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ON THE STABILITY OF THE HOMOGRAPHIC POLYGON CONFIGURATION IN THE MANY-BODY PROBLEM

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ABSTRACT. In this paper the stability of a new class of exact symmetrical solutions in the Newtonian gravitational $(n + 1)$-body problem is studied. This class of solution follows from a suitable geometric distribution of the $(n + 1)$-bodies, and initial conditions, so that the solution is represented geometrically by an oscillating regular polygon with n sides rotating non-uniformly about its center. The body having a mass m_0 is at the center of the polygon, while n bodies having the same mass m are at the vertices of the polygon and move about the central body in identical elliptic orbits. It is proved that for $n = 2$ and for regular polygons $3 \leq n \leq 6$ each corresponding solution is unstable for any value of the central mass m_0. For $n \geq 7$ the solution is linearly stable if both $\mu = m_0/m > 141.477$ and the eccentricity of the particles' orbits e is sufficiently small.

1. Introduction

The differential equations of the many-body problem are not integrable in general case. Some further progress in this field can be obtained by seeking for exact particular solutions of the equations of motion and investigating their stability.

We consider a many-body problem where n bodies, with the same mass m, are at the vertices of a regular n-polygon and move on a closed orbit about its center, while a body having a mass m_0 is at the center of the polygon (see Fig.1).

In [1,2] it was shown that a periodic solution exists on a circle. The polygon remains invariant with time and rotates uniformly about its center. Stability of the invariant polygon in linear and nonlinear approximations was investigated in [4,5]. More in general, E. A. Grebenikov showed in [3] that under the same hypotheses also elliptic orbits are admissible. The symmetry of the system is conserved, but polygon rotates non-uniformly and its dimension oscillate. The corresponding homographic polygon is shown in Fig. 1, for $n = 7$. Stability of the oscillating polygon with respect to the perturbations, being perpendicular to the plane of the particles' orbits, is analyzed in [6]. In this paper we investigate stability of the exact symmetrical solutions of the Newtonian gravitational problem of $(n+1)$ bodies in the more general elliptic case. In comparison with the circular

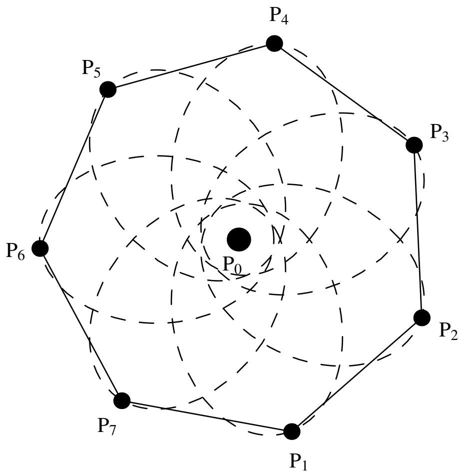

FIGURE 1. The homographic polygon for $(n = 7)$

case this problem is essentially complicated because the equations of the disturbed motion become non-autonomous, and it is necessary to calculate the characteristic exponents for these equations. According to the computed numerical values of the coefficients, there follows that for $n \geq 7$ the solution is linearly stable if both $\mu = m_0/m > 141.477$ and the eccentricity of the particles' orbits e is sufficiently small.

2. Equations of the disturbed motion

The particle P_0 having a mass m_0 and n particles $P_1,\ P_2, \ldots\ P_n$ having the same mass $m_1 = m_2 = \ldots = m$ move under their mutual gravitational forces $F_{jk} = G\dfrac{m_j m_k}{r_{jk}}$ where r_{jk} is the distance between P_j and P_k $(j, k = \overline{1, n})$ and G is the newtonian gravitational constant. Using the relative cylindrical coordinates $\rho,\ \varphi,\ z$ with the particle P_0 being in the origin we can write [6] the equations of motion for the particles $P_1,\ P_2,\ \ldots,\ P_n$ in the form

$$\frac{d^2\rho_j}{dt^2} - \rho_j\left(\frac{d\varphi_j}{dt}\right)^2 + G(m_0+m)\frac{\rho_j}{{r_j}^3} =$$

$$= G\,m \sum_{k=1\ (\neq j)}^{n} \left(\frac{\cos(\varphi_j - \varphi_k)\rho_k - \rho_j}{{r_{j,k}}^3} - \frac{\cos(\varphi_j - \varphi_k)\rho_k}{{r_k}^3}\right),$$

$$\rho_j\,\frac{d^2\varphi_j}{dt^2} + 2\frac{d\rho_j}{dt}\,\frac{d\varphi_j}{dt} = G\,m \sum_{k=1\ (\neq j)}^{n} \left(\frac{1}{{r_k}^3} - \frac{1}{{r_{j,k}}^3}\right)\,\sin(\varphi_j - \varphi_k)\,\rho_k\,, \tag{1}$$

$$\frac{d^2 z_j}{dt^2} + G(m_0+m)\frac{z_j}{r_j^3} = -G\,m \sum_{k=1\ (\neq j)}^{n} \left(\frac{z_j - z_k}{{r_{j,k}}^3} + \frac{z_k}{{r_k}^3}\right)\,, \qquad (j = \overline{1, n})$$

where

$$r_k^2 = \rho_k^2 + z_k^2,\ {r_{j,k}}^2 = \rho_j^2 + \rho_k^2 - 2\rho_j\rho_k\ \cos(\varphi_j - \varphi_k) + (z_j - z_k)^2.$$

In order to consider the elliptic case of the symmetrical solutions in the $(n+1)$-body problem, it is expedient to analyze the motion of the particles in the Nechvil's configurational space [7], with the coordinate transformation

$$\rho_j(t) \to \frac{p}{1+e\ \cos\nu}\,\rho_j(\nu),\ \ \varphi_j(t) \to \varphi_j(\nu),\ \ z_j(t) \to \frac{p}{1+e\ \cos\nu}\,z_j(\nu),$$

where p and e are the parameter and eccentricity of the elliptic orbit respectively and a polar angle ν is used as a new independent variable. Time derivative is transformed as

$$\frac{d}{dt} \to \frac{c}{p^2}(1+e\ \cos\nu)^2\,\frac{d}{d\nu}$$

where c is a constant. Then we obtain the equations of motion in the form

$$\frac{d^2\rho_j}{d\nu^2} - \rho_j\Big(\frac{d\varphi_j}{d\nu}\Big)^2 + \frac{e\ \cos\nu}{1+e\ \cos\nu}\,\rho_j + \frac{Gp\,(m_0+m)}{c^2(1+e\ \cos\nu)}\,\frac{\rho_j}{{r_j}^3} =$$

$$= -\frac{Gp\,m}{c^2(1+e\ \cos\nu)}\sum_{k=1\ (\neq j)}^{n}\Big(\frac{\rho_j - \rho_k\cos(\varphi_j-\varphi_k)}{{r_{j,k}}^3} + \frac{\rho_k\cos(\varphi_j-\varphi_k)}{{r_k}^3}\Big),$$

$$\rho_j\,\frac{d^2\varphi_j}{d\nu^2} + 2\frac{d\rho_j}{d\nu}\,\frac{d\varphi_j}{d\nu} = \frac{Gpm}{c^2(1+e\ \cos\nu)}\sum_{k=1\ (\neq j)}^{n}\Big(\frac{\rho_k}{{r_k}^3} - \frac{\rho_k}{{r_{j,k}}^3}\Big)\,\sin(\varphi_j-\varphi_k),$$

$$\frac{d^2z_j}{d\nu^2} + \frac{e\ \cos\nu}{1+e\ \cos\nu}\,z_j + \frac{Gp(m_0+m)}{c^2(1+e\ \cos\nu)}\,\frac{z_j}{r_j^3} =$$

$$= -\frac{Gpm}{c^2(1+e\ \cos\nu)}\sum_{k=1\ (\neq j)}^{n}\Big(\frac{z_j-z_k}{{r_{j,k}}^3} + \frac{z_k}{{r_k}^3}\Big)\ (j=\overline{1,n}), \tag{2}$$

It is easy to verify that equations (2) have the solution

$$\rho_j(\nu) = 1,\ \varphi_j(\nu) = \nu + \frac{2\pi}{n}\,j,\ z_j(\nu) = 0\ \ (j=\overline{1,n}). \tag{3}$$

Indeed, substituting solution (3) into equations (2) we obtain that the second and the third equations are satisfied identically, while the first one gives for the constant c the following expression

$$c^2 = Gp\Big(m_0 + \frac{m}{4}\sum_{k=1}^{n-1}\frac{1}{\sin(\frac{\pi}{n}k)}\Big).$$

Thus, in the Nechvil's configurational space, (3) determines the equilibrium points of the particles being at the vertices of a regular polygon inscribed into a unit circle. At the same time, the identical elliptic trajectories of the particles $P_1, P_2, \ldots, P_n$ are determined, with the parameter p and eccentricity e which are situated in the xP_0y plane of the barycentric inertial frame of reference.

In order to investigate equations (2) in the vicinity of solution (3) let us make a substitution

$$\rho_j(\nu) \to 1 + u_j(\nu),\ \ \varphi_j(\nu) \to \nu + \frac{2\,\pi}{n}\,j + \gamma_j(\nu).$$

Then we obtain the differential equations of the disturbed motion which are essentially nonlinear. Usually, the first step in studying such equations is an analysis of the corresponding linearized system. Considering the functions $u_j(\nu), \gamma_j(\nu),\ z_j(\nu)$ as small perturbations of solution (3) we can expand equations (2) in Taylor series in powers of $u_j, \gamma_j,\ z_j$ and neglect all terms of order superior or equal to 2. As a result we obtain a system of linearized equations in the form

$$\frac{d^2u_j}{d\nu^2} - 2\,\frac{d\gamma_j}{d\nu} = \frac{8+12\mu+S_1}{(1+e\ \cos\nu)(4\mu+S_1)}\ u_j +$$
$$+\frac{1}{4(1+e\cos\nu)(4\mu+S_1)}\sum_{k=1\ (\neq j)}^{n}\frac{1}{|\sin(\frac{\pi}{n}\ (j-k))|^3}\Big((1-3\cos(\frac{2\pi}{n}(j-k)))u_j +$$
$$+(3-\cos(\frac{2\pi}{n}(j-k))+32\cos(\frac{2\pi}{n}(j-k))|\sin(\frac{\pi}{n}(j-k))|^3)u_k +$$
$$+(1+16|\sin(\frac{\pi}{n}(j-k))|^3)\sin(\frac{2\pi}{n}(j-k))(\gamma_j-\gamma_k)\Big),$$

$$\frac{d^2\gamma_j}{d\nu^2} + 2\frac{du_j}{d\nu} = \frac{1}{4(1+e\ \cos\nu)(4\mu+S_1)}\sum_{k=1\ (\neq j)}^{n}\frac{1}{|\sin(\frac{\pi}{n}\ (j-k))|^3}\Big((1-$$
$$-32|\sin(\frac{\pi}{n}(j-k))|^3)\sin(\frac{2\pi}{n}(j-k))u_k +$$
$$+(3+\cos(\frac{2\pi}{n}(j-k))(1+16|\sin(\frac{\pi}{n}(j-k))|^3))(\gamma_j-\gamma_k)\Big),$$
$$\frac{d^2z_j}{d\nu^2} + \frac{4(1+\mu)+(4\mu+S_1)e\ \cos\nu}{(1+e\ \cos\nu)(4\mu+S_1)}\ z_j =$$
$$=\frac{1}{2(1+e\ \cos\nu)(4\mu+S_1)}\sum_{k=1(\neq j)}^{n}\frac{1}{|\sin(\frac{\pi}{n}\ (i-j))|^3}\Big((1-$$
$$-8|\sin(\frac{\pi}{n}(j-k))|^3)\ z_k - z_j\Big), \tag{4}$$

where

$$\mu = \frac{m_0}{m},\ \ S_1\ = \sum_{k=1}^{n-1}\frac{1}{\sin(\frac{\pi}{n}\ k)}.$$

The stability problem for the solution of (3) is reduced to the stability analysis of the trivial solution of the system (4), which is a linear system of differential equations with periodic coefficients. Let us introduce the n-dimensional vectors $U(\nu)\equiv\{u_j(\nu)\}_{j=\overline{1,n}}$, $V(\nu)\equiv\{\gamma_j(\nu)\}_{j=\overline{1,n}}$ and $Z(\nu)\equiv\{z_j(\nu)\}_{j=\overline{1,n}}$ and the $n\times n$ matrixes B, C, S, D_1, D_2 and K whose components are defined as follows

$$B_{jk} = \frac{1}{|\sin(\frac{\pi}{n}\ (j-k)|^3}\quad for\ \ j\neq k\ \ and\ \ B_{jj} = -\sum_{k=1(\neq j)}^{n} B_{jk};$$
$$C_{jk} = \frac{\cos(\frac{2\pi}{n}\ (j-k))}{|\sin(\frac{\pi}{n}\ (j-k)|^3}\quad for\ \ j\neq k\ \ and\ \ C_{jj} = -\sum_{k=1(\neq j)}^{n} C_{jk};$$

$$S_{jk} = \frac{\sin(\frac{2\pi}{n}\ (j-k))}{|\sin(\frac{\pi}{n}\ (j-k)|^3}\ \ for\ \ j\neq k\ \ and\ \ S_{jj} = -\sum_{k=1(\neq j)}^{n} S_{jk} = 0;$$

$$D_{1jk} = \cos(\frac{2\pi}{n}\ (j-k))\ \ for\ \ j\neq k\ \ and\ \ D_{1jj} = -\sum_{k=1(\neq j)}^{n} D_{1jk} = 1;$$

$$D_{2jk} = \sin(\frac{2\pi}{n}\ (j-k))\ \ for\ \ j\neq k\ \ and\ \ D_{2jj} = -\sum_{k=1(\neq j)}^{n} D_{2jk} = 0;$$

$$K_{jk} = 1\quad (j,k=\overline{1,n}).$$

Then equations (4) can be written as

$$\frac{d^2U}{d\nu^2} - 2\frac{dV}{d\nu} = \frac{1}{4(1+e\ \cos\nu)(4\mu+S_1)}\Big(12(4\mu+S_1)U+$$
$$+(3B-C+32D_1)U-(S+16D_2)V\Big), \tag{5}$$

$$\frac{d^2V}{d\nu^2} + 2\frac{dU}{d\nu} = \frac{1}{4(1+e\ \cos\nu)(4\mu+S_1)}\Big((S-32D_2)U-$$
$$-(3B+C+16D_1)V\Big), \tag{6}$$

$$\frac{d^2Z}{d\nu^2} = \frac{1}{2(1+e\ \cos\nu)(4\mu+S_1)}\ (B-8K-8\mu I_n-2(S_1+4\mu)e\ \cos\nu I_n)Z, \tag{7}$$

where I_n is an $n\times n$ identity matrix. The derivatives of vector-valued functions $U(\nu)$, $V(\nu)$ and $Z(\nu)$ are defined to be the vectors whose components are the corresponding derivatives of $u_j(\nu)$, $\gamma_j(\nu)$, $z_j(\nu)$.

3. Diagonalization of the linearized system

The vectors $E_r \equiv \{E_{r,k}\}_{k,r=\overline{1,n}}$ with

$$E_{r,k} = \frac{1}{\sqrt{n}}\ e^{i\ \frac{2\pi r}{n}\ k}\quad (k,r=\overline{1,n}) \tag{9}$$

and $i=\sqrt{-1}$ is the imaginary unit, are the eigenvectors of the matrixes B, C, S, D_1, D_2 and K in the sense that

$$B\cdot E_r = -\lambda_r\ E_r,$$
$$C\cdot E_r = (2S_1 - \frac{1}{2}\ (\lambda_{r+1}+\lambda_{r-1}))E_r,$$
$$S\cdot E_r = \frac{i}{2}\ (\lambda_{r-1}-\lambda_{r+1})E_r,$$
$$D_1\cdot E_r = \frac{n}{2}\ (\delta_{r,1}+\delta_{r,n-1})E_r,$$
$$D_2\cdot E_r = -\frac{in}{2}\ (\delta_{r,1}-\delta_{r,n-1})E_r,$$
$$K\cdot E_r = n\ \delta_{r,n}\ E_r\quad (r=\overline{1,n})$$

where

$$\lambda_r = 2 \sum_{k=1}^{n-1} \frac{\sin^2(\pi r k/n)}{\sin^3(\pi k/n)}$$

and $\delta_{j,k}$ is the Kronecker delta. For each matrix $F \equiv \{F_{j,k}\}_{j,k=\overline{1,n}}$ the corresponding eigenvalues are

$$\Lambda_r = \sum_{k=1}^{n} F_{1,k}\, e^{i\, \frac{2\pi r}{n}\, (k-1)}. \tag{10}$$

The vectors E_r are linearly independent and satisfy the normalizing condition

$$E_j^+ \cdot E_r = \frac{1}{n} \sum_{k=1}^{n} e^{i\, \frac{2\pi}{n}(r-j)\, k} = \delta_{j,r}.$$

So that we can introduce the matrix Q with components $Q_{k,r} = E_{r,k}$, reducing the matrixes B, C, S, D_1, D_2, K to their diagonal forms through a transformation

$$F \to\ Q^+ F Q = diag[\Lambda_1,\ \Lambda_2,\ \ldots,\ \Lambda_n].$$

Thus we can rewrite equations (5)-(7) in the normal form

$$\frac{d^2 U_r}{d\nu^2} - 2\frac{dV_r}{d\nu} = \frac{1}{1+e\ \cos\nu}(a_r U_r + b_r V_r)\ , \tag{11}$$

$$\frac{d^2 V_r}{d\nu^2} + 2\frac{dU_r}{d\nu} = \frac{1}{1+e\ \cos\nu}(c_r U_r + d_r V_r), \tag{12}$$

$$\frac{d^2 Z_r}{d\nu^2} = -\frac{p_r + e\ \cos\nu}{1+e\ \cos\nu}\, Z_r, \quad (r = \overline{1,n}) \tag{13}$$

where

$$a_r = \frac{1}{4(4\mu+S_1)}\left[-3\lambda_r + \frac{1}{2}(\lambda_{r+1}+\lambda_{r-1}) + 10S_1 + 48\mu + 16n(\delta_{r,1}+\delta_{r,n-1})\right],$$

$$b_r = \frac{i}{8(4\mu+S_1)}\left[\lambda_{r+1} - \lambda_{r-1} + 16n(\delta_{r,1}-\delta_{r,n-1})\right],$$

$$c_r = \frac{i}{8(4\mu+S_1)}\left[\lambda_{r-1} - \lambda_{r+1} + 32n(\delta_{r,1}-\delta_{r,n-1})\right],$$

$$d_r = \frac{1}{4(4\mu+S_1)}\left[3\lambda_r + \frac{1}{2}(\lambda_{r+1}+\lambda_{r-1}) - 2S_1 - 8n(\delta_{r,1}+\delta_{r,n-1})\right],$$

$$p_r = \frac{\lambda_r + 8\mu + 8n\delta_{r,n}}{2(4\mu+S_1)}\ .$$

The coefficients a_r, b_r, c_r, d_r satisfy the following relations

$$a_{n-r} = a_r,\ \ d_{n-r} = d_r,\ \ b_{n-r} = -b_r,\ \ c_{n-r} = -c_r,$$

$$a_n = 3,\ \ b_n = c_n = d_n = 0,\ \ p_n = \frac{4(\mu+n)}{4\mu+S_1}. \tag{14}$$

4. Studying stability of the trivial solution

Equation (13) is independent on equations (11), (12). Thus, in the linear approximation, we may separately study the stability of solution (3) with respect to the orthogonal perturbations Z_r and to the perturbations U_r and V_r contained in the plane of particles' orbits xP_0y.

4.1. Stability of the system (11)-(12). Equations (11), (12), determining the disturbed motion of the particles in xP_0y plane, are second order differential equations with periodic coefficients which are analytic functions of the parameter e in the domain $|e| < 1$. According to the general theory [8], the behavior of the solutions of such system for sufficiently small e is determined by its characteristic exponents calculated for $e = 0$. And the system may be linearly stable only if for $e = 0$ all its characteristic exponents $\alpha_{r,\sigma}$ are various and pure imaginary and satisfy the following inequality

$$\alpha_{r,\sigma} \pm \alpha_{k,\tau} \neq iN \ , \ (k, r = \overline{1, n}) \ , \ (\sigma, \tau = 1, \ 2, \ 3, \ 4; \ N = 0, \ \pm 1, \ \pm 2, \ \dots) \qquad (15)$$

So, first of all we have to calculate the characteristic exponents of system (11), (12) for $e = 0$.

If $e = 0$ then we have two linear differential equations with constant coefficients. Their solution may be sought in the form

$$U_r = \ e^{\alpha_{r,\sigma} \ \nu} x_r, \ V_r = \ e^{\alpha_{r,\sigma} \ \nu} y_r. \qquad (16)$$

Substituting (16) into (11), (12) we obtain the homogeneous system of two algebraic equations

$$(a_r - \alpha^2_{r,\sigma}) \ x_r + (b_r + 2\alpha_{r,\sigma}) \ y_r = 0,$$

$$(c_r - 2\alpha_{r,\sigma}) \ x_r + (d_r - \alpha^2_{r,\sigma}) \ y_r = 0.$$

A non-trivial solution exists, if the corresponding determinant of the coefficients of $x_r, \ y_r$ is equal to zero, i.e.,

$$\alpha^4_{r,\sigma} + q_{2r} \ \alpha^2_{r,\sigma} + i \ q_{1r}\alpha_{r,\sigma} + q_{0r} = 0 \ , \qquad (17)$$

where

$$q_{2r} = 4 - a_r - d_r, \ \ q_{1r} = -2i \ (b_r - c_r), \ \ q_{0r} = a_r d_r - b_r c_r \ .$$

In consequence of (14) we get

$$q_{2n-r} = q_{2r}, \ \ q_{1n-r} = -q_{1r}, \ \ q_{0n-r} = q_{0r} \ .$$

Thus, the necessary condition of stability of a trivial solution of equations (11),(12) is the existence of $4n$ various and pure imaginary roots $\alpha_{r,\sigma}$ of the characteristic equation (17).

4.2. Stability dependence on the number of bodies.
Solutions for $r = n$. For $r = n$ we have $q_{2n} = 1, \ q_{1,n} = q_{0,n} = 0$ and equation (17) takes the form

$$\alpha^2_{n,\sigma}(\alpha^2_{n,\sigma} + 1) = 0. \qquad (18)$$

It has a double root $\alpha_{n,1} = \alpha_{n,2} = 0$ and two simple pure imaginary roots $\alpha_{n,3} = -\alpha_{n,4} = i$ for all μ.

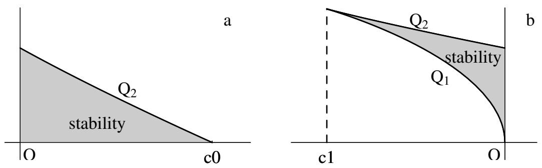

FIGURE 2. Boundary curves for stability domain: a) Eq. (22) $c0 = \frac{1}{4}q_{r2}^2$, b) Eq. (23) $c1 = -\frac{1}{12}q_{r2}^2$

Solutions for $r < n$. Since $q_{2r},\ q_{1r},\ q_{0r}$ are real-valued functions of n and μ, then $\alpha_{r,\sigma}$ and $\alpha_{n-r,\sigma} = \overline{\alpha}_{r,\sigma}$ will be a complex-conjugate pair of the roots of equation (17). So it is sufficient to analyze equation (17) for $r = 1, \ldots,\ [n/2]$, where [ ] is the entire part of $n/2$. Making a substitution $\alpha_{r,\sigma} \to i\beta_{r,\sigma}$ in (17) we rewrite it as

$$\left(\beta_{r,\sigma}^2 - \frac{q_{2r}}{2}\right)^2 = \frac{q_{2r}^2}{4} + q_{1r}\beta_{r,\sigma} - q_{0r}. \tag{19}$$

And this equation should have four various real roots $\beta_{r,\sigma}$ for all $r = 1, \ldots, [n/2]$. It means that there should exist four points of intersection of the curve

$$y = \left(x^2 - \frac{q_{2r}}{2}\right)^2 \tag{20}$$

and the line

$$y = \frac{q_{2r}^2}{4} + q_{1r}x - q_{0r}. \tag{21}$$

Standard analysis shows that the curve (20) and the line (21) have four points of intersection only if $q_{2r} > 0$ and the additional following conditions are fulfilled (see Fig. 2)

$$0 < q_{1r} < Q_2,\ \ 0 \le q_{0r} < \frac{1}{4}\, q_{2r}^2 \tag{22}$$

or

$$Q_1 < q_{1r} < Q_2,\ \ -\frac{1}{12}\, q_{2r}^2 < q_{0r} < 0 \tag{23}$$

where

$$Q_1 = \frac{1}{3}\sqrt{\frac{2}{3}}\ \frac{q_{2r}^2 - 12q_{0r} - q_{2r}\sqrt{q_{2r}^2 + 12q_{0r}}}{(q_{2r} - \sqrt{q_{2r}^2 + 12q_{0r}})^{1/2}},$$

$$Q_2 = \frac{1}{3}\sqrt{\frac{2}{3}}\ \frac{q_{2r}^2 - 12q_{0r} + q_{2r}\sqrt{q_{2r}^2 + 12q_{0r}}}{(q_{2r} + \sqrt{q_{2r}^2 + 12q_{0r}})^{1/2}}.$$

As a consequence, equation (17) will have four various pure imaginary roots for all $r = 1, \ldots,\ [n/2]$, if coefficients q_{2r}, q_{1r}, q_{0r} satisfy the conditions (22), (23).

As $q_{2r},\ q_{1r},\ q_{0r}$ depend only on the number of particles n and the ratio of masses $\mu = m_0/m$, the characteristic exponents $\alpha_{r,\sigma}$ for $e = 0$ are entirely determined by the parameters μ and n. But conditions (22), (23) are essentially nonlinear and can not give us the values of μ, corresponding to the stable behaviour of a trivial solution of equations

(11), (12) for all n, in analytical form. So we can analyze them only numerically. By fixing the value n we have, for instance, for $n = 2$

$$
\begin{aligned}
q_{r0} =& \frac{(1-\cos(\pi r) - 16\,\delta_{r,1})(2+12\mu+\cos(\pi r)+16\,\delta_{r,1})}{2(1+4\mu)^2}\\
q_{r1} =& 0\\
q_{r2} =& \frac{3+8\mu-\cos(\pi r)-16\,\delta_{r,1}}{2+8\,\mu}
\end{aligned}
$$

for $n = 3$

$$
\begin{aligned}
q_{r0} =& \frac{11+24\sqrt{3}\,\mu}{48(1+\mu\sqrt{3})^2} - \frac{5(7+9\mu\sqrt{3})}{72(1+\mu\sqrt{3})^2}\cos\left(\frac{2\pi r}{3}\right) - \frac{31+45\mu\sqrt{3}}{72(1+\mu\sqrt{3})^2}\cos\left(\frac{4\pi r}{3}\right) -\\
&- \frac{1}{72(1+\mu\sqrt{3})^2}\left(198\sqrt{3}+972\mu+153\sqrt{3}\,\cos(\pi r)\,\cos\left(\frac{\pi r}{3}\right)\right)(\delta_{r,1}+\delta_{r,2}) -\\
&- \frac{1}{72(1+\mu\sqrt{3})^2}\left(27\cos(\pi r)\sin\left(\frac{\pi r}{3}\right)(\delta_{r,1}-\delta_{r,2}) + 3888\,\delta_{r,1}\,\delta_{r,2}\right)\\
q_{r1} =& -\frac{\sqrt{3}}{2(1+\mu\sqrt{3})}\left(\frac{2}{3}\cos(\pi r)\,\sin(\frac{\pi\, r}{3}) + 6(\delta_{r,1}-\delta_{r,2})\right)\\
q_{r2} =& \frac{1}{12(1+\mu\sqrt{3})}\left(20+12\mu\sqrt{3}-2\cos(\pi r)\,\cos(\frac{\pi r}{3}) - 18\sqrt{3}\,(\delta_{r,1}+\delta_{r,2})\right).
\end{aligned}
$$

Analogously for higher values of n. Such analysis shows that for $2 \le n \le 6$ the conditions (22), (23) can not be fulfilled for any value of μ. It means that for $2 \le n \le 6$ equations (11),(12) have at least one characteristic exponent with a positive real part and their trivial solution is unstable for $e = 0$ and any value of μ. This conclusion coincides with the corresponding results obtained in [4]. As the characteristic exponents are continuous functions of e there exists at least one characteristic exponent with a positive real part for sufficiently small e as well. Therefore, according to Liapunov's theorem on linearized stability [9,10], we can conclude that in case of $2 \le n \le 6$ the solution (3) is unstable with respect to the perturbations, contained in the plane of particles' orbits, for sufficiently small value of eccentricity e and for any value of mass of the central particle m_0.

For $n \ge 7$ the conditions (22), (23) are fulfilled if the parameter μ is sufficiently large. For example, $\mu > 139.852$ and $\mu > 212.261$ for $n = 7$ and $n = 8$ respectively. Let us consider the case $n = 7$ in detail. Numerical calculations show that for $\mu > 139.852$ there are 12 various real roots $\beta_{r,\sigma} \ne \pm 1$ of equation (19) for $r = 1, 2, 3$. It means there are 24 various pure imaginary characteristic exponents of the system (11), (12) $\alpha_{r,\sigma} \ne \pm i$ $(r = \overline{1,6};\ \sigma = 1, 2, 3, 4)$. Besides, there are two characteristic exponents $\alpha_{7,3} = -\alpha_{7,4} = i$ and one double characteristic exponent $\alpha_{7,1} = \alpha_{7,2} = 0$. So the general solution of equations (11), (12) contains secular terms. But they appear only for some very special perturbations of the system and can be eliminated by a suitable choice of the integration constants.

4.3. Stability dependence on the mass ratio μ. Studying the dependence of the characteristic exponents $\alpha_{r,\sigma}$ on the parameter μ we obtain that there are eight values of μ when the resonance condition

$$\alpha_{r,\sigma} \pm \alpha_{k,\tau} = \pm i\ , \ (k, r = \overline{1,6})\ , \ (\sigma, \tau = 1,\ 2,\ 3,\ 4) \tag{24}$$

is fulfilled. They are

$$\mu = 141.477;\ 149.378;\ 154.86;\ 157.935;\ 162.72; 173.969;\ 178.158;\ 182.126. \quad (25)$$

Consequently, in the vicinity of these points in the μe plane the domains of instability of a trivial solution may arise for $e > 0$, where some characteristic exponents get a positive real part. To investigate behaviour of the characteristic exponents in the vicinity of points (25) for small values of e let us calculate the fundamental matrix of the system (11), (12) using the Liapunov-Poincare method of a small parameter [8,10].

The system (11), (12) can be written in the form

$$\frac{dx}{d\nu} = P(\nu, e)x, \tag{26}$$

where $x = (U_r,\ V_r,\ \frac{dU_r}{d\nu},\ \frac{dV_r}{d\nu})$ is a vector with four components and $P(\nu, e)$ is an 4×4 matrix function that can be represented as

$$P(\nu, e) = P_0 + \sum_{k=1}^{\infty} P_k(\nu)e^k, \tag{27}$$

and

$$P_0 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ a_r & b_r & 0 & 2 \\ c_r & d_r & -2 & 0 \end{pmatrix}, \quad P_k(\nu) = (-\cos\nu)^k \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ a_r & b_r & 0 & 0 \\ c_r & d_r & 0 & 0 \end{pmatrix}.$$

The series (27) converges for any ν in the domain $|e| < 1$ and $P_k(\nu)$ are continuous finite functions. Then the fundamental matrix $X(\nu, e)$ for the system (26), normalized by the condition $X(0, e) = I_4$, may be represented in the form

$$X(\nu, e) = \sum_{k=0}^{\infty} X_k(\nu)e^k. \tag{28}$$

The series (28) converges in the domain $|e| < 1$ for any ν and $X_k(\nu)$ are continuous matrices satisfying the following recurrence relation

$$\frac{dX_k}{d\nu} = \sum_{j=0}^{k} P_j(\nu)\ X_{k-j}(\nu) \tag{29}$$

and initial conditions

$$X_0(0) = I_4,\ \ X_k(0) = 0\ \ (k \geq 1).$$

A solution of equation (29) may be written in the form

$$X_0 = \exp(-P_0\nu),$$

$$X_k = \exp(P_0\nu) \int_0^{\nu} \exp(-P_0\tau) \sum_{j=1}^{k} P_j(\tau)X_{k-j}(\tau)\, d\tau\ \ (k \geq 1). \tag{30}$$

When the fundamental matrix $X(\nu, e)$ is found we can calculate the coefficients $\Delta_k(\rho)$ in the expansion of the characteristic polynomial $\Delta(\rho,\ e)$ in powers of e of the form

$$\Delta(\rho,\ e) \equiv \det(X(2\pi, e) - \rho I_4) = \sum_{k=0}^{\infty}\ \Delta_k(\rho)\ e^k. \tag{31}$$

Then the roots $\rho(e)$ of the characteristic equation $\Delta(\rho, e) = 0$ can be calculated in the vicinity of each point (25) as power series in e. The corresponding characteristic exponents $\alpha_{r,\sigma}(e)$ are determined then as

$$\alpha_{r,\sigma}(e) = \frac{1}{2\pi}\ \ln\rho(e) = \sum_{k=0}^{\infty} \phi_k e^k. \tag{32}$$

With the above method we can successively calculate the coefficients ϕ_k in expansion (32) in the vicinity of each point (25). We made a numerical computation for $n = 7$ in the vicinity of resonance points (25) with accuracy of e^2. For the first resonance point $\mu = 141.477$, for instance, characteristic polynomial (32) was found in the form

$$\begin{gathered}
\rho^4 + \rho^3(0.579035 \mp 0.551048i - (0.016994 \mp 0.007327i)\mu_1 e + \\
+e^2(19.1474 \mp 11.37446i + (0.0000193 \mp 0.0000729i)\mu_1^2 - \\
-(0.016994 \mp 0.007328i)\mu_2)) + \rho^2(-0.841931 - 0.014533e\mu_1 + \\
+e^2(-30.921 + 0.0006388\mu_1^2 - 0.014533\mu_2)) + \rho(0.579035 \pm 0.551048i - \\
-(0.016994 \pm 0.007327i)\mu_1 e + e^2(19.1474 \pm 11.3745i + \\
+(0.000019 \pm 0.000073i)\mu_1^2 - (0.016994 \pm 0.007328i)\mu_2)) + 1,
\end{gathered}$$

where $\mu_1,\ \mu_2$ are the coefficients in the expansion $\mu = 141.477 + \mu_1 e + \mu_2 e^2$. The corresponding characteristic exponents (32) satisfying resonance condition (24) are

$$\begin{gathered}
\alpha(e) = \mp 0.50000i \pm 0.00281ie\mu_1 \pm \\
\pm ie^2(-9.99552 - 0.000029\mu_1^2 + 0.00281\mu_2).
\end{gathered} \tag{33}$$

It should be emphasized that according to the general theory [8,10] a positive real part can arise only in the characteristic exponents satisfying the resonance condition (15). For $\mu = 141.477$ there are only two such characteristic exponents $\alpha = \pm 0.5i$. And we see from (33) that these characteristic exponents remain to be pure imaginary in the vicinity of the point $\mu = 141.477$. Similar calculations made in the vicinity of each point (25) show that all characteristic exponents are pure imaginary for $\mu > 139.852$ and sufficiently small e.

Note that characteristic exponents $\alpha_{7,3} = -\alpha_{7,4} = i$ satisfy the resonance condition

$$\alpha_{7,3} - \alpha_{7,4} = 2i$$

for any value of μ. But analysis of the equations (11), (12) for $r = n = 7$ shows that $\alpha_{7,3}, \alpha_{7,4}$ also remains to be pure imaginary for small $e > 0$. Thus, we may conclude that for $\mu > 139.852$ and sufficiently small e all characteristic exponents of the system (11), (12) are pure imaginary. It means that exact symmetrical solutions (3) in the problem of 8 bodies are stable in linear approximation in respect to the perturbations contained in the plane of the particles' orbits.

Stability of the equation (13). Equation (13) determining the orthogonal perturbations Z_r is just the Hill's equation and it was investigated in details in [6]. It was shown there that the domains of instability of its trivial solution in the μe plane are only in the vicinity of the points $p_r = (2k-1)^2/4 \ \ (k = 1, 2, \ldots)$ and their boundaries are

$$p_r = \frac{1}{4} \mp \frac{3}{8}\, e + \frac{15}{128}\, e^2 \mp \frac{45}{2048}\, e^3 + \frac{885}{32768}\, e^4,$$

$$p_r = \frac{9}{4} - \frac{135}{256}\, e^2 \mp \frac{45}{2048}\, e^3 - \frac{34695}{262144}\, e^4, \ \ldots \tag{34}$$

Numerical analysis of the coefficient p_r shows that for $\mu > 1.45152$ they satisfy the following inequality

$$1 < p_r < 9/4.$$

So a trivial solution of equation (13) is stable for $\mu > 1.45152$ and sufficiently small value of e.

5. Conclusion

We have analyzed the stability of the exact particular solutions (3) of the Newtonian problem of $(n+1)$ bodies found in [3]. It was shown that in case of the circular orbits of the particles ($e = 0$) the characteristic exponents of the linearized equations of the disturbed motion may be found as roots of a fourth degree polynomial whose coefficients depend both on n and on the mass ratio $\mu = m_0/m$. The conditions for the coefficients of the polynomial giving existence of four various pure imaginary roots were explicitly obtained. The results are in agreement with similar results already known [4]. Numerical calculations made for $2 \le n \le 8$ have shown that for $2 \le n \le 6$ solutions (3) are unstable for any value of μ in both circular ($e = 0$) and elliptic cases.

The proposed algorithm of calculation of the characteristic exponents for the system of linear differential equations with periodic coefficients is discussed. Using this algorithm we have calculated the characteristic exponents for the system of two second order differential equations of the disturbed motion of the particles in the vicinity of the resonance points for $n = 7$. We have proved that all characteristic exponents are various and pure imaginary and solutions (3) are linearly stable if $\mu > 139.852$ and eccentricity of the elliptic orbits of the particles e is sufficiently small.

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