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ON GENERALIZED SEMISYMMETRIC RIEMANNIAN MANIFOLDS

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ABSTRACT. There are several generalizations of the concept of semi-symmetric Riemannian manifolds. In the present paper, we consider some special types of generalized semi-symmetric Riemannian manifolds with positive or negative defined curvature operator or sectional curvature. As applications of our theory we prove some propositions about semi-symmetric, Ricci-semi-symmetric and birecurrent Riemannian manifolds.

1. Introduction

An n-dimensional $(n \geq 3)$ Riemannian manifold (M,g) with the Levi-Civita connection ∇ is a *semisymmetric manifold* if its Riemannian curvature tensor R satisfies the following condition

$$R(X,Y) \circ R = 0. \tag{1}$$

Here $\circ$ denotes tensor derivations and X,Y are arbitrary vector fields on the manifold M.

Semisymmetric manifolds have been investigated by E. Cartan and are the generalization of symmetric Riemannian manifolds in the sense that the curvature tensor of any locally symmetric Riemannian manifolds satisfies (1). However, there exist examples of semisymmetric but not locally symmetric spaces. There are many papers where authors consider these manifolds (see, for example, Boeckx, Kowalski, and Vanhecke 1996; Deszcz and Hotloś 1998; Kowalczyk 2001; Lumiste 1996; Mikeš 1988, 1996; Mikeš, Hinterleitner, and Vanzurová 2009; Sinyukov 1979; Szabo 1982).

These manifolds have been locally classified by Szabo (1982). The author proves that for every Riemannian semi-symmetric space there exists an everywhere dense open subset U such that, around every point of U, the space is locally isometric to a space which is the direct product of symmetric spaces, 2-dimensional Riemannian spaces, elliptic, hyperbolic, Euclidean and Kählerian cones, and spaces foliated by $(n-2)$-dimensional Euclidean spaces (i.e. Riemannian spaces (M,g) with index of nullity $\nu(p) = n-2$ for any $p \in M$).

The theory of Riemannian semisymmetric manifolds has been presented in the monograph by Boeckx, Kowalski, and Vanhecke (1996).

In the present paper, we consider some special types of generalized semisymmetric Riemannian manifolds with a non-degenerate curvature operator or nonzero sectional curvature. As applications of our theory we show vanishing theorems for semisymmetric, Ricci-semisymmetric and birecurrent Riemannian manifolds.

2. T-semisymmetric Riemannian manifolds

There are several generalizations of the concept of semisymmetric Riemannian manifolds. For example (see Mikeš 1980, 1988; Mikeš, Hinterleitner, and Vanzurová 2009; Mikeš and Rachtlnek 2000, 2002), a Riemannian manifold (M,g) is called a *T-semi-symmetric manifold* if (M,g) admits a tensor field T of the type (p,q) such that the following condition

$$R(X,Y)\circ T = 0 \tag{2}$$

is true for the Riemannian curvature tensor R of (M,g), where $R(X,Y)$ acts as a derivation on T and for any vector fields X,Y on M.

We recall that at each point x of M, the quadratic form g_x induces a canonical isomorphism $T_xM \to T_x^*M$ and more generally, a canonical isomorphism between any spaces $T^{(q,p)}M$ and $T^{(r,h)}M$ for $r+h = p+q$. These isomorphisms correspond to lowering (resp. raising) indices in classical tensor notation. For example, we have the following $T_{i_1\cdots i_q k_1\cdots k_p} = g_{k_1 j_1}\cdots g_{k_p j_p} T^{j_1\cdots j_p}_{i_1\cdots i_q}$ by local expressions $T = (T^{j_1\cdots j_p}_{i_1\cdots i_q})$ and $g = (g_{ij})$ for the tensor T and the metric tensor g respectively. This fact and also that ∇ is natural connection of metric g (i.e. $\nabla g = 0$) allows us to consider only covariant tensors T. In addition, note that g defines a positive define quadratic form on the space $T_x^{(q,0)}M$ of covariant tensors of type $(q,0)$ and consequently on any subspace of $T_x^{(q,0)}M$ such as the space $\Lambda^q M$ of skew-symmetric and space $S^q M$ of symmetric covariant tensors of type $(q,0)$.

On the other hand, there is a well-known point-wise orthogonal decomposition (see Besse 1987, p. 45)

$$T^{(2,0)}M = T^*M \otimes T^*M = S^2M \oplus \Lambda^2 M,$$

from which we obtain the following orthogonal decomposition $T = ST + \Lambda T$ for any tensor field T of type $(2,0)$ where $ST = Pr_{S^2M}T$ and $\Lambda T = Pr_{\Lambda^2 M}T$ are orthogonal projections on the tensor spaces S^2M and $\Lambda^2 M$, respectively.

Next, we note that if any Riemannian manifold (M,g) is a T-semi-symmetric manifold with a tensor field T of the type $(2,0)$ then (M,g) is a ST and ΛT-semi-symmetric manifold. Moreover, the converse proposition is true.

3. Statement of the main results

3.1. Conditions for non-existence of T-semisymmetric manifold. In accordance with above results, we consider two special types of T-semisymmetric manifolds. Firstly, it is a T-semi-symmetric n-dimensional Riemannian manifold with a covariant skew-symmetric tensor field of type $(2,0)$, i.e. $T \in \Lambda^2 M$. In this case the following lemma is true.

Lemma 1. *Let (M,g) be an n-dimensional $(n \geq 3)$ Riemannian manifold with a positive (negative) define Riemannian curvature operator $\mathscr{R} : \Lambda^2 M \to \Lambda^2 M$, then (M,g) can not be a T-semisymmetric Riemannian manifold with a nonzero covariant skew-symmetric tensor field $T \in S^2M$.*

Lemma 1 implies in particular that for a Riemannian manifold of constant positive (negative) sectional curvature, or for a conformally flat Riemannian manifold with positive definite (negative definite) Ricci tensor, we have $T = 0$.

Secondary, we consider a T-semisymmetric manifold with a covariant symmetric tensor field of type $(2,0)$, i.e. $T \in S^2M$. In this case we will prove the following lemma.

Lemma 2. *Let (M,g) be a T-semisymmetric n-dimensional $(n \geq 2)$ Riemannian manifold with a covariant symmetric tensor field $T \in S^2M$ and $\{e_1,\ldots,e_n\}$ be an orthonormal basis at an arbitrary point $x \in M$ which consists of eigenvectors of T such that $T(e_i,e_j) = \lambda_i \delta_{ij}$ for the eigenvalue λ_i of T and the Kronecker delta δ_{ij}. If the sectional curvature $sec(e_i \wedge e_j) \neq 0$ then $\lambda_i = \lambda_j$. In particular, if (M,g) has nonzero sectional curvatures at each point of (M,g), then $T = \lambda \cdot g$ for some smooth scalar function λ.*

Next, from Lemma 1 and Lemma 2 we conclude that the following theorem holds.

Theorem 1. *Let (M,g) be a T-semisymmetric n-dimensional $(n \geq 3)$ Riemannian manifold with a tensor field T of the type $(2,0)$ and a positive (negative) Riemannian curvature operator $\mathscr{R}$: $\Lambda^2M \to \Lambda^2M$, then $T = \lambda \cdot g$ for some smooth scalar function λ.*

For the proof of this theorem, we note that from positive or negative definiteness of the Riemannian curvature operator $\mathscr{R}$ of (M,g) we can conclude that the sectional curvature of (M,g) is positive or negative definite, respectively. In dimension 3, the converse is true.

3.2. Condition for non-existence of Ricci-semisymmetric manifolds. In particular, if $T = Ric$ for the Ricci tensor Ric of (M,g) then (M,g) is called a *Ricci-semisymmetric manifold* which was defined (see Mikeš 1980; Mirzoyan 1992) by the following condition $R(X,Y) \circ Ric = 0$. Remark here, that though $R(X,Y) \circ R = 0$ implies $R(X,Y) \circ Ric = 0$, but the converse is not true, in general. The main structure theorem of the Mirzoyan (1992) paper says that a smooth Riemannian manifold is Ricci-semisymmetric if and only if it is locally a product of two-dimensional Riemannian manifolds, Einstein spaces and semi-Einstein spaces. Some results by Z.I. Szabó are used in the proof of this theorem. On the other hand, we have following corollary.

Corollary 1. *Let (M,g) be a Ricci-semisymmetric n-dimensional $(n \geq 3)$ Riemannian manifold and $\{e_1,\ldots,e_n\}$ be an orthonormal basis at an arbitrary point $x \in M$ which consists of the Ricci principal directions such that $Ric(e_i,e_j) = \lambda_i \delta_{ij}$ for the eigenvalue λ_i of Ric and the Kronecker delta δ_{ij}. If the sectional curvature $sec(e_i \wedge e_j) \neq 0$ then $\lambda_i = \lambda_j$. In particular, a Ricci-semi-symmetric Riemannian manifold (M,g) with nonzero sectional curvatures at each point of (M,g) is an Einstein manifold (i.e. $Ric = const \cdot g$) and, moreover, in dimension 3 is a Riemannian manifold of constant curvature.*

Geometrical interpretation of the Ricci principal directions has been present in the monograph by Eisenhart (1997).

4. T-birecurrent Riemannian manifold

Lichnerowicz has defined (see Mirzoyan 1992) a *recurrent of second order* (or briefly a *birecurrent*) Riemannian manifold (M,g) by the equation $\nabla^2 R = a \otimes R$, where R is the Riemannian curvature tensor of (M,g) and a is a covariant tensor field of order 2. He proved that if a birecurrent (M,g) is compact and the scalar curvature does nowhere vanish it is recurrent in the ordinary sense: $\nabla R = b \otimes R$ where b is a 1-form on (M,g). Wakakuwa in

Vrănceanu (1985) proved the same result for non-compact irreducible Riemannian manifold (M,g) with dimension $n \geq 3$.

There are several generalizations of the concept of birecurrent Riemannian manifold. For example (see Ewert-Krzemieniewski 1991; Nakagawa 1966), a tensor field T of type (p,q) is called a birecurrent tensor if it satisfies the equation

$$\nabla^2 T = a \otimes T. \tag{3}$$

In addition Mirzoyan (1992) has proved that for any covariant birecurrent tensor T its associated tensor field a is a symmetric tensor field. Therefore using the standard commutation formulas (Ricci identities)

$$(\nabla^2 T)(X,Y) - (\nabla^2 T)(Y,X) = R(X,Y) \circ T$$

we obtain from (3) the equation (2). Hence if a Riemannian manifold (M,g) admits a birecurrent tensor T then (M,g) is a T-semi-symmetric manifold. Therefore, from the Lemma 2 we obtain automatically the following corollary.

Corollary 2. *Let T be a birecurrent covariant symmetric tensor field of the type $(2,0)$ on an n-dimensional $(n \geq 2)$ Riemannian manifold (M,g) and $\{e_1,\dots,e_n\}$ be an orthonormal basis at an arbitrary point $x \in M$ which consists of eigenvectors of T such that $T(e_i,e_j) = \lambda_i \delta_{ij}$ for the eigenvalue λ_i of T and the Kronecker delta δ_{ij}. If the sectional curvature $sec(e_i \wedge e_j) \neq 0$ then $\lambda_i = \lambda_j$. In particular, if (M,g) has nonzero sectional curvatures at each point of (M,g), then $T = \lambda \cdot g$ for some smooth scalar function λ.*

On the other hand, from the Theorem 1 we obtain another corollary.

Corollary 3. *Let T be a birecurrent covariant tensor field of the type $(2,0)$ on an n-dimensional $(n \geq 3)$ Riemannian manifold (M,g) with positive (negative) Riemannian curvature operator $\mathscr{R} : \Lambda^2 M \to \Lambda^2 M$, then $T = \lambda \cdot g$ for some smooth scalar function λ.*

5. Proof of the results

Proof of Lemma 1. Let $\{x^1, x^2, \dots, x^n\}$ be a local coordinate system on the chart (U,ϕ). We denote by $g = (g_{ij})$, $R = (R^h_{ijk})$, $Ric = (R_{ij} := R^k_{ikj})$ and $T = (T_{i_1 i_2 \cdots i_q})$, respectively, the metric, the curvature, the Ricci tensors and a covariant tensor field T of type $(q,0)$ with respect to the local coordinate system $\{x^1, x^2, \dots, x^n\}$. Then the formula (2) becomes

$$T_{l i_2 \cdots i_q} R^l_{i_1 jk} + T_{i_1 l i_3 \cdots i_q} R^l_{i_2 jk} + \cdots + T_{i_1 i_2 \cdots i_{q-1} l} R^l_{i_q jk} = 0 \tag{4}$$

by local expression for the covariant tensor field T and the curvature tensor R.

Let T be a skew-symmetric tensor field $T \in \Lambda^2 M$ then from (4) we obtain

$$T_{lk} R^l_{ist} + T_{il} R^l_{kst} = 0. \tag{5}$$

If we transvect (5) with $g^{kt} T^{is}$ we get (see Yano and Bochner 1953, Sec. 4 of Chapter III; Yano 1970, p. 70)

$$F_2(T) := R_{ij} T^i_k T^{jk} - \frac{1}{2} R_{ijkl} T^{ij} T^{kl} = 0. \tag{6}$$

where F_2: $\Lambda^2 M \otimes \Lambda^2 M \to \mathbb{R}$ is the well known (see Besse 1987, p. 53; Petersen 2006, p. 211) quadratic form from the classic formula of Bochner-Weitzenböck $g(\Delta T, T) = g(\nabla^* \nabla T, T) + F_q(T,T)$ for $q = 2$. Here Δ is the *Hodge Laplacian* and $\nabla^* \nabla$ is the *rough*

(Bochner) Laplacian. Moreover, (6) is equals to the following equations (Petersen 2006, pp. 220, 221)

$$\sum_{\alpha} r_{\alpha} \| [\theta_{\alpha}, T] \|^2 = 0 \tag{7}$$

where r_α are the eigenvalues and θ_α the duals of the eigenvectors of the standard symmetric Riemannian curvature operator $\mathscr{R}$: $\Lambda^2 M \to \Lambda^2 M$ given by the identity (Sinyukov 1979, p. 36)

$$g(\mathscr{R}(X \wedge Y), V \wedge W) = R(X \wedge Y, V \wedge W) = R(X, Y, V, W)$$

for arbitrary vector fields X, Y, V and W. Furthermore, we say that (M, g) has *positive* (*negative*) *Riemannian curvature operator* if all eigenvalues of $\mathscr{R}$ are positive (negative).

Next we suppose that (M, g) has the positive (negative) Riemannian curvature operator, then $r_\alpha > 0$ ($r_\alpha < 0$) and the formula (7) implies that $[\theta_\alpha, T] = 0$ for all α. In this case T must vanish (see Petersen 2006, p. 221). This completes the proof of Lemma 1.

Proof of Lemma 2. The formula (2) becomes

$$T_{lk} R^l_{ist} + T_{il} R^l_{kst} = 0. \tag{8}$$

by local expression for the symmetric tensor field T of type $(2, 0)$ and the curvature tensor R. Let $\{e_1, \ldots, e_n\}$ be an orthogonal basis in $T_x M$ at an arbitrary point $x \in M$ such that $T(e_i, e_j) = \lambda_i \, \delta_{ij}$ for the Kronecker delta δ_{ij}. Then we can rewrite the identities (8) as

$$(\lambda_i - \lambda_j) \cdot sec(e_i \wedge e_j) = 0 \tag{9}$$

where $sec(e_i \wedge e_j) = g(\mathscr{R}(e_i \wedge e_j), e_i \wedge e_j)$ is called a *sectional curvature of two-plane* (see Petersen 2006, p. 36). If we suppose that $sec(e_i \wedge e_j) \neq 0$ then from (9) we obtain $\lambda_i = \lambda_j$. In particular, if sectional curvatures of all two-planes $\pi \subset T_x M$ at each point $x \in M$ are not zero then from (9) we obtain $\lambda_1 = \cdots = \lambda_n = \lambda$. Hence Lemma 2 is proved.

Proof of Corollary 1. The Corollary 1 follows from Lemma 2 automatically. If $n = 3$, then (M, g) is Einstein if and only if it has constant sectional curvature too (see Besse 1987, p. 44).

Remark. We note that if $n = 2$, then at each point x in M, we have $Ric = \frac{1}{2} s \cdot g$ for the scalar curvature s, and the condition $R(X, Y) \circ Ric = 0$ is always satisfied.

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