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ASYMPTOTIC WAVES FROM THE POINT OF VIEW OF DOUBLE-SCALE METHOD

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ABSTRACT. In this paper the asymptotic waves (smooth solutions of nonlinear hyperbolic partial differential equations (PDEs)) are introduced from the point of view of double scale method, giving a physical interpretation of the new (fast) variable, related to the surface across which the derivatives of the solution vary steeply.

1. Introduction

The motion of a large number of media is described by nonlinear hyperbolic PDEs. Their solutions are referred to as *waves*. Some of them present various types of discontinuities, some others not. In the first case, as some surface is crossed, the solution or/and its derivatives undergo a jump. In this case it is said that the solution presents a *shock*, or it is a *shock wave* or that we are in presence of a discontinuity wave (jump of the first order derivatives). In the second case, instead of the jump there is a steep variation. In this case it is said that the solution is an *asymptotic wave*. Both these types of solutions are called *nonlinear waves* because they satisfy nonlinear hyperbolic PDEs.

The theoretical interest in nonlinear waves was manifest as early as the years '50 and '60 of the last century, leading to basic results in the field. Subsequently, a lot of applications to various equations from elasticity, fluid mechanics and other branches of physics were carried out ([1]-[8]).

The mathematical aspects involved into the study of asymptotic waves belong to the singular perturbation theory, namely the double-scale method. This approach was initiated in the papers of H. Poincaré and Krylov and Bogoliubov ([9]-[10]), treating nonlinear oscillations governed by ordinary differential equations. Then, it was very much applied in [10] and studied in [11], [12], [13], first in the mechanics of discrete points framework and subsequently ([14]-[19]), almost simultaneously with the theory of asymptotic waves, in a more and more rigorous way and for more and more general settings.

The multiple-scale method, and, in particular, the double-scale approach, is appropriate to phenomena which possess qualitatively distinct aspects at various scales. For instance, at some well-determined times or space coordinates, the characteristics of the motion vary steeply, while at larger scale the characteristics are slow and describe another type of motion. As the phenomena of large and small scales occur simultaneously, the nondimensionalization must take into account more than one characteristic time and/or length and the asymptotic approximations of the solutions of the governing equations are more than one too. In addition, the scales are defined by some small parameters.

Since the asymptotic variables are the small parameters, the governing equations are of perturbation. Moreover, due to the existence of several asymptotic approximations of the solution for various domains of variation of (some) independent variables, the perturbations are singular.

The double-scale method can be viewed as a method of matched asymptotic expansions ([18]), another important method of singular perturbations. Also, the multiple-scale method is a multivariable method, it implying the introduction of several new independent variables.

In the context of rheological media, a series of studies on linear waves were carried out in [20]-[22] (Ciancio-Restuccia 1985, 1987). In this paper the asymptotic smooth waves are introduced from the point of view of double scale method (see [23], Georgescu 1995). In Section 2 we define the new (fast)variable related to the surface across which the derivatives of the solution vary steeply. Section 3 concerns the hyperbolic equations and the relevance of their characteristics to the study of asymptotic waves.

2. Application of double-scale method to nonlinear hyperbolic PDEs

In this section first we recall an application of double-scale method to non-linear hyperbolic PDEs, providing the behaviour of the Maxwell rheological media, developed in [20], where the propagation of asymptotic waves was studied. In particular, in [24]-[28], using the methods of classical thermodynamics of irreversible processes with internal variables, the stress-strain relations for viscoanelastic media of order n with memory were derived. It was assumed that the strain tensor ε_{ik} is given by an elastic part and an inelastic part, n different types of microscopic phenomena occur which give rise to inelastic strains and the total inelastic deformation is additively composed of n contributions due to these phenomena. These contributions were introduced as internal degrees of freedom (internal tensorial variables) in the state vector. In the case that only one microscopic phenomenon gives rise to inelastic strain (only one internal variable of mechanical origin is taken into consideration) and no viscous effects and memory effects occur the stress-strain relations for *anelastic media of order one without memory* or also *Maxwell media* were derived. In [20] the balance equations for the mass and momentum density together with the constitutive relations for Maxwell media were written, respectively, in the form

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x^i}(\rho v_i) = 0, \tag{1}$$

$$\rho\left(\frac{\partial}{\partial t}v_i + v_k\frac{\partial}{\partial x^k}v_i\right) + \frac{\partial}{\partial x^k}\tilde{P}_{ik} + \frac{\partial}{\partial x^i}P = 0, \tag{2}$$

where $v_i = \frac{du_i}{dt}$ is the i-th component of the velocity field,

(3)
$$a^{(1,1)}\eta^{(1,1)}_{(s)}\tilde{P}_{ik}+\frac{\partial}{\partial t}\tilde{P}_{ik}+v_p\frac{\partial}{\partial x^p}\tilde{P}_{ik}+\frac{1}{2}a^{(1,1)}\left(\frac{\partial}{\partial x^k}v_i+\frac{\partial}{\partial x^i}v_k\right)-\frac{1}{3}a^{(1,1)}\frac{\partial}{\partial x^p}v_p\delta_{ik} = 0,$$

$$b^{(1,1)}\eta^{(1,1)}_{(v)}P' + \frac{\partial}{\partial t}P' + v_p\frac{\partial}{\partial x^p}P' + \frac{1}{3}b^{(1,1)}\frac{\partial}{\partial x^p}v_p = 0, \tag{4}$$

where $(i,k = 1,2,3)$, the relation $\frac{d\varepsilon_{ik}}{dt} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x^k} + \frac{\partial v_k}{\partial x^i}\right)$ is used and $\tilde{P}_{ik}$ is the deviator of the mechanical pressure tensor P_{ik} .
P_{ik} is defined in terms of the symmetric Cauchy tensor $P_{ik} = -\tau_{ik}$. Moreover,

$$\tilde{P}_{ik} = P_{ik} - \frac{1}{3}P_{ss}\delta_{ik}, \quad P = \frac{1}{3}P_{ss}, \quad P_{ss} = trP,$$

$$P_{ik} = \tilde{P}_{ik} + P\delta_{ik}, \quad \tilde{P}_{ss} = 0, \quad P' = P - P_0 = -(\tau - \tau_0),$$

where P is the scalar part of the mechanical pressure tensor P_{ik} and τ_0 and P_0 are the scalar parts of τ_{ik} and P_{ik} , respectively, of the medium in a state of thermodynamic equilibrium. This equilibrium state plays the role of a reference state. Moreover, the coefficients in (3) and (4) satisfy the relations

$$a^{(1,1)}\eta^{(1,1)}_{s} \geq 0, \quad a^{(1,1)} \geq 0,$$

$$b^{(1,1)}\eta^{(1,1)}_{v} \geq 0, \quad b^{(1,1)} \geq 0, \tag{5}$$

where $a^{(1,1)}$ and $b^{(1,1)}$ are scalar constants which occur in the equation of state, while the coefficients $\eta^{(1,1)}_s$ and $\eta^{(1,1)}_v$ are called *phenomenological coefficients* and represent fluidities.

The stress strain relations (3) and (4) are valid in the isothermal and isotropic case. Equations. (1)-(4) form a system of ten quasi-linear first order PDEs for mass density, three components of the velocity field, five independent components of $\tilde{P}_{ik}$ and the scalar part of the stress tensor P' , that can be written in the form (see [20])

$$\mathbf{A}^{\alpha}(\mathbf{U})\mathbf{U}_{\alpha} = \mathbf{B}(\mathbf{U}) \qquad (\alpha = 0,1,2,3), \tag{6}$$

where $\mathbf{U}_\alpha = \frac{\partial \mathbf{U}}{\partial x^\alpha}$, $x^0 = t$ (time), x^1, x^2, x^3 are the space coordinates, $\mathbf{U}$ is a column vector (with 10 components) of the unknown functions (which depend on x^α), $\mathbf{A}^\alpha$ are square matrices 10×10 and $\mathbf{B}$ is a column vector with 10 components (four of them are null and the others are proportional to the components of $\mathbf{U}$).
In particular

$$\mathbf{U} = \begin{pmatrix} \rho, & v_1, & v_2, & v_3, & \tilde{P}_{11}, & \tilde{P}_{12}, & \tilde{P}_{13}, & \tilde{P}_{22}, & \tilde{P}_{23}, & P' \end{pmatrix}^T,$$

$$\mathbf{B} = \begin{pmatrix} 0, & 0, & 0, & 0, & \tilde{P}^*_{11}, & \tilde{P}^*_{12}, & \tilde{P}^*_{13}, & \tilde{P}^*_{22}, & \tilde{P}^*_{23}, & P'^* \end{pmatrix}^T, \tag{7}$$

$$with \quad \tilde{P}^*_{ik} = -a^{(1,1)}\eta^{(1,1)}_{(s)}\tilde{P}_{ik}, \quad P'^* = -b^{(1,1)}\eta^{(1,1)}_{(v)}P', \qquad (i,k = 1,2,3).$$

In [20], following [4]-[8], the propagation of asymptotic waves was studied using a general method devised to oscillatory approximate solutions for first order quasilinear hyperbolic systems.

Equations (6) are semilinear (the highest order derivatives $\mathbf{U}_t$ and $\mathbf{U}_\alpha$ occur linearly).

In semilinear hyperbolic PDEs (see Section 3) the solution $\mathbf{U}(x^\alpha)$, $\alpha = 0,1,...,n$ or/and some of its derivatives undergo *steep changes* across the so-called *interior layers*, situated in the neighborhood of a family of moving in E^n surfaces $S(t\ )$ (parametrized by the time t)

$$\varphi(t, x^i) = \bar{\xi}, \quad \bar{\xi} = const, \quad (i = 1,...,n). \tag{8}$$

Along these surfaces the variation of $\mathbf{U}$ is slow. In this case it is said that $\mathbf{U}$ *evolves in progressive waves* and that the surfaces (8) are (improperly called) the wave surfaces or, simply, waves. Usually, the dimension of φ is a phase. Let $\omega \gg 1$ be a large parameter; usually its dimension is a frequency.

Asymptotically, this means that both "slow" (old) and "fast" (new) variables are necessary to characterize completely the behavior of the solution as some parameter tends to zero (see [23]).

Each variable (slow or fast) has a characteristic scale. This is why, an approximate solution is looked for as depending on the old as well as on the new variables, where the new variables are thought as independent of the old variables.

When no interior layers are present, all characteristic quantities have the same scale. If only one new variable of a different scale is imposed, we say that the problem has a *double scale*.

In this case, the appropriate method, used to derive solutions of asymptotic approximation is called the *double-scale method*. Usually, the new variable is a space variable x^i or the time t multiplied by a function of the small parameter.

In the application to hyperbolic PDEs the choice of the new variable has a physical meaning. Namely, in the case of the wavelike solutions, the new variable is related to the wavefront

$$\varphi(t, x^i) = 0, \;\; i.e. \; the \; new \; variable \; is \;\;\; \xi = \frac{\varphi(x^\alpha)}{\omega^{-1}}, \tag{9}$$

where (i = 1, 2, 3), $(\alpha = 0, 1, 2, 3)$ and $\omega \gg 1$, hence ω^{-1} is a small parameter. In fact, ω is written only in order to make easier the determination of the order of magnitude of the quantities. Moreover, it is understood that $\varphi(x^\alpha)$ is of the same order of magnitude as ω^{-1} . Therefore ξ is *asymptotically fixed*, i.e. not too large, not too small. All other quantities in equations are also understood to be asymptotically fixed. Then, $U = U(x^\alpha, \xi)$ and x^α and ξ are considered independent. In this way, with respect to the variables (x^α, ξ), the order of magnitude is given only by the powers of ω^{-1} and a formal computation can proceed.

Introduce the quantities

$$\lambda = -\frac{\frac{\partial \varphi}{\partial t}}{|grad \varphi|}, \qquad \mathbf{n} = \frac{grad \varphi}{|grad \varphi|}, \tag{10}$$

which are closely related to the quantities occurring in the higher-order asymptotic approximations of the solution of hyperbolic PDEs. Relation (10) shows that λ has the dimension of the velocity, while $\mathbf{n}$ is the normal to the wavefront surface $(9)_1$.

First, the double-scale method consists in expressing the derivatives with respect to x^α in terms of the derivatives with respect to x^α and ξ.

Then, we look for the solution of the equations as an asymptotic series of powers of the small parameter, say ϵ, namely with respect to the asymptotic sequence $\left\{1, \varepsilon^{a+1}, \varepsilon^{a+2}, ..., \right\}$ or $\left\{1, \varepsilon^{\frac{1}{p}}, \varepsilon^{\frac{2}{p}}, ..., \right\}$, as $\varepsilon \to 0$. In [20] it is considered $p = 1$, and $\varepsilon = \omega^{-1}$, such that

$$\mathbf{U}(x^\alpha, \xi) \sim \mathbf{U}^0(x^\alpha, \xi) + \omega^{-1}\mathbf{U}^1(x^\alpha, \xi) + \omega^{-2}\mathbf{U}^2(x^\alpha, \xi) + ... \tag{11}$$

The third point of the method consists in introducing the asymptotic expansion of the solution into the equations in order to obtain the sets of equations of various order of asymptotic approximation.

Each such set has as a solution one approximation $\mathbf{U}^r (r = 1, 2, ...)$ of $\mathbf{U}$.
Let us sketch these steps for the equation (6). Thus, introducing (11) in the derivatives $\mathbf{U}_\alpha = \frac{\partial \mathbf{U}}{\partial x^\alpha}$ we have

$$\frac{\partial \mathbf{U}}{\partial x^\alpha} \sim \omega^{-1}\left(\frac{\partial \mathbf{U}^1}{\partial x^\alpha} + \omega \frac{\partial \mathbf{U}^1}{\partial \xi}\frac{\partial \varphi}{\partial x^\alpha}\right) + \omega^{-1}\frac{\partial^2 \mathbf{U}^2}{\partial \xi}\frac{\partial \varphi}{\partial x^\alpha} + O(\omega^{-2}),\ as\ \omega^{-1} \to 0, \tag{12}$$

where we took into account that the first approximation $\mathbf{U}^0$ was a constant. Further, taking into account the form of $\mathbf{A}^\alpha$ and $\mathbf{B}$, the following asymptotic expansions were deduced

$$\mathbf{A}^\alpha(\mathbf{U}) \sim \mathbf{A}^\alpha(\mathbf{U}^0) + \omega^{-1}\nabla \mathbf{A}^\alpha(\mathbf{U}^0)\mathbf{U}^1 + O(\omega^{-2}), \quad as \quad \omega^{-1} \to 0, \tag{13}$$

$$\mathbf{B}^\alpha(\mathbf{U}) \sim \mathbf{B}^\alpha(\mathbf{U}^0) + \omega^{-1}\nabla \mathbf{B}^\alpha(\mathbf{U}^0)\mathbf{U}^1 + O(\omega^{-2}), \quad as \quad \omega^{-1} \to 0. \tag{14}$$

Introducing (12) - (14) into (6) and matching the obtained series, it follows

$$\mathbf{A}^\alpha(\mathbf{U}^0)\frac{\partial \varphi}{\partial x^\alpha}\frac{\partial \mathbf{U}^1}{\partial \xi} = 0, \tag{15}$$

$$\mathbf{A}^\alpha(\mathbf{U}^0)\frac{\partial \varphi}{\partial x^\alpha}\frac{\partial \mathbf{U}^2}{\partial \xi} = -\mathbf{A}^\alpha(\mathbf{U}^0)\frac{\partial \mathbf{U}^1}{\partial x^\alpha} - \nabla \mathbf{A}^\alpha(\mathbf{U}^0)\mathbf{U}^1\frac{\partial \mathbf{U}^1}{\partial \xi}\frac{\partial \varphi}{\partial x^\alpha} + \left(\nabla \left[\mathbf{B}(\mathbf{U}^0)\right]\right)\mathbf{U}^1. \tag{16}$$

Equation (15) is linear in $\mathbf{U}^1$, while (16) is affine in $\mathbf{U}^2$. These are the equations of first and second order asymptotic approximation respectively.

Of course, equations of asymptotic approximation of higher order can be written and they are affine, but their solution is very difficult. Just to solve the linear equation (15), the only one dealt with in [20], the special Lax- Boillat method of constructing the eigenvalues and eigenvectors of (15) in terms of $\mathbf{n}$ (given by $(10)_2$) and the coefficients in $\mathbf{A}^\alpha$ was used ([1]-[8]).

Remaind that the wavefront φ is still an unknown function. In order to determine it, some other peculiarities of the hyperbolicity of (6) were taken into account. Of course, they have nothing to do with the double-scale method, but enables one to solve (15).

Remark, now, an interesting fact related to the variation of $\mathbf{U}$ along a curve *C*, defined by $x = x^\alpha(s)$, where s is the parameter along *C*. We have

$$\frac{d\mathbf{U}}{ds} = \omega \frac{\partial \mathbf{U}}{\partial \xi}\frac{\partial \varphi}{\partial s} + \frac{\partial \mathbf{U}^\alpha}{\partial x^\alpha}\frac{dx^\alpha}{ds}. \tag{17}$$

This relation shows that $\mathbf{U}$ does not vary too much if the curve is situated on the wavefront $(9)_1$, while it undergoes large variations (due to the presence of the large parameter ω) if *C* crosses the wavefront. Therefore, the choice of the new variable ξ in the double- scale method is appropriate to the study of asymptotic waves.

For various other physical and related mathematical aspects of asymptotic waves the reader is kindly referred to [29]-[33].

3. Hyperbolicity of Euclidean spaces and of semilinear PDEs

In order to relate the characteristics of motion with hyperbolicity of equations governing this motion, and then to describe the mathematical and physical meaning of the relevant quantities in the previous sections, we recall the definition of the nonlinear hyperbolic PDEs.

Following [8], we define these equations and show how some physical properties are introduced in the methods used to solve them. Let E^{n+1} be an Euclidean space, let $P \in E^{n+1}$ be a current point, let $\mathbf{U} = \mathbf{U}(P)$ be the unknown vector function $\mathbf{U} = (U_1, U_2, ..., U_N)$, solution of the first-order semilinear PDE in a more general form reads

$$\mathbf{G}^{\alpha}(\mathbf{U}(P), P)\frac{\partial \mathbf{U}}{\partial y^{\alpha}} = \mathbf{g}(\mathbf{U}(P), P), \qquad (\alpha = 0, 1, 2, ..., n), \tag{18}$$

where $\mathbf{g} = (g_1, g_2, ..., g_N)^T$ is a column vector, y^{α} are the Cartesian coordinates of P and $\mathbf{G}^{\alpha}$ are n+1 square matrices of the $N \times N$ type. Denote by $\mathbf{G}^{\alpha A}_{B}$, $(A, B = 1, 2, ..., N)$ a real function defined on E^{n+1} which is a current entry of $\mathbf{G}^{\alpha}$.

We say that (18) is a *nonlinear hyperbolic equation* if the n+1 matrices $\mathbf{G}^{\alpha}$ endow E^{n+1} with a hyperbolic structure at the current point $P \in E^{n+1}$, i.e. if the following two conditions are satisfied ([29], [8]):

(i) there exists a direction $\mathbf{v} \equiv v_{\alpha}(P)$ such that $det\ \mathbf{A}^0 \neq 0$, where

$$\mathbf{A}^0 = \mathbf{G}^{\alpha} v_{\alpha};$$

(ii) $if\ \mathbf{v},\ \mathbf{e}_i,\ (i = 1, 2, ..., n;\ \mathbf{e}_i \equiv e_{i\alpha};\ v^{\alpha}v_{\alpha} = 1;\ v^{\alpha}e_{i\alpha} = 0;\ e^{\alpha}_{i}e_{j\alpha} = \delta_{ij})$ is an orthonormal base of E^{n+1} at P for every direction $\mathbf{n} \equiv n_j$ of the n-dimensional subspace of E^{n+1} generated by the base $\mathbf{e}_i$ (orthogonal to $\mathbf{v}$), then the matrix

$$\mathbf{A}_n = \mathbf{A}^{0^{-1}}\mathbf{G}^{\alpha} e_{i\alpha} n_i \tag{19}$$

possesses N linearly-independent left and right eigenvectors $\mathbf{l}_A$ and $\mathbf{r}_A$, respectively, corresponding to the real eigenvalue λ^A of multiplicity m^A , i.e.

$$(\mathbf{A}_n(\mathbf{U}) - \lambda^A I)\mathbf{r}_A = 0, \qquad \mathbf{l}_A(\mathbf{A}_n(\mathbf{U}) - \lambda^A I) = 0. \tag{20}$$

Since $\mathbf{A}_n$ depends on $\mathbf{U}$ and $\mathbf{n}$ ($\mathbf{n}$ being arbitrary in E^n), this means that the eigenvalues and eigenvectors also depend on $\mathbf{U}$ and $\mathbf{n}$. This is why we write e.g. $\lambda^A(\mathbf{U}, \mathbf{n})$. The index n of $\mathbf{A}_n$ remainds the vector $\mathbf{n}$ and not the dimension of the Euclidean space E^n . The superscript A in λ^A is not related to the matrix $\mathbf{A}$ but to the multiplicity m^A . The dummy index convention is understood.

The above definition of nonlinear hyperbolicity generalizes the definition for the case of linear or affine hyperbolic PDEs, where $\mathbf{G}^{\alpha}$ and $\mathbf{g}$ do not depend on $\mathbf{U}$. It preserves the fact that the Cauchy problem for (18) is well-posed. Indeed, condition (i) ensures the possibility to write (18) in the form

$$\mathbf{U}_t + \mathbf{A}^i\mathbf{U}_{x^i} = \mathbf{B}(\mathbf{U}, x^{\alpha}), \tag{21}$$

where

$$x^0 = t \equiv y^{\alpha}v_{\alpha},\ x^i = y^{\alpha}e_{i\alpha},\ \mathbf{A}^i(\mathbf{U},\ x^{\alpha}) = \mathbf{A}^{0^{-1}}\mathbf{G}^{\alpha}e_{i\alpha},\ \mathbf{B}(\mathbf{U}, x^{\alpha}) = \mathbf{A}^{0^{-1}}\mathbf{g}.$$

The form (21) is obtained by multiplying (18) by $\mathbf{A}^{0^{-1}}$ (which exists by virtue of condition (i)). Also, from (6) it is possible to get (21) by the left multiplication by $\mathbf{A}^{0^{-1}}$. In this way, (6) and (21) are two equivalent mathematical models for the same physical problem.

In our concrete case (21), e. g. (6), we have:

- $x^0 = t$ (the time), x^1, x^2, x^3 are the space variables,
- $\mathbf{A}^0$ is the matrix coefficient of $\frac{\partial \mathbf{U}}{\partial t}$, $(N = 10,\ n = 3)$,
- $\mathbf{v} = \mathbf{e}_0$ and has coordinates $e_{o\alpha}$, $\mathbf{e}_i$ have coordinates

$$e_{i\alpha}, \quad e_{\alpha\beta} = \delta_{\alpha\beta}, \quad \delta_{\alpha\beta} = 0 \quad for \quad \alpha \neq \beta,$$

$$\delta_{\alpha\beta} = 1, \quad for \quad \alpha = \beta; \quad (\alpha, \beta = 0, 1, 2, 3); \quad (i = 1, 2, 3).$$

In this way, $\{\mathbf{e}_0,\ \mathbf{e}_1,\ \mathbf{e}_2,\ \mathbf{e}_3\}$ is a basis in the space E^4, and, in the Euclidean space E^3, it corresponds to the canonical basis. Hence for our concrete situation (21) is a nonlinear hyperbolic PDFs system. The matrix $\mathbf{A}_n$ follows defined by

$$\mathbf{A}_n = \mathbf{A}^{0^{-1}}\mathbf{G}^1 n_1 + \mathbf{A}^{0^{-1}}\mathbf{G}^2 n_2 + \mathbf{A}^{0^{-1}}\mathbf{G}^3 n_3,$$

or, taking into account (21), equivalently by

$$\mathbf{A}_n = \mathbf{A}^1 n_1 + \mathbf{A}^2 n_2 + \mathbf{A}^3 n_3. \tag{22}$$

In our example (6), it was found ([20]) that, indeed, the eigenvalues are real and the eigenvectors are linearly independent.

Let us now take as $\mathbf{n}$ in condition (ii) just the unit vector normal to that hypersurface S (improperly called wavefront) (i. e. as in $(9)_1$)

$$\varphi(t, x^1, x^2, x^3) = 0, \tag{23}$$

which was supposed to be characterized by the fact that the solution $\mathbf{U}$ of (6) varies steeply across it. In fact, equation (23) is identical to $(9)_1$. Then (23) implies that along S we have $\frac{d\varphi}{dt} = 0$, implying

$$\frac{\partial \varphi}{\partial t} + \mathbf{v} \cdot grad\varphi = 0, \quad \textit{or equivalently}, \quad \frac{\frac{\partial \varphi}{\partial t}}{|grad\varphi|} + \mathbf{v} \cdot \frac{grad\varphi}{|grad\varphi|} = 0. \tag{24}$$

Obviously $\frac{grad\varphi}{|grad\varphi|} = \mathbf{n}$, such that the previous equality reads

$$\frac{\frac{\partial \varphi}{\partial t}}{|grad\varphi|} + \mathbf{v} \cdot \mathbf{n} = 0. \tag{25}$$

Introducing the notation $\lambda = -\frac{\frac{\partial \varphi}{\partial t}}{|grad\varphi|}$ (as in (10)) we have

$$\lambda(\mathbf{U}, \mathbf{n}) = \mathbf{v} \cdot \mathbf{n}. \tag{26}$$

Introduce also the notation

$$\Lambda^i \equiv \frac{dx^i}{dt}, \quad and \quad \Psi(x^\alpha, \frac{\partial \varphi}{\partial x^\alpha}) \equiv \frac{\partial \varphi}{\partial t} + \lambda |grad\varphi|, \tag{27}$$

such that $\lambda = v^i n_i$ where λ is called the *velocity normal to the progressive wave* and $\mathbf{\Lambda}$, of coordinates Λ^i , the *radial velocity*. In this way various vectors in the definition of hyperbolic equations were related to the characteristics of the motion. In addition, the theory in [5] enables us to deduce the equation for φ by using (24) and (25).

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