DOI: 10.1478/AAPP.901A4

AAPP | Atti della Accademia Peloritana dei Pericolanti
Classe di Scienze Fisiche, Matematiche e Naturali
ISSN 1825-1242

Vol. 90, No. 1, A4 (2012)

CONSTRUCTION OF $\mathcal{B}-$FOCAL CURVES OF SPACELIKE BIHARMONIC $\mathcal{B}-$SLANT HELICES ACCORDING TO BISHOP FRAME IN $\mathbb{E}(1,1)$

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(Communicated by Mauro Francaviglia)

ABSTRACT. In this paper we find parametric equations of $\mathcal{B}-$focal curves of spacelike biharmonic $\mathcal{B}-$slant helices according to Bishop frame in terms of Bishop curvatures in the Lorentzian group of rigid motions $\mathbb{E}(1,1)$.

1. Introduction

A smooth map $\phi : N \longrightarrow M$ is said to be biharmonic if it is a critical point of the bienergy functional:

$$E_2(\phi) = \int_N \frac{1}{2} |\mathcal{T}(\phi)|^2 \, dv_h,$$

where $\mathcal{T}(\phi) := \mathrm{tr}\nabla^{\phi} d\phi$ is the tension field of ϕ

The Euler–Lagrange equation of the bienergy [1–5] is given by $\mathcal{T}_2(\phi) = 0$. Here the section $\mathcal{T}_2(\phi)$ is defined by

$$\mathcal{T}_2(\phi) = -\Delta_\phi \mathcal{T}(\phi) + \mathrm{tr}R(\mathcal{T}(\phi), d\phi)\, d\phi, \tag{1.1}$$

and called the bitension field of ϕ. Non-harmonic biharmonic maps are called proper biharmonic maps.

This study is organised as follows: First, we study $\mathcal{B}-$focal curves of spacelike biharmonic $\mathcal{B}-$slant helices. Finally, we find parametric equations of $\mathcal{B}-$focal curves of spacelike biharmonic $\mathcal{B}-$slant helices according to Bishop frame in terms of Bishop curvatures in the Lorentzian group of rigid motions $\mathbb{E}(1,1)$.

2. Preliminaries

Let $\mathbb{E}(1,1)$ be the group of rigid motions of Euclidean 2-space. This consists of all matrices of the form

$$\begin{pmatrix} \cosh x & \sinh x & y \\ \sinh x & \cosh x & z \\ 0 & 0 & 1 \end{pmatrix}.$$

Topologically, $\mathbb{E}(1,1)$ is diffeomorphic to $\mathbb{R}^3$ under the map

$$\mathbb{E}(1,1) \longrightarrow \mathbb{R}^3 : \begin{pmatrix} \cosh x & \sinh x & y \\ \sinh x & \cosh x & z \\ 0 & 0 & 1 \end{pmatrix} \longrightarrow (x,y,z).$$

It's Lie algebra has a basis consisting of

$$\mathbf{X}_1 = \frac{\partial}{\partial x},\ \mathbf{X}_2 = \cosh x \frac{\partial}{\partial y} + \sinh x \frac{\partial}{\partial z},\ \mathbf{X}_3 = \sinh x \frac{\partial}{\partial y} + \cosh x \frac{\partial}{\partial z},$$

for which

$$[\mathbf{X}_1, \mathbf{X}_2] = \mathbf{X}_3,\ \ [\mathbf{X}_2, \mathbf{X}_3] = 0,\ \ [\mathbf{X}_1, \mathbf{X}_3] = \mathbf{X}_2.$$

Put

$$x^1 = x,\ x^2 = \frac{1}{2}(y+z),\ x^3 = \frac{1}{2}(y-z).$$

Then, we get

$$\mathbf{X}_1 = \frac{\partial}{\partial x^1},\ \mathbf{X}_2 = \frac{1}{2}\left(e^{x^1}\frac{\partial}{\partial x^2} + e^{-x^1}\frac{\partial}{\partial x^3}\right),\ \mathbf{X}_3 = \frac{1}{2}\left(e^{x^1}\frac{\partial}{\partial x^2} - e^{-x^1}\frac{\partial}{\partial x^3}\right). \quad (2.1)$$

The bracket relations are

$$[\mathbf{X}_1, \mathbf{X}_2] = \mathbf{X}_3,\ \ [\mathbf{X}_2, \mathbf{X}_3] = 0,\ \ [\mathbf{X}_1, \mathbf{X}_3] = \mathbf{X}_2.$$

We consider left-invariant Lorentzian metrics which has a pseudo-orthonormal basis $\{\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3\}$. We consider left-invariant Lorentzian metric [6], given by

$$g = -\left(dx^1\right)^2 + \left(e^{-x^1}dx^2 + e^{x^1}dx^3\right)^2 + \left(e^{-x^1}dx^2 - e^{x^1}dx^3\right)^2, \qquad (2.2)$$

where

$$g(\mathbf{X}_1, \mathbf{X}_1) = -1,\ g(\mathbf{X}_2, \mathbf{X}_2) = g(\mathbf{X}_3, \mathbf{X}_3) = 1.$$

Let coframe of our frame be defined by

$$\theta^1 = dx^1,\ \theta^2 = e^{-x^1}dx^2 + e^{x^1}dx^3,\ \theta^3 = e^{-x^1}dx^2 - e^{x^1}dx^3.$$

3. Spacelike biharmonic $\mathcal{B}-$slant helices in the Lorentzian group of rigid motions $\mathbb{E}(1,1)$

Let $\gamma : I \longrightarrow \mathbb{E}(1,1)$ be a non geodesic spacelike curve on the $\mathbb{E}(1,1)$ parametrized by arc length. Let $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ be the Frenet frame fields tangent to the $\mathbb{E}(1,1)$ along γ defined as follows:

$\mathbf{T}$ is the unit vector field γ' tangent to γ, $\mathbf{N}$ is the unit vector field in the direction of $\nabla_{\mathbf{T}}\mathbf{T}$ (normal to γ), and $\mathbf{B}$ is chosen so that $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ is a positively oriented orthonormal basis. Then, we have the following Frenet formulas [7]:

$$\begin{aligned} \nabla_{\mathbf{T}}\mathbf{T} &= \kappa\mathbf{N}, \\ \nabla_{\mathbf{T}}\mathbf{N} &= \kappa\mathbf{T} + \tau\mathbf{B}, \\ \nabla_{\mathbf{T}}\mathbf{B} &= \tau\mathbf{N}, \end{aligned} \qquad (3.1)$$

where κ is the curvature of γ and τ is its torsion and

$$\begin{aligned} g(\mathbf{T},\mathbf{T}) &= 1,\ g(\mathbf{N},\mathbf{N}) = -1,\ g(\mathbf{B},\mathbf{B}) = 1, \\ g(\mathbf{T},\mathbf{N}) &= g(\mathbf{T},\mathbf{B}) = g(\mathbf{N},\mathbf{B}) = 0. \end{aligned}$$

The Bishop frame or parallel transport frame is an alternative approach to defining a moving frame that is well defined even when the curve has vanishing second derivative. The Bishop frame is expressed as [8–14]:

$$\begin{aligned} \nabla_{\mathbf{T}}\mathbf{T} &= k_1\mathbf{M}_1 - k_2\mathbf{M}_2, \\ \nabla_{\mathbf{T}}\mathbf{M}_1 &= k_1\mathbf{T}, \\ \nabla_{\mathbf{T}}\mathbf{M}_2 &= k_2\mathbf{T}, \end{aligned} \tag{3.2}$$

where

$$\begin{aligned} g(\mathbf{T},\mathbf{T}) &= 1,\ g(\mathbf{M}_1,\mathbf{M}_1) = -1,\ g(\mathbf{M}_2,\mathbf{M}_2) = 1, \\ g(\mathbf{T},\mathbf{M}_1) &= g(\mathbf{T},\mathbf{M}_2) = g(\mathbf{M}_1,\mathbf{M}_2) = 0. \end{aligned}$$

Here, we shall call the set $\{\mathbf{T},\mathbf{M}_1,\mathbf{M}_2\}$ as Bishop trihedra, k_1 and k_2 as Bishop curvatures and $\tau(s) = \psi'(s)$, $\kappa(s) = \sqrt{|k_2^2 - k_1^2|}$. Thus, Bishop curvatures are defined by

$$\begin{aligned} k_1 &= \kappa(s)\sinh\psi(s), \\ k_2 &= \kappa(s)\cosh\psi(s). \end{aligned}$$

With respect to the orthonormal basis $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ we can write

$$\begin{aligned} \mathbf{T} &= T^1\mathbf{e}_1 + T^2\mathbf{e}_2 + T^3\mathbf{e}_3, \\ \mathbf{M}_1 &= M_1^1\mathbf{e}_1 + M_1^2\mathbf{e}_2 + M_1^3\mathbf{e}_3, \\ \mathbf{M}_2 &= M_2^1\mathbf{e}_1 + M_2^2\mathbf{e}_2 + M_2^3\mathbf{e}_3. \end{aligned} \tag{3.3}$$

Definition 3.2. (see Ref. [3]) *A regular spacelike curve $\gamma : I \longrightarrow \mathbb{E}(1,1)$ is called a $\mathcal{B}-$slant helix provided the timelike unit vector $\mathbf{M}_1$ of the curve γ has constant angle θ with some fixed timelike unit vector u, that is*

$$g(\mathbf{M}_1(s), u) = \cosh\wp \text{ for all } s \in I. \tag{3.4}$$

Lemma 3.3. (see Ref. [3]) *Let $\gamma : I \longrightarrow \mathbb{E}(1,1)$ be a unit speed spacelike curve with non-zero natural curvatures. Then γ is a $\mathcal{B}-$slant helix if and only if*

$$\frac{k_1}{k_2} = \tanh\wp. \tag{3.5}$$

4. $\mathcal{B}-$focal curves of spacelike biharmonic $\mathcal{B}-$slant helices in the Lorentzian group of rigid motions $\mathbb{E}(1,1)$

Denoting the focal curve by $\mathfrak{focal}_\gamma$, we can write

$$\mathfrak{focal}_\gamma(s) = (\gamma + \mathfrak{f}_1^{\mathcal{B}}\mathbf{M}_1 + \mathfrak{f}_2^{\mathcal{B}}\mathbf{M}_2)(s), \tag{4.1}$$

where the coefficients $\mathfrak{f}_1^{\mathcal{B}}$, $\mathfrak{f}_2^{\mathcal{B}}$ are smooth functions of the parameter of the curve γ, called the first and second focal curvatures of γ, respectively.

To separate a focal curve according to Bishop frame from that of Frenet- Serret frame, in the rest of the paper, we shall use notation for the focal curve defined above as $\mathcal{B}$-focal curve.

Theorem 4.1. *Let* $\gamma : I \longrightarrow \mathbb{E}(1,1)$ *is a non geodesic spacelike biharmonic* $\mathcal{B}-$*slant helix with timelike* $\mathbf{M}_1$ *and* $\mathfrak{focal}_\gamma$ *its focal curve in the Lorentzian group of rigid motions* $\mathbb{E}(1,1)$*. Then, the vector equation of* $\mathfrak{focal}_\gamma^B(s)$ *is*

$$
\begin{aligned}
&(-\sinh\wp s + a_1 + \mathfrak{p}\cosh\wp)\mathbf{X}_1 + [-\mathcal{A}e^{-\sinh\wp s + a_1}[(\sinh\wp - \mathcal{A}_1)\cos[\mathcal{A}_1 s + \mathcal{A}_2] \\
&+ (\sinh\wp + \mathcal{A}_1)\sin[\mathcal{A}_1 s + \mathcal{A}_2]] + a_2 e^{\sinh\wp s - a_1} \\
&-\mathcal{A}e^{\sinh\wp s - a_1}[(\sinh\wp - \mathcal{A}_1)\cos[\mathcal{A}_1 s + \mathcal{A}_2] \\
&+ (\sinh\wp + \mathcal{A}_1)\sin[\mathcal{A}_1 s + \mathcal{A}_2]] + a_3 e^{-\sinh\wp s + a_1} + \mathfrak{p}\sinh\wp\cos[\mathcal{A}_1 s + \mathcal{A}_2] \\
&+\frac{1+\mathfrak{p}k_1}{k_2}\sin[\mathcal{A}_1 s + \mathcal{A}_2]]\mathbf{X}_2 + [-\mathcal{A}e^{-\sinh\wp s + a_1}[(\sinh\wp - \mathcal{A}_1)\cos[\mathcal{A}_1 s + \mathcal{A}_2] \\
&+ (\sinh\wp + \mathcal{A}_1)\sin[\mathcal{A}_1 s + \mathcal{A}_2]] + a_2 e^{\sinh\wp s - a_1} \\
&+\mathcal{A}e^{\sinh\wp s - a_1}[(\sinh\wp - \mathcal{A}_1)\cos[\mathcal{A}_1 s + \mathcal{A}_2] \\
&+ (\sinh\wp + \mathcal{A}_1)\sin[\mathcal{A}_1 s + \mathcal{A}_2]] + a_3 e^{-\sinh\wp s + a_1} + \mathfrak{p}\sinh\wp\sin[\mathcal{A}_1 s + \mathcal{A}_2] \\
&+\frac{-1-\mathfrak{p}k_1}{k_2}\cos[\mathcal{A}_1 s + \mathcal{A}_2]]\mathbf{X}_3, \qquad (4.2)
\end{aligned}
$$

where $a_1, \mathfrak{p}, \mathcal{A}_1, \mathcal{A}_2$ *are constants of integration and*

$$\mathcal{A} = \frac{\cosh\wp}{2\left(\mathcal{A}_1^2 + \sinh^2\wp\right)}.$$

Proof. Assume that γ is a unit speed spacelike biharmonic $\mathcal{B}-$slant helix with timelike $\mathbf{M}_1$ and $\mathfrak{focal}_\gamma$ its focal curve in the Lorentzian group of rigid motions $\mathbb{E}(1,1)$.

So, by differentiating of the formula (2.1), we get

$$\mathfrak{focal}_\gamma^{\mathcal{B}}(s)' = (1 + \mathfrak{f}_1^{\mathcal{B}}k_1 + \mathfrak{f}_2^{\mathcal{B}}k_2)\mathbf{T} + \left(\mathfrak{f}_1^{\mathcal{B}}\right)'\mathbf{M}_1 + \left(\mathfrak{f}_2^{\mathcal{B}}\right)'\mathbf{M}_2. \qquad (4.3)$$

On the other hand, from Definition 3.2, we obtain

$$\mathbf{M}_1 = \cosh\wp\mathbf{X}_1 + \sinh\wp\cos[\mathcal{A}_1 s + \mathcal{A}_2]\mathbf{X}_2 + \sinh\wp\sin[\mathcal{A}_1 s + \mathcal{A}_2]\mathbf{X}_3. \qquad (4.4)$$

Using (2.1) in (4.4), we may be written as

$$\mathbf{M}_2 = -\sin[D_1 s + D_2]\mathbf{X}_2 + \cos[\mathcal{A}_1 s + \mathcal{A}_2]\mathbf{X}_3. \qquad (4.5)$$

Furthermore, from above equations we get

$$\mathbf{T} = -\sinh\wp\mathbf{X}_1 - \cosh\wp\cos[\mathcal{A}_1 s + \mathcal{A}_2]\mathbf{X}_2 - \cosh\wp\sin[\mathcal{A}_1 s + \mathcal{A}_2]\mathbf{X}_3. \qquad (4.6)$$

On the other hand, the first 2 components of Eq.(4.3) vanish, we get

$$
\begin{aligned}
\mathfrak{f}_1^{\mathcal{B}}k_1 + \mathfrak{f}_2^{\mathcal{B}}k_2 &= -1, \\
\left(\mathfrak{f}_1^{\mathcal{B}}\right)' &= 0.
\end{aligned}
$$

Considering second equation above system, we chose

$$\mathfrak{f}_1^{\mathcal{B}} = \mathfrak{p} = \text{constant} \neq 0. \qquad (4.7)$$

Then, it holds that

$$\mathfrak{f}_2^{\mathcal{B}} = \frac{-1-\mathfrak{p}k_1}{k_2}. \tag{4.8}$$

By means of obtained equations, we express

$$\mathfrak{F}_\gamma^{B}(s) = (\gamma + \mathfrak{p}\mathbf{M}_1 + \frac{-1-\mathfrak{p}k_1}{k_2}\mathbf{M}_2)(s), \tag{4.9}$$

where $\mathfrak{p}$ is a constant.

Considering equations (4.5) and (4.6) by the (4.9), we get (4.2). This completes the proof.

Corollary 4.2. *Let $\gamma : I \longrightarrow \mathbb{E}(1,1)$ is a non geodesic spacelike biharmonic $\mathcal{B}-$slant helix with timelike $\mathbf{M}_1$ and $\mathfrak{focal}_\gamma$ its focal curve in the Lorentzian group of rigid motions $\mathbb{E}(1,1)$. Then, the focal curvatures of $\mathfrak{focal}_\gamma$ are*

$$\mathfrak{f}_2^{\mathcal{B}} = \frac{-1-\mathfrak{f}_1^{\mathcal{B}}k_2\tanh\wp}{k_2} = \text{constant} \neq 0. \tag{4.10}$$

Proof. Suppose that γ is a non geodesic spacelike biharmonic $\mathcal{B}-$slant helix with timelike $\mathbf{M}_1$ and $\mathfrak{focal}_\gamma$ its focal curve. From (3.5) and (4.8) the focal curvature of $\mathfrak{focal}_\gamma$ takes the form (4.10). This completes the proof.

Then, we give the following theorem.

Theorem 4.3. *Let $\gamma : I \longrightarrow \mathbb{E}(1,1)$ is a non geodesic spacelike biharmonic $\mathcal{B}-$slant helix with timelike $\mathbf{M}_1$ and $\mathfrak{focal}_\gamma$ its focal curve in the Lorentzian group of rigid motions $\mathbb{E}(1,1)$. Then, the vector equation of $\mathfrak{focal}_\gamma^B(s)$ is*

$$\begin{aligned}
x_{\mathfrak{f}}^1(s) &= -\sinh\wp s + a_1 + \mathfrak{p}\cosh\wp, \\
x_{\mathfrak{f}}^2(s) &= \frac{1}{2}\exp[-\sinh\wp s + a_1 + \mathfrak{p}\cosh\wp] \\
&\quad [-\mathcal{A}e^{-\sinh\wp s + a_1}[(\sinh\wp - \mathcal{A}_1)\cos[\mathcal{A}_1 s + \mathcal{A}_2] \\
&\quad + (\sinh\wp + \mathcal{A}_1)\sin[\mathcal{A}_1 s + \mathcal{A}_2]] + a_2 e^{\sinh\wp s - a_1} \\
&\quad -\mathcal{A}e^{\sinh\wp s - a_1}[(\sinh\wp - \mathcal{A}_1)\cos[\mathcal{A}_1 s + \mathcal{A}_2] \\
&\quad + (\sinh\wp + \mathcal{A}_1)\sin[\mathcal{A}_1 s + \mathcal{A}_2]] + a_3 e^{-\sinh\wp s + a_1} \\
&\quad +\mathfrak{p}\sinh\wp\cos[\mathcal{A}_1 s + \mathcal{A}_2] + \frac{1+\mathfrak{p}k_1}{k_2}\sin[\mathcal{A}_1 s + \mathcal{A}_2]] \\
&\quad +\frac{1}{2}\exp[\sinh\wp s - a_1 - \mathfrak{p}\cosh\wp][-\mathcal{A}e^{-\sinh\wp s + a_1}[(\sinh\wp - \mathcal{A}_1) \\
&\quad \cos[\mathcal{A}_1 s + \mathcal{A}_2] + (\sinh\wp + \mathcal{A}_1)\sin[\mathcal{A}_1 s + \mathcal{A}_2]] + a_2 e^{\sinh\wp s - a_1} \\
&\quad +\mathcal{A}e^{\sinh\wp s - a_1}[(\sinh\wp - \mathcal{A}_1)\cos[\mathcal{A}_1 s + \mathcal{A}_2] \\
&\quad + (\sinh\wp + \mathcal{A}_1)\sin[\mathcal{A}_1 s + \mathcal{A}_2]] + a_3 e^{-\sinh\wp s + a_1} \\
&\quad +\mathfrak{p}\sinh\wp\sin[\mathcal{A}_1 s + \mathcal{A}_2] + \frac{-1-\mathfrak{p}k_1}{k_2}\cos[\mathcal{A}_1 s + \mathcal{A}_2]],
\end{aligned} \tag{4.11}$$

$$
\begin{aligned}
x_{\mathfrak{f}}^{3}(s) = {} & \frac{1}{2}\exp[-\sinh\wp s + a_1 + \mathfrak{p}\cosh\wp] \\
& [-\mathcal{A}e^{-\sinh\wp s + a_1}[(\sinh\wp - \mathcal{A}_1)\cos[\mathcal{A}_1 s + \mathcal{A}_2] \\
& + (\sinh\wp + \mathcal{A}_1)\sin[\mathcal{A}_1 s + \mathcal{A}_2]] + a_2 e^{\sinh\wp s - a_1} \\
& -\mathcal{A}e^{\sinh\wp s - a_1}[(\sinh\wp - \mathcal{A}_1)\cos[\mathcal{A}_1 s + \mathcal{A}_2] \\
& + (\sinh\wp + \mathcal{A}_1)\sin[\mathcal{A}_1 s + \mathcal{A}_2]] + a_3 e^{-\sinh\wp s + a_1} \\
& +\mathfrak{p}\sinh\wp\cos[\mathcal{A}_1 s + \mathcal{A}_2] + \frac{1 + \mathfrak{p}k_1}{k_2}\sin[\mathcal{A}_1 s + \mathcal{A}_2]] \\
& -\frac{1}{2}\exp[\sinh\wp s - a_1 - \mathfrak{p}\cosh\wp][-\mathcal{A}e^{-\sinh\wp s + a_1}[(\sinh\wp - \mathcal{A}_1) \\
& \cos[\mathcal{A}_1 s + \mathcal{A}_2] + (\sinh\wp + \mathcal{A}_1)\sin[\mathcal{A}_1 s + \mathcal{A}_2]] + a_2 e^{\sinh\wp s - a_1} \\
& +\mathcal{A}e^{\sinh\wp s - a_1}[(\sinh\wp - \mathcal{A}_1)\cos[\mathcal{A}_1 s + \mathcal{A}_2] \\
& + (\sinh\wp + \mathcal{A}_1)\sin[\mathcal{A}_1 s + \mathcal{A}_2]] + a_3 e^{-\sinh\wp s + a_1} \\
& +\mathfrak{p}\sinh\wp\sin[\mathcal{A}_1 s + \mathcal{A}_2] + \frac{-1 - \mathfrak{p}k_1}{k_2}\cos[\mathcal{A}_1 s + \mathcal{A}_2]],
\end{aligned}
$$

where $\mathfrak{p}$,$\mathcal{A}_1, \mathcal{A}_2$ *are constants of integration and*

$$
\mathcal{A} = \frac{\cosh\wp}{2\left(\mathcal{A}_1^2 + \sinh^2\wp\right)}.
$$

Proof. Assume that γ is a non geodesic spacelike biharmonic $\mathcal{B}-$slant helix and its focal curve is $\mathfrak{focal}_\gamma$. Substituting (2.1) to (4.2), we have (4.11) as desired. This concludes the proof of theorem.

5. Conclusions

Consider a curve in a space and suppose that the curve is sufficiently smooth so that the Bishop Frame adapted to it is defined; the curvatures k_1 and k_2 then provide a complete characterization of the curve. In this paper we have found parametric equations of $\mathcal{B}$-focal curves of spacelike biharmonic $\mathcal{B}$-slant helices according to Bishop frame in terms of Bishop curvatures in the Lorentzian group of rigid motions.

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Received 14 February 2012; communicated 30 May 2012; published online 4 September 2012