text{conv}(A_n)) \leq \tau'_n$ . Suppose  $\dim(\text{conv}(A_n)) < \tau'_n$ . This implies that there exists a hyperplane  $CY = c_0$  with  $(C, c_0) \in R^{\tau'_n+1}$  such that every vertex of  $\text{conv}(A_n)$  lies on this hyperplane. Let  $C = (\mathbf{c}_4, \dots, \mathbf{c}_n)$ , where  $\mathbf{c}_k$  denotes the component corresponding to  $(c_k(e), e \in E'_{k-1})$ .

**Claim 1:**  $c_0 = 0$ .

Proof of Claim 1: Consider the pedigree  $(e^4, \dots, e^n)$ . Then the corresponding  $Y \in A_n$  is  $\mathbf{0}$ . So  $CY = 0$  which implies  $c_0 = 0$  as required.

**Claim 2:**  $\mathbf{c}_4 = \mathbf{0}$ .

Proof of Claim 2: Consider  $Y$  corresponding to the pedigree given by

$$((1, 2), (1, 4), e^6, \dots, e^n).$$

Now  $CY = 0$  implies  $c_4((1, 2)) + c_5((1, 4)) = 0$ . But the pedigree  $((1, 3), e^5, \dots, e^n)$  yields  $c_4((1, 3)) = 0$ ; and the pedigree  $((1, 3), (1, 4), e^6, \dots, e^n)$  yields  $c_4((1, 3)) + c_5((1, 4)) = 0$ . Together we have  $c_5((1, 4)) = 0$ , and so  $c_4((1, 2)) = 0$ . Hence  $\mathbf{c}_4 = \mathbf{0}$ .

This forms the basis for the proof by induction on  $k$ . Assuming  $\mathbf{c}_4, \dots, \mathbf{c}_k$  are all zero vectors we shall show that  $\mathbf{c}_{k+1} = \mathbf{0}$ .

Let  $e = (i, j) \in E'_k$ . Form Lemma 3.3 we have a pedigree in  $P_n$  such that  $x_{k+1}(e) = x_{k+2}(i, k+1) = 1$  and  $x_l(e^l) = 1, k+3 \leq l \leq n$ . For the corresponding  $Y$  we then have  $CY = 0 + c_{k+1}(e) + c_{k+2}((i, k+1)) + 0 = 0$ . This implies

$$c_{k+1}(e) + c_{k+2}((i, k+1)) = 0. \tag{2}$$

Consider  $Q$  in  $P_k$  such that a generator of  $(i, k)$  is available in  $Q$ . From  $Y \in A_n$  corresponding to the pedigree  $(Q, (i, k), e^{k+2}, \dots, e^n)$ , we get

$$c_{k+1}((i, k)) = 0. \tag{3}$$

From  $Y \in A_n$  corresponding to the pedigree  $(Q, (i, k), (i, k+1), e^{k+3}, \dots, e^n)$ , we get

$$c_{k+1}((i, k)) + c_{k+2}((i, k+1)) = 0. \tag{4}$$

Now using Equations (3 and 4) we have

$$c_{k+2}((i, k+1)) = 0. \tag{5}$$

Equations (2 and 5) yield  $c_{k+1}(e) = 0$ . Thus we have  $\mathbf{c}_{k+1} = \mathbf{0}$ .

Hence  $C = \mathbf{0}$  and  $c_0 = 0$  implying  $\dim(\text{conv}(A_n)) = \tau'_n$ .  $\square$

Now any  $Y \in \text{conv}(A_n)$  can be extended to a vector in  $R^{\tau_n}$  by augmenting the last coordinate  $y_k(e^k)$  to each component  $\mathbf{y}_k$  of  $Y$  corresponding to  $k \in V_n \setminus V_3$ . It is easy to

see, if  $y_k(e^k) = 1 - y_k(E'_{k-1})$  for each  $k$ , then such a vector is in  $\text{conv}(P_n)$ . Therefore  $\dim(\text{conv}(P_n)) \geq \tau'_n$ . Thus we have proved Theorem 3.2.

**Theorem 3.2.** *The dimension of the pedigree polytope  $\text{cov}(P_n)$  is  $\tau_n - (n - 3)$ .*

#### 4. Optimization, separation and membership problems

Here we repeat the preliminary material from Arthanari (2008), dealing with problems relating to polytopes, and their relative complexity of computation.

$P \subset \mathbb{R}^d$  is called a  $v$ -polytope, if  $P$  is the convex hull of finitely many points  $X_1, \dots, X_r$  in  $\mathbb{R}^d$ .  $P \subset \mathbb{R}^d$  is called a  $\mathcal{H}$ -polyhedron, if  $P$  is the intersection of finitely many half-spaces,  $\mathbf{a}_i X \leq a_0$ , for  $(\mathbf{a}_i, a_0) \in \mathbb{R}^{d+1}$ , for  $i = 1, \dots, s$ . It is well known that a bounded  $\mathcal{H}$ -polyhedron is indeed a  $v$ -polytope. The *affine rank* of a polytope  $P$  (denoted by  $\text{arank}(P)$ ) is defined as the maximum number of affinely independent vectors in  $P$ . The *dimension* of a polytope  $P$  is (denoted  $\dim(P)$ ) and defined to be  $\text{arank}(P)$  minus 1.

Let  $P \subset \mathbb{R}^d$  be a polytope. The *barycentre* of  $P$  is defined as

$$\bar{X} = 1/p \sum_{X^i \in \text{vert}(P)} X^i,$$

where  $p$  is the cardinality of  $\text{vert}(P)$ , the vertex set of  $P$ . (for introduction to polytopes see Ziegler 2002).

Given  $n \in \mathbb{Z}$  the *input size* of  $n$  is the number of digits in the binary expansion of the number  $n$  plus 1 for the sign if  $n$  is non zero. We write,

$$\langle n \rangle = 1 + \lceil \log_2(n + 1) \rceil.$$

Input size of  $n$ ,  $\langle n \rangle$ , is also known as the *digital size* of  $n$ .

Given  $r = p/q$  a rational number, where  $p$  and  $q$  are mutually prime, that is,  $\text{gcd}(p, q) = 1$ , we have input size of  $r$  given by

$$\langle r \rangle = \langle p \rangle + \langle q \rangle.$$

For every rational  $r$  we have  $|r| \leq 2^{\langle r \rangle - 1} - 1$ .

**Definition 4.1.** [Rationality Guarantee] Let  $P \subset \mathbb{R}^d$  be a polytope and  $\phi$  and  $v$  positive integers. We say that  $P$  has *facet complexity* at most  $\phi$  if  $P$  can be described as the solution set of a system of linear inequalities each of which has input size  $\leq \phi$ . We say,  $P$  has *vertex complexity* at most  $v$  if  $P$  can be written as  $P = \text{conv}(V)$ , where  $V \subset \mathbb{Q}^d$  is finite and each vector in  $V$  has input size  $\leq v$ .

We have Lemma 4.1 from Grötschel and Schrijver (1988) connecting facet and vertex complexities.

**Lemma 4.1.** Let  $P \subset \mathbb{R}^d$  be a non empty, full dimensional polytope. If  $P$  has vertex complexity at most  $v$ , then  $P$  has facet complexity at most  $3d^2v$ .

Given a polytope  $P \subset \mathbb{Q}^d$ , and a  $Y \in \mathbb{Q}^d$ , the problem to decide whether  $Y \in P$  or not, is called the *membership problem* for  $P$ . Let  $P \subset \mathbb{Q}^d$ , be a polytope with facet complexity at most  $\phi$ . Let  $\text{MemAl}(P, Y, \text{Answer})$  be an algorithm<sup>1</sup> to solve the membership problem,

<sup>1</sup> The term *oracle* or *subroutine* is generally used to mean that this algorithm is called by another algorithm as a procedure.

where  $P$  is known to *MemAl* not necessarily explicitly, and on input of  $Y \in Q^d$  having input size  $\langle Y \rangle$ , *MemAl* halts with *Answer* = *yes* if  $Y \in P$  and *Answer* = *no* otherwise. If the membership checking time of *MemAl* is polynomially bounded above by a function of  $(d, \phi, \langle Y \rangle)$  we say *MemAl* is an efficient oracle.

Given a polytope  $P \subset Q^d$ , and a  $Y \in Q^d$ , the problem to decide whether  $Y \in P$ , and if  $Y \notin P$  then identifying a hyperplane that separates  $P$  and  $Y$ , is called the *separation problem* for  $P$ . Identifying a hyperplane is achieved through finding a vector  $a \in Q^d$  such that  $aX < aY$  for all  $X \in P$

Given a nonempty polytope  $P \subset Q^d$ , and a  $C \in Q^d$ , the problem of finding a  $X^* \in P \ni CX^* \leq CX$  for all  $X \in P$  is called the *linear optimization problem* for  $P$ .

Recently, Maurras (2002) shows that an intuitively appealing construction is possible for the separation problem of a polytope, by finding a hyperplane separating the polytope and a point not in the polytope, after a polynomial number of calls to a membership oracle. The conditions under which this is possible are same as that of Yudin and Nemirovsky (1976), namely,

**Assuption 4.1** (Maurras’s Conditions).

- 1 The polytope  $P$  is well defined in the  $d$ -dimensional space of  $Q^d$  of rational vectors. (There is a bound on the encoding length of any vertex of  $P$ . The polytope is rationality guaranteed.)
- 2  $P$  has non-empty interior.
- 3  $a \in \text{int}(P)$  is given.

In the book *Geometric Algorithms* (Grötschel and Schrijver 1988) we find a construction due to Yudin and Nemirovsky (1976) to devise a polynomial algorithm for finding a separating plane using a membership oracle, when a convex set  $K$  instead of the polytope  $P$  is considered. But this algorithm requires in addition the radii of the inscribed and circumscribed balls, also uses Ellipsoid algorithm twice.

Next we check that the Assumption 4.1 is met for a projection of the pedigree polytope.

**4.1. Checking  $\text{conv}(A_n)$  meets Maurras’s conditions.** Recall that  $e^k = (k-2, k-1)$ , and  $E'_{k-1} = E_{k-1} \setminus \{e^k\}$ , for  $k \in V_n \setminus V_3$  and  $\tau'_n = \tau_n - (n-3)$ .

**Theorem 4.1.** [Maurras’s Conditions Met] Given  $n \geq 4$ , and  $A_n$  as defined earlier in Sec. 3, we have,

- (1) The polytope  $\text{conv}(A_n)$  is full dimensional, that is  $\dim(\text{conv}(A_n)) = \tau'_n$
- (2) The barycentre of  $\text{conv}(A_n)$  is given by,

$$\bar{Y} = [2/(n-1)!] \underbrace{(1/p_3, 1/p_3, \dots, 1/p_{n-1}, \dots, 1/p_{n-1})}_{2\text{-times}} \underbrace{\phantom{(1/p_3, 1/p_3, \dots, 1/p_{n-1}, \dots, 1/p_{n-1})}}_{p_{n-1}-1 \text{ times}}$$

- (3)  $\bar{Y} \in \text{int}(\text{conv}(A_n))$ .

*Proof.* Part 1 of the theorem is from Theorem 3.1.

Part 2 can be verified by noticing that the cardinality of  $\text{vert}(\text{conv}(A_n))$ , is  $(n-1)!/2$ , and in any  $X$  in  $P_n$ , the  $(k-3)^{\text{rd}}$  component has  $p_{k-1}$  coordinates, and exactly one of the coordinates is a 1. In  $P_n$  for any component the 1 appears equally likely among the coordinates. And for any  $Y \in A_n$  we have deleted the last coordinate in each component of the corresponding  $X$ .

Proof of part 3 of the theorem follows from the fact that  $\bar{Y}$  does not lie on any facet defining hyperplane  $CY = c_0$ , for  $(C, c_0) \in Q^{\tau'_n+1}$ . Suppose, it lies on some facet defining hyperplane  $CY = c_0$  ( that is  $CY \leq c_0$  for all  $Y \in \text{conv}(A_n)$ ). Then

$$C\bar{Y} - c_0 = [2/(n - 1)!] \sum_{X \in P_n} (CY^X - c_0) = 0,$$

where  $Y^X$  is the element of  $A_n$  corresponding to a  $X \in P_n$ . Thus for all  $X \in P_n$ , we have  $CY^X = c_0$ ,

$$\implies \dim(\text{conv}(A_n)) \leq \tau'_n - 1.$$

This contradicts the fact  $\dim(\text{conv}(A_n))$  is  $\tau'_n$ .

Therefore  $\bar{Y} \in \text{int}(\text{conv}(A_n))$ . Hence the theorem. □

**Theorem 4.2.** [*Facet - Complexity of  $\text{conv}(A_n)$* ]  $\text{conv}(A_n)$  has facet complexity at most  $\phi = 3\tau_n'^3 + 3\tau_n'^2(n - 3)$ . That is  $\text{conv}(A_n)$  is rationality guaranteed.

*Proof.* Each vertex  $Y$  of  $\text{conv}(A_n)$  is a 0 – 1 vector of length  $\tau'_n$ . So  $Y$  can be encoded with input size

$$\langle Y \rangle \leq \tau'_n + (n - 3) = v.$$

(This follows from the fact that there are at most  $n - 3$  1's in any  $Y$  and  $\langle 0 \rangle = 1$  &  $\langle 1 \rangle = 1 + \lceil \log_2 2 \rceil = 2$ .)

Therefore,  $\text{conv}(A_n)$  has vertex complexity  $\leq v$ .

Using lemma(4.1), we have, facet complexity of  $\text{conv}(A_n)$ ,

$$\begin{aligned} &\leq 3\tau_n'^2 v \\ &= 3\tau_n'^2 (\tau'_n + (n - 3)) \\ &= 3\tau_n'^3 + 3\tau_n'^2 (n - 3). \end{aligned}$$

Hence,  $\text{conv}(A_n)$  is rationality guaranteed. □

Thus we find  $\text{conv}(A_n)$  satisfies all the requirements of Maurras’s conditions (Assumption 4.1). Therefore if we have a membership oracle for  $\text{conv}(A_n)$  we can call that a polynomial number of times to separate a  $Y \in Q^{\tau'_n}$  from  $\text{conv}(A_n)$ . With this in view we direct our attention to the membership problem of the pedigree polytope, since  $Y$  is in  $\text{conv}(A_n)$  if and only if the corresponding pedigree  $X$  is in  $\text{conv}(P_n)$ .

The membership problem for the pedigree polytope is studied by Arthanari (2008) and a necessary condition that can be checked in polynomial time is given. Some sufficient conditions for membership are also discussed there.

### 5. Some valid inequalities for the pedigree polytope

Let  $X = (\mathbf{x}_4, \dots, \mathbf{x}_n) \in P_n$  correspond to the pedigree  $W = (e_4, \dots, e_n)$ . Though we have defined the pedigree polytope  $\text{conv}(P_n)$  using the pedigrees it is of interest to explore the equalities and inequalities that define the pedigree polytope. With this in view consider a polytope whose extreme points contain all the pedigrees, though not all extreme points of this polytope are integral.

**Definition 5.1** (Valid inequality). An inequality  $d^T x \leq d_0$  is *valid* for a set  $S \subset R^n$  if  $d^T x \leq d_0$  for all  $x \in S$ . The inequalities  $d^T x \leq d_0$  and  $d'^T x \leq d'_0$  are equivalent if  $(d, d_0) = \lambda(d', d'_0)$  for some  $\lambda > 0$ . If they are not equivalent but there is  $\lambda > 0$  such that  $d' \leq \lambda d$  and  $d'_0 \geq \lambda d_0$ , then  $\{x : x \geq 0, d'^T x \leq d'_0\} \subset \{x : x \geq 0, d^T x \leq d_0\}$  and we say that  $d'^T x \leq d'_0$  *dominates*  $d^T x \leq d_0$ . If a valid inequality is not dominated by any other valid inequality, it is called a *maximal* valid inequality.

The following lemmas give some valid inequalities for  $conv(P_n)$ .

**Lemma 5.1.**  $X \in P_n$  implies

$$\sum_{k=4}^n x_k(e) \leq 1, e \in E_3. \tag{6}$$

**Proof:** If  $e \neq e_k, \forall k$  then  $\sum_{k=4}^n x_k(e) = 0$ , so the inequality (6) is satisfied. Otherwise, let  $e = e_k$  for some  $k$ . Then  $x_k(e) = 1$ . Since  $e_k$  are all distinct,  $e_l \neq e, l \neq k$ . Therefore,  $x_l(e) = 0, l \neq k$ . Hence,  $\sum_{k=4}^n x_k(e) = 1$ , and the inequality (6) is satisfied.  $\square$

**Lemma 5.2.**  $X \in P_n$  implies

$$-x_j(\delta(i) \cap E_{j-1}) + \sum_{k=j+1}^n x_k(e) \leq 0, e = (i, j) \in E_{n-1} \setminus E_3. \tag{7}$$

**Proof:** Consider any  $e = (i, j) \in E_{n-1} \setminus E_3$ . Either  $e \neq e_k, k \geq j + 1$ , or  $e = e_l$  for some  $l \geq j + 1$ . In the first case,  $\sum_{k=j+1}^n x_k(e) = 0$ . So, inequality (7) is automatically satisfied for  $e$ . In the second case, as  $x_l(e_l) = 1$ , and the  $e_k$  are all distinct,  $\sum_{k=j+1}^n x_k(e) = 1$ . Now,  $\delta(i) \cap E_{j-1}$  is the set of generators of  $e = (i, j)$ . Since  $W$  is a pedigree, there exists a  $\acute{e} \in \delta(i) \cap E_{j-1}$  such that  $\acute{e} = e_j$ . (From Def. 2.2.) Therefore  $x_j(e_j) = 1$ . So  $x_j(\delta(i) \cap E_{j-1}) = 1$ . Hence,

$$-x_j(\delta(i) \cap E_{j-1}) + \sum_{k=j+1}^n x_k(e) = -1 + 1 = 0.$$

Hence, inequality (7) is satisfied, as an equality. Hence the lemma.  $\square$

Next we define the polytope given by the eq. 1 and inequalities 6, and 7.

**Definition 5.2.** [ $P_{MI}(n)$  – Polytope] Consider  $X \in R^{e_n}$  satisfying the non negativity restrictions,  $X \geq 0$  and the eq. (1) and the inequalities (6) and (7).

The set of all such  $X$  is a polytope, as we have defined it using linear equalities and inequalities. We call this polytope,  $P_{MI}(n)$ .

As every pedigree satisfies the equalities and inequalities that define  $P_{MI}(n)$ , we can now conclude that  $conv(P_n) \subset P_{MI}(n)$ . In addition we have,

**Theorem 5.1.**  $X \in P_n$  implies  $X$  is an extreme point of  $P_{MI}(n)$ .

**Proof:** We have shown by Lemmas (3.1), (5.1) and (5.2), that  $X \in P_{MI}(n)$ . Assume that the equations and then the inequalities are given in that order. Now consider the first  $(n - 3)$  rows of the submatrix formed by the columns corresponding to positive components of  $X$ , that is,  $x_k(e_k), k \in V_n \setminus V_3$ . Since this is an identity matrix of size  $n - 3$ , the columns corresponding to the positive components of  $X$  are linearly independent. Now these  $n - 3$

columns with the identity columns ( $p_{n-1}$  in all) corresponding to the slack variables of the inequality constraints, form a basis for the  $P_{MI}(n)$ - in standard form. Hence,  $X \in P_n$  corresponds to an extreme point  $P_{MI}(n)$ .  $\square$

*Remark 5.1.* The subscript  $MI$  in  $P_{MI}$  is used to refer to the fact that precisely these constraints are used in the  $MI$  formulation of  $STSP$  given by Arthanari (1983).

So while studying the membership problem for the pedigree polytope it is sufficient to consider  $X \in P_{MI}(n)$ . We study necessary and sufficient conditions for a  $X \in P_{MI}(n)$  to be in  $conv(P_n)$  through some characterisation theorems proved in Sec. 7.

## 6. The multistage insertion formulation

In this section we give the  $MI$ -formulation given by Arthanari (1983) and briefly outline some properties and its connection to the pedigrees, and the  $STSP$ . (Arthanari 2005, 2006, 2008). The standard formulation of the  $STSP$  is due to Dantzig, Fulkerson, and Johnson (1954) (DFJ). As every tour is a 2 matching, and the converse is not true, as a 2 matching could correspond to a subtour, they give a 0–1 programming formulation having  $n$  equalities and exponentially many inequalities that ensure subtours are eliminated from consideration. Given  $n$ , relaxing the integer restriction of the  $DFJ$  formulation we obtain the subtour elimination polytope, ( $SEP(n)$ ).

The  $MI$ -formulation is based on constructing  $STSP$  tours by sequentially inserting nodes into the initial tour of three nodes 1, 2 and 3. Given graph  $K_n$ , starting with tour  $T_3 = [1, 2, 3, 1]$ , nodes from 4 to  $n$  are inserted sequentially between the nodes of this tour until a complete tour of size  $n$  is achieved. For all  $1 \leq i < j \leq k-1$  and  $4 \leq k \leq n$ , the decision variables of the  $MI$ -formulation are defined as follows:

$$x_{ijk} = \begin{cases} 1, & \text{if node } k \text{ is inserted between nodes } i \text{ and } j, \\ 0, & \text{otherwise.} \end{cases}$$

We also use the equivalent notation  $x_k(e)$  for  $x_{ijk}$  when  $e = (i, j) \in E_{k-1}$ .

Let  $c_{ij}$  be the cost of an edge  $(i, j) \in E_n$ . The insertion of some node  $k$  between nodes  $i$  and  $j$ , would replace the edge  $(i, j)$  with two new edges of  $(i, k)$  and  $(j, k)$  in the tour. This will increase the total cost of the tour by  $C_{ijk} = c_{ik} + c_{jk} - c_{ij}$ . The objective function of the  $MI$ -formulation is to minimize the total incremental cost. The  $MI$ -formulation (Arthanari 1983) is:

$$\min \sum_{k=4}^n \sum_{(1 \leq i < j \leq k-1)} C_{ijk} x_{ijk}$$

subject to:

$$\sum_{1 \leq i < j \leq k-1} x_{ijk} = 1, \quad 4 \leq k \leq n, \tag{8}$$

$$\sum_{k=4}^n x_{ijk} \leq 1, \quad 1 \leq i < j \leq 3, \tag{9}$$

$$-\sum_{r=1}^{i-1} x_{rij} - \sum_{s=i+1}^{j-1} x_{isj} + \sum_{k=j+1}^n x_{ijk} \leq 0, \quad 1 \leq i < j, 4 \leq j \leq n-1, \tag{10}$$

$$x_{ijk} \in \{0, 1\}, \quad 1 \leq i < j \leq k-1, 4 \leq k \leq n. \tag{11}$$

Constraint (8) of the model guarantees that each node from 4 to  $n$  is inserted in some edge. Constraint (9) ensures that at most one node is inserted in each of the edges of  $T_3$ . Constraint (10) makes sure that a node is inserted into an edge of the subtour only if that edge has been generated through previous insertions and is available.

By relaxing the integer constraint from  $MI$ -formulation and also adding the redundant constraints

$$-\sum_{r=1}^{i-1} x_{rin} - \sum_{s=i+1}^{n-1} x_{isn} \leq 0, i = 1, \dots, n-1, \tag{12}$$

we obtain the  $MI$ -relaxation problem, Problem 2 of Arthanari and Usha (2000).

Let the slack variables corresponding to the  $MI$ -relaxation problem be denoted by  $u_{ij}$ , corresponding to the inequality for edge  $(i, j)$ . It is shown by Arthanari and Usha (2000) that for any  $MI$  problem instance of size  $n$ , if for some  $i, j \in V_n$  the slack variable  $u_{ij} = 1$ , then the edge  $(i, j)$  is present in the tour given by the optimal solution of the problem. Notice that, the polytope corresponding to the  $MI$ -relaxation problem is indeed the  $P_{MI}(n)$ . The affine transformation of  $P_{MI}(n)$ , projecting out  $x_{ijk}$  variables, is denoted by  $\mathcal{U}(n)$ . Arthanari and Usha (2000) compared  $\mathcal{U}(n)$  with the  $SEP(n)$ , and proved that  $\mathcal{U}(n) \subseteq SEP_n$ . Thus the  $MI$  formulation is as tight as the standard  $DFJ$  formulation and has only polynomially many constraints. Different formulations of  $TSP$  varying in size and in the strength of the LP relaxations have been compared in the literature (Langevin, Soumis, and Desrosiers 1990; Orman and Williams 2007; Padberg and Sung 1991).

### 7. Characterising membership in $conv(P_n)$

In this section, given a  $X \in P_{MI}(n)$  we wish to develop procedures to check whether  $X$  is indeed in  $conv(P_n)$ . Let  $|P_k|$  denote the cardinality of  $P_k$ .

**Definition 7.1.** Given  $X \in P_{MI}(n)$  and  $X/k \in conv(P_k)$ , consider  $\lambda \in R_+^{|P_k|}$  that can be used as a weight to express  $X/k$  as a convex combination of  $X^r \in P_k$ . Let  $I(\lambda)$  denote the index set of positive coordinates of  $\lambda$ . Let  $\Lambda_k(X)$  denote the set of all possible *weight vectors*, for a given  $X$  and  $k$ , that is,

$$\Lambda_k(X) = \left\{ \lambda \in R_+^{|P_k|} \mid \sum_{r \in I(\lambda), X^r \in P_k} \lambda_r X^r = X/k, \sum_{r \in I} \lambda_r = 1 \right\}.$$

**7.1. Some results from bipartite flow feasibility.** Next, we give the definition of a flow feasibility problem in bipartite networks, called the forbidden arcs transportation problem.

**Definition 7.2.** Forbidden Arcs Transportation Problem (FAT): The FAT problem can be defined as a variation of a capacitated transportation problem in a bipartite network, with

some of the arcs marked as forbidden. Given positive values for the supply (demand) of each origin (destination), the FAT problem seeks to find a feasible flow from the origins to the destinations.

For details on graph related terms see any standard text on graph theory such as Bondy and Murthy (2008) or on combinatorial optimization such as Korte and Vygen (2012).

## 7.2. Flow problems used in membership checking.

**Definition 7.3.** Consider a  $X \in P_{MI}(n)$  such that  $X/k \in \text{conv}(P_k)$ . We denote the  $k$ -tour corresponding to a pedigree  $X^\alpha$  by  $H^\alpha$ . Given a weight vector  $\lambda \in \Lambda_k(X)$ , we define a FAT problem with the following data:

$$\begin{array}{ll} O -- & \text{Origins] : } \alpha, \alpha \in I(\lambda) \\ a -- & \text{Supply] : } a_\alpha = \lambda_\alpha \\ D -- & \text{Destinations] : } \beta, e_\beta \in E_k, x_{k+1}(e_\beta) > 0 \\ b -- & \text{Demand] : } b_\beta = x_{k+1}(e_\beta) \\ \mathcal{A} -- & \text{Arcs] : } \{(\alpha, \beta) \in O \times D | e_\beta \in H^\alpha\} \end{array}$$

We designate this problem as  $FAT_k(\lambda)$ . Notice that arcs  $(\alpha, \beta)$  not satisfying  $e_\beta \in H^\alpha$  are the forbidden arcs. We also say  $FAT_k$  is feasible if problem  $FAT_k(\lambda)$  is feasible for some  $\lambda \in \Lambda_k(X)$ .

Equivalently, the arcs in  $\mathcal{A}$  can be interpreted as follows: If  $W^\alpha$  is the pedigree corresponding to  $X^\alpha \in P_k$  for an  $\alpha \in I(\lambda)$  then the arcs  $(\alpha, \beta) \in \mathcal{A}$  are such that the  $(W^\alpha, e_\beta)$  is an extension of  $W^\alpha$ .

**Example 7.1.** Consider  $X = (0 \frac{1}{3} \frac{2}{3}, 0 \frac{1}{6} 0 \frac{1}{6} \frac{1}{3} \frac{1}{3})$ . We wish to check whether  $X$  is in  $\text{conv}(P_5)$ . It is easy to check that  $X$  indeed satisfies the constraints of  $P_{MI}(5)$ . Also  $X/4 = (0 \frac{1}{3} \frac{2}{3})$  is obviously in  $\text{conv}(P_4)$ . And  $\Lambda_4(X) = \{(0 \frac{1}{3} \frac{2}{3})\}$ . Assume that the pedigrees in  $\mathcal{P}_4$  are numbered such that,  $X^1 = (1 0 0)$ ,  $X^2 = (0 1 0)$  and  $X^3 = (0 0 1)$  and the edges in  $E_4$  are numbered according to their edge labels. Then  $I(\lambda) = \{2, 3\}$ . Here  $k = 4$  and the  $FAT_4(\lambda)$  is given by a problem with origins,  $O = \{2, 3\}$  with supply  $a_2 = \frac{1}{3}, a_3 = \frac{2}{3}$  and destinations,  $D = \{2, 4, 5, 6\}$  with demand  $b_2 = b_4 = \frac{1}{6}, b_5 = b_6 = \frac{1}{3}$ . Corresponding to origin 2 we have the pedigree  $W^2 = ((1, 3))$ . And the edge corresponding to destination 2 is  $e_2 = (1, 3)$ . As  $(W^2, e_2)$  is not an extension of  $W^2$ , we do not have an arc from origin 2 to destination 2. Similarly,  $(W^3, e_4)$ ,  $(W^2, e_5)$  are not extensions of  $W^3$  and  $W^2$  respectively, so we do not have arcs from origin 3 to destination 4 and origin 2 to destination 5. We have the set of arcs given by,

$$\mathcal{A} = \{(2, 4), (2, 6), (3, 1), (3, 5) \text{ and } (3, 6)\}.$$

Notice that  $f$  given by  $f_{24} = f_{26} = f_{32} = f_{36} = \frac{1}{6}, f_{35} = \frac{1}{3}$  is feasible to  $FAT_4(\lambda)$ . (See Figure 1). This  $f$ , in fact, gives a weight vector to express  $X$  as a convex combination of the vectors in  $P_5$ , which are the extensions corresponding to arcs with positive flow. This role of  $f$  is in general true and we state this as Theorem 7.1.

It is easy to check that  $f$  is the unique feasible flow in this example, so no other weight vector exists to certify  $X$  in  $\text{conv}(P_5)$ . Thus we have expressed  $X$  as a convex combination of the incidence vectors of the pedigrees  $W^7 = ((1, 3)(1, 4)), W^8 =$

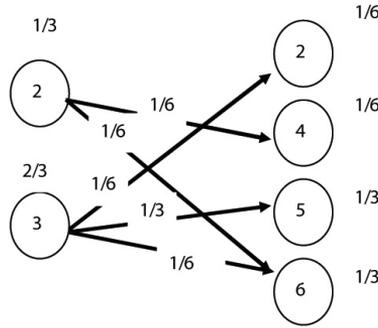


FIGURE 1.  $FAT_4$  Problem for Example 7.1.

$((1, 3)(3, 4)), W^{10} = ((2, 3)(1, 3)), W^{11} = ((2, 3)(3, 4))$  (each of them receive a weight of  $\frac{1}{6}$ ) and  $W^{12} = ((2, 3)(2, 4))$  (which receives a weight of  $\frac{1}{3}$ ).

**Theorem 7.1.** *Let  $k \in V_{n-1} \setminus V_3$ . Suppose  $\lambda \in \Lambda_k(X)$  is such that  $FAT_k(\lambda)$  is feasible. Consider any feasible flow  $f$  for the problem. Let  $W^{\alpha\beta}$  denote the extension  $(W^\alpha, e_\beta)$ , a pedigree in  $\mathcal{P}_{k+1}$ , corresponding to the arc  $(\alpha, \beta)$ . Let  $\mathcal{W}_f$  be the set of such pedigrees,  $W^{\alpha\beta}$  with positive flow  $f_{\alpha\beta}$ . Then  $f$  provides a weight vector to express  $X/k + 1$  as a convex combination of pedigrees in  $\mathcal{W}_f$ .*

Next we observe that  $conv(P_n)$  can be characterized using a sequence of flow feasibility problems as stated in the following theorems:

**Theorem 7.2.** *If  $X \in conv(P_n)$  then  $FAT_k$  is feasible  $\forall k \in V_{n-1} \setminus V_3$ .*

**Theorem 7.3.** *Let  $k \in V_{n-1} \setminus V_3$ . If  $\lambda \in \Lambda_k(X)$  is such that  $FAT_k(\lambda)$  is feasible, then  $X/(k + 1) \in conv(P_{k+1})$ .*

The proofs of these theorems are given by Arthanari (2006). In general we do not have to explicitly give the set  $\Lambda_k(X)$ . The set is used in the proofs. Thus, for a given  $X \in P_{MI}(n)$  the condition

$$\forall k \in V_{n-1} \setminus V_3, \exists \lambda \in \Lambda_k(X) \text{ such that } FAT_k(\lambda) \text{ is feasible}$$

is both *necessary and sufficient* for  $X$  to be in  $conv(P_n)$ .

In Theorem 7.3 we have a procedure to check whether a given  $X \in P_{MI}(n)$ , is in the pedigree polytope,  $conv(P_n)$ . Since feasibility of a  $FAT_k(\lambda)$  problem for a weight vector  $\lambda$  implies  $X/(k + 1)$  is in  $conv(P_{k+1})$ , we can sequentially solve  $FAT_k(\lambda_k)$  for each  $k = 4, \dots, n - 1$  and if  $FAT_k(\lambda_k)$  is feasible we set  $k = k + 1$  and while  $k < n$  we repeat; at any stage if the problem is infeasible we stop. So if we have reached  $k = n$  we have a proof that  $X \in conv(P_n)$ . However if for a  $\lambda \in \Lambda_k(X)$  the problem is infeasible we can not conclude that  $X \notin conv(P_n)$ . Example 7.2 illustrates this.

**Example 7.2.** Consider  $X$  given by

$$\begin{aligned} \mathbf{x}_4 &= (1/2, 1/2, 0); \\ \mathbf{x}_5 &= (0, 0, 1/2, 1/2, 0, 0); \\ \mathbf{x}_6 &= (0, 0, 0, 1/2, 1/2, 0, 0, 0, 0). \end{aligned}$$

$FAT_4(\lambda)$  for the unique  $\lambda = \mathbf{x}_4$  is feasible.  $f$  given by  $f((1,2), (2,3)) = f((1,3), (1,4)) = 1/2$  with flow along other arcs zero is a feasible flow for  $FAT_4(\lambda)$ .

Now the problem  $FAT_5(\lambda)$  corresponding to the  $\lambda$  given by  $f$  is infeasible as the maximum flow in the corresponding network is only  $1/2$ .

We are not able to conclude whether  $X \in \text{conv}(P_6)$ . But we can check that  $X = 1/2(X^1 + X^2)$  where  $X^1$  is given by  $x_4^1((1,2)) = x_5^1((1,4)) = x_6^1((2,4)) = 1$  and  $X^2$  is given by  $x_4^2((1,3)) = x_5^2((2,3)) = x_6^2((1,4)) = 1$ .

However if we have chosen the alternative  $f^*$ , feasible solution for  $FAT_4(\lambda)$ , given by  $f^*((1,2), (1,4)) = f^*((1,3), (2,3)) = 1/2$  with flow along other arcs zero, we have the problem  $FAT_5(\lambda^*)$  corresponding to  $f^*$ . And this problem is feasible and so we conclude  $X \in \text{conv}(P_6)$ .

Current research is directed towards devising methods to find a suitable  $\lambda$  for which the  $FAT_k(\lambda)$  problem is feasible or to show that for no  $\lambda \in \Lambda_k(X)$  the problem is feasible.

## 8. Some properties of interest relating to polytopes

Next we give some definitions and properties relating to polytopes, wherein  $F$  refers to the vertices of the polytope,  $P = \text{conv}(F)$ .

**Definition 8.1** (Adjacency).  $\mathbf{x}, \mathbf{y} \in F$  are *adjacent* vertices of  $P = \text{conv}(F)$  if and only if, for every  $\lambda, 1 > \lambda > 0$ ,  $\lambda\mathbf{x} + (1 - \lambda)\mathbf{y}$  cannot be expressed as a convex combination of elements of  $F \setminus \{\mathbf{x}, \mathbf{y}\}$ . In other words, the line segment  $[\mathbf{x}, \mathbf{y}]$  is an edge of the polytope, that is, it is an one dimensional face of  $P$ .

Similarly, we can define *nonadjacency* in polytopes. It is easy to observe that in the def. 8.1 of adjacency, if we are considering a  $0 - 1$  polytope, it is sufficient to consider convex combinations of vertices that agree with  $\mathbf{x}$  and  $\mathbf{y}$  on coordinates in which they themselves agree. We can have an equivalent definition of adjacency of  $\mathbf{x}, \mathbf{y}$  in  $P$  as: any point  $\mathbf{x}^0$  in the line segment  $(\mathbf{x}, \mathbf{y})$  can be expressed as a convex combination of vertices of  $P$  in an unique manner. And so we have the easy to show equivalent definition of nonadjacency of vertices:

**Definition 8.2** (Nonadjacency).  $\mathbf{x}, \mathbf{y} \in F$  are *nonadjacent* in  $\text{conv}(F)$  if and only if there exist a  $S \subset F$  and a weight vector  $\mu$  such that

- $S \cap \{\mathbf{x}, \mathbf{y}\} \neq \{\mathbf{x}, \mathbf{y}\}$ ,
- $\sum_{\mathbf{w} \in S} \mu(\mathbf{w})\mathbf{w} = (\mathbf{x} + \mathbf{y})/2$ ,  $\sum_{\mathbf{w} \in S} \mu(\mathbf{w}) = 1$ ,  $\mu(\mathbf{w}) > 0$ ,  $\mathbf{w} \in S$ .

Such a  $S$  is called a *witness for nonadjacency* of the given vertices, or *witness* for short.

Given two vertices of a polytope, the problem of determining whether they are non-adjacent vertices of the polytope is called the nonadjacency testing problem. It is well known that such a problem with respect to tours in the *STSP* polytope is *NP-Complete* (Papadimitriou 1978). Such a problem for the pedigree polytope is discussed by Arthanari (2006). Interestingly nonadjacency testing of pedigrees can be done in strongly polynomial time.

Given a polytope  $P$  we define the *graph of P*,  $G(P)$  as the graph whose vertex set is the set of vertices of  $P$  and an edge exists in  $G(P)$  between two vertices if and only if the vertices are adjacent vertices of the polytope  $P$ .

**Property 8.1** (Separability). (Naddef and Pulleyblank 1981) For any  $S \subseteq \mathbf{E}$ , for any  $\mathbf{x} \in F$  we let  $\mathbf{x}[S] \equiv (x_j : j \in S)$  and we let  $F[S] \equiv \{\mathbf{x}[S] : \mathbf{x} \in F\}$  we say that  $S \subseteq \mathbf{E}$  is a *separator* of  $F$  if and only if for every  $\mathbf{x}' \in F[S]$ , every  $\mathbf{x}'' \in F[\mathbf{E} \setminus S]$ , the concatenation  $\mathbf{x}$  of  $\mathbf{x}'$  and  $\mathbf{x}''$  defined by

$$\begin{aligned} x_j &\equiv x'_j : j \in S \\ &\equiv x''_j : j \in \mathbf{E} \setminus S \end{aligned}$$

belongs to  $F$ .  $F$  is said to be *nonseparable* if no proper separator  $S$  of  $F$ , (with  $\emptyset \neq S \neq F$ ) exists.

If a proper separator exists for  $F$  then the coordinates in  $S$  and outside  $S$  can take values freely, independent of each other. Depending on whether  $F$  is separable or not the corresponding polytope  $P = \text{conv}(F)$  is called separable or nonseparable respectively. Next we give some properties studied by Matsui and Tamura (1995).

A sequence  $\rho = (\mathbf{x}^0, \mathbf{x}^1, \dots, \mathbf{x}^K)$  of distinct vertices of  $P$  is called a *vertex sequence* (of  $P$  from  $\mathbf{x}^0$  to  $\mathbf{x}^K$ ). When a vertex sequence  $\rho$  contains  $K + 1$  vertices, we say the length of  $\rho$  is  $K$ .

**Definition 8.3** (Monotone Vertex Sequence). (Matsui and Tamura 1995)

A vertex sequence  $\rho = (\mathbf{x}^0, \mathbf{x}^1, \mathbf{x}^2, \dots, \mathbf{x}^K)$  is called a *monotone vertex sequence*, when it satisfies the condition that: for each index  $j$ , either

$$x_j^0 \leq x_j^1 \leq x_j^2 \leq \dots \leq x_j^K \text{ or } x_j^0 \geq x_j^1 \geq x_j^2 \geq \dots \geq x_j^K$$

.

**Property 8.2** (A). (Matsui and Tamura 1995) If two vertices  $\mathbf{x}^1$  and  $\mathbf{x}^2$  of  $P$  are not adjacent, then there exists a vertex  $\mathbf{x}'$  of  $P$  such that  $(\mathbf{x}^1, \mathbf{x}', \mathbf{x}^2)$  is a monotone vertex sequence of  $P$ .

For  $0 - 1$  polytopes, using def. 8.2, we have a vertex  $\mathbf{y}$  in  $S$  which is different from both  $\mathbf{x}^1$  and  $\mathbf{x}^2$  and  $\mathbf{y}$  agrees on coordinates in which the other two vertices themselves agree. It is easy to verify that  $(\mathbf{x}^1, \mathbf{y}$  and  $\mathbf{x}^2)$  is a monotone vertex sequence. This fact appears as Lemma 2.1 in Matsui and Tamura (1995).

**Property 8.3** (B). (Matsui and Tamura 1995) If  $(\mathbf{x}^1, \mathbf{x}^2, \mathbf{x}^3)$  is a monotone vertex sequence of  $P$ , then the vector  $\mathbf{x}^1 - \mathbf{x}^2 + \mathbf{x}^3$  is a vertex of  $P$ .

**Property 8.4** (Combinatorial). (Naddef and Pulleyblank 1981) If  $\mathbf{x}^1$  and  $\mathbf{x}^2$  are nonadjacent vertices of  $P$  a  $0 - 1$  polytope, then there exist two other vertices  $\mathbf{y}^1$  and  $\mathbf{y}^2$  of  $P$  such that  $\mathbf{x}^1 + \mathbf{x}^2 = \mathbf{y}^1 + \mathbf{y}^2$ . We say the vertex set  $F$  is a *combinatorial set*. The corresponding graph and polytope are called *combinatorial graph* and *combinatorial polytope* respectively.

**Property 8.5** (Hirsch). (Naddef 1989) Let  $P$  be any polytope in  $R^n$ . For any  $\mathbf{c} \in R^n$  and for any vertex  $\mathbf{x}^0$  of  $P$ , the following holds. If the problem of minimizing  $\mathbf{c}\mathbf{x}$  over  $P$  has an optimum then there exists a vertex sequence  $\rho = (\mathbf{x}^0, \mathbf{x}^1, \dots, \mathbf{x}^K)$  of  $P$  satisfying

- (1)  $\mathbf{x}^K$  is an optimal solution of the problem,
- (2)  $\mathbf{x}^{i-1}$  and  $\mathbf{x}^i$  are adjacent for all  $i \in \{1, 2, \dots, K\}$ ,
- (3)  $K \leq f(P) - d(P)$  where  $K$  is the length of the sequence,  $f(P)$  is number of facets of  $P$  and  $d(P)$  is the dimension of  $P$ .

See Korte and Vygen (2012) for definitions of the terms like facet, dimension.

In addition to the property **8.5**, if we have  $\mathbf{c}\mathbf{x}^0 \geq \mathbf{c}\mathbf{x}^1 \geq \dots \geq \mathbf{c}\mathbf{x}^K$  we have the monotone version of the Hirsch property.

Naddef (1989) showed that 0 – 1 polytopes have Hirsch property. If a polytope satisfies properties **8.2** and **8.3** then it has monotone Hirsch property is shown by Matsui and Tamura (1995). They also prove monotone Hirsch property is true for all 0 – 1 polytopes.

**Property 8.6 (P1).** (Ikebe, Matsui, and Tamura 1993) If  $\mathbf{x}^1$  and  $\mathbf{x}^2$  are nonadjacent vertices of  $P$  then there exist other vertices  $\mathbf{y}^1, \dots, \mathbf{y}^r$  of  $P$  and positive integers  $\lambda_1, \dots, \lambda_r$  such that  $\mathbf{x}^2 - \mathbf{x}^1 = \sum_{i=1}^r \lambda_i(\mathbf{y}^i - \mathbf{x}^1)$ .

**Property 8.7 (Strong Adjacency).** (Ikebe, Matsui, and Tamura 1993) Let  $P$  be any polytope in  $R^n$ . Consider the problem of minimizing  $\mathbf{c}\mathbf{x}$  over  $P$ , for  $\mathbf{c} \in R^n$ . If every best valued (optimal) vertex of  $P$  is adjacent to some second best valued vertex of  $P$  for each  $\mathbf{c}$ , we say  $P$  has strong adjacency property.

*Remark 8.1.* We have the following implications among the properties discussed above:

- Every 0 – 1 polytope has properties( **8.2**, **8.5**); and in fact 0 – 1 polytopes possess the monotone Hirsch property (see for proofs Matsui and Tamura 1995; Naddef 1989, respectively).
- Properties( **8.2** and **8.3**) imply property( **8.4**) (see Matsui and Tamura 1995).
- Property( **8.4**) implies property( **8.6**) (easily follows from the definitions).
- Property( **8.6**) implies property( **8.7**). (Proved by Ikebe, Matsui, and Tamura (1993))

## 9. The pedigree polytope is a combinatorial polytope

The main result of this section is the observation that the pedigree polytopes are combinatorial polytopes. The implications of this and other results relating to some of the properties listed earlier in Section **8** are also discussed.

**9.1. Graph of rigidity and its implications on adjacency of pedigrees.** Given  $X^{[1]}, X^{[2]} \in P_n$ , let the corresponding pedigrees be  $W^{[i]}, i = 1, 2$ . Let the  $2 \times (n - 3)$  array  $L = (e_{ij})$  denote the edges in  $W^{[1]}, W^{[2]}$  as rows, respectively. That is,  $x_j^{[i]}(e_{ij}) = 1, i = 1, 2$ , and  $4 \leq j \leq n$ . We also informally say,  $e_{ij}$  is in  $X^{[i]}$ , if the corresponding edge is the  $i^{th}$  element of  $L$ .

It is shown in Arthanari (2006) that we have the following fact about *inheritance* of the adjacency property.

**Theorem 9.1.** Given  $X^{[1]}, X^{[2]} \in P_n$ , suppose  $X^{[1]}/k, X^{[2]}/k$  are adjacent/nonadjacent in  $\text{conv}(P_k)$ , for some  $k, 4 \leq k < n$ , and  $x_{k+1}^{[i]}(e) = 1, i = 1, 2$  for some  $e \in E_k$ , then  $X^{[1]}/k + 1, X^{[2]}/k + 1$  are adjacent/nonadjacent in  $\text{conv}(P_{k+1})$ , accordingly.

So we consider only the components of the given pedigrees where they differ.

**Definition 9.1.** Given  $X^{[1]}, X^{[2]} \in P_n$ , we call  $D = \{q | \mathbf{x}_q^{[1]} \neq \mathbf{x}_q^{[2]}, 4 \leq q \leq n\}$  the set of discordant components or discords. This means, in terms of  $L, e_{1q} \neq e_{2q}, q \in D$ .

In this subsection we define the graph of rigidity for a given pair of pedigrees. And show that the connectedness of the graph answers the adjacency question.

**Definition 9.2.** Let  $\bar{i} = 3 - i$ . Given the pedigrees  $W^{[i]}, i = 1, 2$ . Let  $D$  be the set of discords. We say  $t \in D$  is welded to  $s, s \in D, s < t$  if either

- for some  $i = 1, 2$ , no generator of  $e_{it} = (u, v)$  is available in the pedigree  $W^{[\bar{i}]}$ , and  $s = \max(4, v)$  or
- for some  $i = 1, 2, e_{i\bar{s}} = e_{it}$ .

**Definition 9.3.** Given a pair of pedigrees in  $\mathcal{P}_n$ , we define the graph of rigidity denoted by  $G_R$ . The vertex set of  $G_R$  is the corresponding set of discords  $D$ , and the edge set is given by  $\{(s, q) | s, q \in D, s < q, \text{ and } q \text{ is welded to } s\}$ .

The graph  $G_R$  expresses the restriction imposed on the elements of  $D$  as far as producing a witness for nonadjacency of  $X^{[1]}$  and  $X^{[2]}$  in  $\text{conv}(P_n)$  is concerned. Any  $Y \in S \subset P_n$ , a witness, has to agree with the components on which both  $X^{[1]}, X^{[2]}$  themselves agree and has to have exactly one edge from  $\{e_{iq}, i = 1, 2\}, q \in D$ . And so we may visualize  $Y$  as the incidence vector of a pedigree obtained from  $X^{[1]}$  or  $X^{[2]}$  by swapping  $(e_{1q}, e_{2q})$ , for some  $q \in D$ . Next we find conditions on  $G_R$  that will ensure nonadjacency of pedigrees.

In Arthanari (2006) the following theorems (Theorems 9.2, 9.3) are proved, we state them without proof here.

**Theorem 9.2.** Given  $X^{[i]} \in P_n, i = 1, 2$ , if  $C$  is a component of  $G_R$ , then swapping  $C$  produces  $Y^{[i]} \in P_n$ .

**Theorem 9.3.** Given  $X^{[i]} \in P_n, i = 1, 2$ , consider the graph of rigidity  $G_R$ . The given pedigrees are nonadjacent in  $\text{conv}(P_n)$  if and only if  $G_R$  is not connected.

Using Theorem 9.2 we can conclude that if  $G_R$  is not a connected graph, then the set of vertices in  $C$  of any connected component of  $G_R$  is a proper subset of  $D$  and swapping them produces a set  $S = \{Y^{[1]}, Y^{[2]}\} \subset P_n$ . And  $S$  is a witness for nonadjacency of the given  $X^{[1]}, X^{[2]}$  in  $\text{conv}(P_n)$ . Notice that all  $q$  in a connected component of  $G_R$  are required to be swapped simultaneously, to obtain a legitimate swap<sup>2</sup>. That is, each connected component of  $G_R$  is minimally legitimate. Thus if  $G_R$  is a connected graph then we have no evidence for nonadjacency or we can declare  $X^{[1]}$  and  $X^{[2]}$  are adjacent in  $\text{conv}(P_n)$ . Thus we have  $\text{conv}(P_n)$  is a combinatorial polytope. We state this fact as theorem 9.4.

**Theorem 9.4.** The pedigree polytope,  $\text{conv}(P_n)$  is a combinatorial polytope. For any pair of nonadjacent vertices  $X^{[i]}, i = 1, 2$  in  $\text{conv}(P_n)$  consider the graph of rigidity  $G_R$ . Any connected component  $C$  of  $G_R$  is minimally legitimate.

In Arthanari (2006) a strongly polynomial algorithm is outlined to test whether a given pair of pedigrees are nonadjacent.

**9.2. Other properties shared by the pedigree polytopes.** Recall that several properties of polytopes were listed in Sec. 8 and their implications and inclusions proved in the literature are summarized in Remark 8.1. Here we study which among them are satisfied by the pedigree polytopes.

Matsui and Tamura (1995) give the following counter example to show that combinatorial property (property (8.4) does not imply property B (property (8.3)):

<sup>2</sup>A swap is legitimate if it produces pedigrees. It is minimal if no subset of the component swapped is legitimate.

**Example 9.1.** Consider the set of 0 – 1 vectors  $V = \{(0,0), (1,0), (1,1)\}$ .  $conv(V)$  is combinatorial in a vacuous sense as the vertices are all mutually adjacent in  $conv(V)$ . Though  $((0, 0), (1, 0), (1, 1))$  is a monotone vertex sequence, the vector  $(0,0) - (1,0) + (1,1) = (0,1) \notin V$ . So it does not satisfy property B.

Notice that for  $n = 4$ ,  $\mathcal{P}_4 = \{(100), (010), (001)\}$ . The vertices are mutually adjacent. And there is no monotone vertex sequence available here. So in a vacuous sense  $conv(P_4)$  has property( **8.3**).

However, we shall show that the pedigree polytopes for  $n > 4$  do not possess property( **8.3**).

**Theorem 9.5.** *The pedigree polytopes,  $conv(P_n)$ ,  $n > 4$  do not satisfy property( **8.3**).*

**Proof:** We prove the result by induction on  $n$ . The basis for induction is provided by  $conv(P_5)$ . Consider the vertex sequence  $\rho = (X^{[1]}, X^{[2]}, X^{[3]})$  corresponding to the pedigrees  $((1,2), (1,3)), ((1,2), (1,4)), ((1,3), (1,4))$  respectively. It is easy to check that  $\rho$  is a monotone vertex sequence (using definition **8.3**). But  $X^{[1]} - X^{[2]} + X^{[3]} = (010, 010000)$  corresponds to  $((1,3), (1,3))$  and it is not a pedigree as it violates the requirement that the edges in a pedigree are distinct. Hence  $conv(P_5)$  does not have property( **8.3**).

Now assume that property( **8.3**) is violated by all polytopes  $conv(P_k)$ , for  $5 \leq k \leq n - 1$ . Now we shall show that so is the case with  $conv(P_n)$ .

From the induction hypothesis  $\exists$  a monotone vertex sequence  $\rho = (X^{[1]}, X^{[2]}, X^{[3]})$  in  $conv(P_{n-1})$  such that  $X^{[1]} - X^{[2]} + X^{[3]}$  is not a vertex of  $conv(P_{n-1})$ . Now consider the last component of these vectors in the sequence, namely,  $\mathbf{x}_{n-1}^{[i]}, i = 1, 2$  and 3 respectively.

Consider the edge  $e = (u, v)$  for which  $\mathbf{x}_{n-1}^{[2]}(e) = 1$ . As  $\rho$  is a monotone vertex sequence, we have three cases to consider.

*Case 1:*  $\mathbf{x}_{n-1}^{[1]}(e) = 1$  and  $\mathbf{x}_{n-1}^{[3]}(e) = 1$ . Consider  $e' = (u, n - 1)$  let  $\mathbf{y}(e')$  be the indicator of  $e'$ . Notice that  $(X^{[i]}, \mathbf{y}(e'))$  is a pedigree in  $\mathcal{P}_n$  for all  $1 \leq i \leq 3$ . And the vertex sequence  $\rho' = ((X^{[1]}, \mathbf{y}(e')), (X^{[2]}, \mathbf{y}(e')), (X^{[3]}, \mathbf{y}(e')))$  in  $conv(P_n)$  is monotone. But fails to satisfy property( **8.3**).

*Case 2:*  $\mathbf{x}_{n-1}^{[1]}(e) = 1$  and  $\mathbf{x}_{n-1}^{[3]}(e) = 0$ . Let  $e' = (u, n - 1)$ . Notice that  $(X^{[i]}, \mathbf{y}(e'))$  is a pedigree in  $\mathcal{P}_n$  for  $i = 1, 2$ . Choose  $e''$  such that  $(X^{[3]}, \mathbf{y}(e''))$  corresponds to a pedigree in  $\mathcal{P}_n$ . Now the vertex sequence  $\rho' = ((X^{[1]}, \mathbf{y}(e')), (X^{[2]}, \mathbf{y}(e')), (X^{[3]}, \mathbf{y}(e'')))$  in  $conv(P_n)$  is monotone. But fails to satisfy property B. *Case 3:*  $\mathbf{x}_{n-1}^{[1]}(e) = 0$  and  $\mathbf{x}_{n-1}^{[3]}(e) = 1$ . Proof is similar to *Case 2*.

Hence  $conv(P_n)$  does not have property( **8.3**).  $\square$

So we have non-vacuous examples of combinatorial polytopes, which do not satisfy property B stipulated by Matsui and Tamura.

Next we observe that the pedigree polytopes are nonseparable. Recall the definition of separability (Definition **8.1**).

**Lemma 9.1.**  $P_n$  is not separable for  $n \geq 4$ .

**Proof:** As observed earlier, the vectors in  $P_4$  are  $(1\ 0\ 0)$ ,  $(0\ 1\ 0)$  and  $(0\ 0\ 1)$ , and obviously  $P_4$  is not separable. So  $S = E_3$  is the only separator for  $P_4$ .

We prove the result by induction on  $n$ . The basis for induction is provided by  $P_4$ . Now assume that all  $P_k$ , for  $5 \leq k \leq n-1$  are nonseparable. We shall show that so is the case with  $P_n$ .

From the induction hypothesis, for  $P_{n-1}$  no proper subset of  $E_3 \times \dots \times E_{n-2}$ , is a separator of  $P_{n-1}$ . And now consider any nonempty separator  $S \subseteq \mathbf{E}$  of  $P_n$ , if one exists, where  $\mathbf{E} = E_3 \times \dots \times E_{n-1}$ . It must be of the form  $S = E_3 \times \dots \times E_{n-2} \times U$  for some  $U$  subset of  $E_{n-1}$ . For any  $X = (\mathbf{x}_4, \dots, \mathbf{x}_n) \in P_n$  the last component  $\mathbf{x}_n$  is the indicator of some  $e \in E_{n-1}$ . So we will not be able to fix the value of  $\mathbf{x}_n(e)$  for any edge  $e$  in  $\mathbf{E} \setminus S$  independent of  $X/n-1$ . Hence  $U$  must be the whole of  $E_{n-1}$ . That is,  $\mathbf{E}$  is the only nonempty separator for  $P_n$ . So  $P_n$  is nonseparable. Hence the result.  $\square$

The nonseparability is rather straight forward, from the way pedigrees are defined. We now turn our attention to the graph of the pedigree polytope,  $G(P_n)$ . We say a graph  $G$  is *nonbipartite*, when there is no way we can partition the vertices of  $G$  into two sets  $A$  and  $B$  such that the edges in  $G$  are between  $A$  and  $B$  only. For example  $G(P_4)$ , is nonbipartite as it is a triangle.

Now an application of the following Lemma 9.2 proved by Naddef and Pulleyblank (1981) (Lemma 2.8) implies that the graphs  $G(P_n)$ , is nonbipartite for  $n \geq 4$ .

**Lemma 9.2.** (Naddef and Pulleyblank 1981) Let  $F \subseteq \{0, 1\}^E$  be a combinatorial set. If  $|F| \geq 3$  and  $F$  is nonseparable then  $G(F)$  is *nonbipartite*.

One of the two main theorems from Naddef and Pulleyblank (1981) (Theorems 2.10) on the graphs of combinatorial sets is stated below for nonseparable sets as Theorem 9.6

**Theorem 9.6.** *Let  $G(P)$  be the graph of a  $0-1$  polytope  $P$  corresponding to a nonseparable combinatorial set  $F$ . Then every pair of distinct nodes of  $G(P)$  is joined by a Hamiltonian path.*

From Theorem 9.4 and Lemma 9.1,  $P_n, n \geq 4$ , is a nonseparable combinatorial set. Applying Theorem 9.6 to the graph of the pedigree polytope,  $G(P_n)$ , we have the result, every pair of distinct pedigrees is joined by a Hamiltonian path. As  $G(P_n), n \geq 4$  has at least 3 vertices, (so not isomorphic to  $K_1$  or  $K_2$ ) this implies every edge in  $G(P_n)$  belongs to a Hamiltonian cycle. (This property is called strong hamiltonicity (see Naddef and Pulleyblank 1981)).

The fact that the pedigree polytopes are combinatorial can be used to derive (from Remark 8.1) the strong adjacency property (property (8.7)) for the pedigree polytopes. Recently, another algorithmic advantage of pedigree polytope being combinatorial was used along with explicit use of the *MI* relaxation to show nonadjacency in the pedigree polytope implies nonadjacency of the corresponding tours in the *STSP* polytope (Arthanari 2013).

**9.3. Other related works.** In Arthanari (2008) a multi flow problem is solved to check a necessary condition for membership in the pedigree polytope. This can be done in polynomial time. In Haerian Ardekani and Arthanari (2008) an illustrative example is given to explain this necessary condition and the membership problem. Ardekani in her doctoral thesis gives a counter example to show that this necessary condition is not sufficient for membership in the pedigree polytope (Haerian Ardekani 2011). In addition Ardekani reports encouraging results comparing different formulations with the *MI* formulation, and other heuristics based on pedigree approach to solve the *STSP*. In Arthanari and Usha

(2001) the equivalence of the *MI* formulation and the cycle shrink formulation (Carr 1995, 1996) due to Carr is established. Usha Mohan in her doctoral thesis studies the structure of small *MI* polytopes and gives the *MI* formulation of the asymmetric *TSP*.

## 10. Conclusions and future research directions

This paper is an expository article on the pedigree polytope, its properties, connections to the *MI* formulation for solving the symmetric traveling salesman problem and the insight we derive from this new polyhedral approach to solving the *STSP*. Apart from summarising the main results on the pedigree polytope the paper also briefly outlines other related works. The computational experience with this approach, studied in the doctoral thesis of Haerian Ardekani (2011) is encouraging. In this thesis several computational experiments with the *MI* relaxation, pedigree related heuristics and branch and bound exact solution process are conducted. The gap between the LP relaxation of different formulations and the integer optimum indicates superior performance of the *MI* relaxation. There is a need to devise new algorithms that can handle large instances of the *MI* relaxation. The pedigree polytope has properties not shared by the *STSP* polytope and this fact can be further explored to devise new solution approaches to solve the *STSP*. The future research directions could include 1] identifying facets of pedigree polytope, 2] developing special algorithms to solve the *MI* relaxation, 3] exploiting the combinatorial property of the pedigree polytope to design adjacent vertex methods, and 4] finding new sufficient conditions for membership in the pedigree polytope.

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