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HARMONIC WAVELET SOLUTION OF POISSON'S PROBLEM WITH A LOCALIZED SOURCE

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ABSTRACT. A method, based on a multiscale (wavelet) decomposition of the solution is proposed for the analysis of the Poisson problem. The solution is approximated by a finite series expansion of harmonic wavelets and is based on the computation of the connection coefficients. It is shown, how a sourceless Poisson's problem, solved with the Daubechies wavelets, can also be solved in presence of a localized source in the harmonic wavelet basis.

1. Introduction

Wavelets are some special functions $\psi_k^n(x)$ [1] which depend on two parameters: n is the scale (refinement, compression, or dilation) parameter and k is the localization (translation) parameter. These functions fulfill the fundamental axioms of multiresolution analysis so that by a suitable choice of the scale and translation parameter one is able to easily and quickly approximate any function (even tabular) with decay to infinity. Due to this multiscale approach the approximation by wavelets is the best one for at least two reasons: a minimum set of coefficients to represent the phenomenon (compression) and a direct projection into a given scale, thus giving a direct physical interpretation of the phenomenon. In each scale the wavelet coefficients and, in particular, the detail coefficients β_k^n describe "local" oscillations. Therefore wavelets seem to be the more expedient tool for studying these problems which are localized (in time or in frequency) and/or have some discontinuities.

Among the many families of wavelets, harmonic wavelets [2–7] are the most expedient tool for studying processes which are localized in Fourier domain, as it happens, e.g. in dispersive wave propagation [8–10].

They are a family of basis functions, which are analytically defined, infinitely differentiable, and band limited [7–9, 11] thus enabling us to easily study, their differentiable properties. Harmonic wavelets are complex function, but there are some advantages with them when we have to compute the corresponding connection coefficients (also called refinable integrals [12–15]). In fact, the connection coefficients of harmonic wavelets can be explicitly (and finitely) computed at any order [2], while in the available literature were given only for a few order of derivatives [6], or like in the case of the Daubechies wavelets, by some approximate formulas [11, 16–18]. In [16–18] the authors focused on the compactly supported wavelets and they based their derivation of the connection coefficients on the dilation equation.

In general the computation of the connection coefficients seems to be a difficult task for at least two reasons: first, the most known (and used) wavelets are not functionally defined by a finite formula and, second, even in presence of a simple formula defining the wavelet family there not exists a simple expression for the corresponding connection coefficients [11], except for the harmonic wavelet basis [2].

By using the Frobenius method and the connection coefficients the Poisson problem can be transformed into an infinite dimensional algebraic system, which can be solved by fixing a finite scale of approximation.

2. Harmonic Wavelets

The harmonic scaling function [7] is the complex function

$$\boxed{\varphi(x) \stackrel{\text{def}}{=} \frac{e^{2\pi i x} - 1}{2\pi i x}} \, . \tag{1}$$

Since

$$e^{\pi i n} = \begin{cases} 1, & n = 2k \, , \qquad k \in \mathbb{Z} \\ -1, & n = 2k+1 \, , \qquad k \in \mathbb{Z} \end{cases} \tag{2}$$

it is, in particular, $\varphi(n) = 0 \, , \; n \in \mathbb{Z}$.

The Fourier transform of the harmonic scaling function (1)

$$\widehat{\varphi}(\omega) = \widehat{\varphi(x)} \stackrel{\text{def}}{=} \frac{1}{2\pi} \int_{-\infty}^{\infty} \varphi(x) e^{-i\omega x} \mathrm{d}x$$

is a function with a compact support in the frequency domain (i.e. with a bounded, or banded, frequency)

$$\boxed{\widehat{\varphi}(\omega) = \frac{1}{2\pi} \chi(2\pi + \omega)} \, , \tag{3}$$

$\chi(\omega)$ being the characteristic function defined as

$$\chi(\omega) \stackrel{\text{def}}{=} \begin{cases} 1 & , \quad 2\pi \le \omega \le 4\pi \, , \\ 0 & , \quad \text{elsewhere} \, . \end{cases} \tag{4}$$

The scaling function is a function very well localized in the frequency domain, despite its slow decay in the space variable.

Starting from the scaling function it is possible to define a filter and to derive the corresponding wavelet function (see e.g. [7])

$$\boxed{\widehat{\psi}(\omega) = \frac{1}{2\pi} \chi(\omega)} \, .$$

By the inverse Fourier transform

$$\int_{-\infty}^{\infty} \frac{1}{2\pi}\chi(\omega)e^{i\omega x}\mathrm{d}\omega = \frac{1}{2\pi}\int_{2\pi}^{4\pi} e^{i\omega x}\mathrm{d}\omega \ ,$$

we get the harmonic wavelet [2–4, 6, 7]

$$\boxed{\psi(x) \stackrel{\text{def}}{=} \frac{e^{4\pi i x} - e^{2\pi i x}}{2\pi i x} = e^{2\pi i x}\varphi(x)} \ . \tag{5}$$

The dilated and translated instances of (1),(5) are (see e.g. [8])

$$\boxed{\left\{\begin{array}{lcl} \varphi^n_k(x) & \stackrel{\text{def}}{=} & 2^{n/2}\dfrac{e^{2\pi i\,(2^n x-k)}-1}{2\pi i(2^n x-k)} \\ \\ \psi^n_k(x) & \stackrel{\text{def}}{=} & 2^{n/2}\dfrac{e^{4\pi i(2^n x-k)}-e^{2\pi i(2^n x-k)}}{2\pi i(2^n x-k)} \end{array}\right.} \ , \tag{6}$$

with $n, k \in \mathbb{Z}$.

It is well known that if $\widehat{f}(\omega)$ is the Fourier transform of $f(x)$ then

$$f\widehat{(ax \pm b)} = \frac{1}{a}e^{\pm i\omega b/a}\widehat{f}(\omega/a) \ , \tag{7}$$

so that we can easily obtain the dilated and translated instances of the Fourier transform of (6), (see e.g. [2]):

$$\boxed{\left\{\begin{array}{lcl} \widehat{\varphi}^n_k(\omega) & = & \dfrac{2^{-n/2}}{2\pi}e^{-i\omega k/2^n}\chi(2\pi+\omega/2^n) \\ \\ \widehat{\psi}^n_k(\omega) & = & \dfrac{2^{-n/2}}{2\pi}e^{-i\omega k/2^n}\chi(\omega/2^n) \end{array}\right.} \ . \tag{8}$$

From the definition of the inner (or scalar or dot) product, of two functions $f(x), g(x)$,

$$\langle f, g\rangle \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} f(x)\,\overline{g(x)}\mathrm{d}x \quad , \tag{9}$$

and taking into account the Parseval equality

$$\langle f, g\rangle = 2\pi\left\langle \widehat{f}, \widehat{g}\right\rangle = 2\pi\int_{-\infty}^{\infty} \widehat{f}(\omega)\,\overline{\widehat{g}(\omega)}\mathrm{d}\omega \quad , \tag{10}$$

it can be shown that

$$\left\{\begin{array}{l} \langle\varphi^n_k(x), \varphi^m_h(x)\rangle = \delta^{nm}\delta_{kh} \ , \ \langle\overline{\varphi}^n_k(x), \overline{\varphi}^m_h(x)\rangle = \delta^{nm}\delta_{kh} \ , \\ \langle\varphi^n_k(x), \overline{\varphi}^m_h(x)\rangle = 0 \ , \ \left\langle\psi^n_k(x), \overline{\psi}^m_h(x)\right\rangle = 0 \ , \\ \langle\psi^n_k(x), \psi^m_h(x)\rangle = \delta^{nm}\delta_{hk} \ , \left\langle\overline{\psi}^n_k(x), \overline{\psi}^m_h(x)\right\rangle = \delta^{nm}\delta_{kh} \ , \\ \left\langle\varphi^n_k(x), \overline{\psi}^m_h(x)\right\rangle = 0 \ , \ \langle\overline{\varphi}^n_k(x), \psi^m_h(x)\rangle = 0 \ , \end{array}\right. \tag{11}$$

where δ^{nm} (δ_{hk}) is the Kronecker symbol. Some simple technical calculations also show that both the harmonic scaling function and the harmonic wavelets fulfill the multiresolution conditions

$$\int_{-\infty}^{\infty} \varphi(x)\mathrm{d}x = 1 \quad , \int_{-\infty}^{\infty} \psi_k^n(x)\mathrm{d}x = 0 \; .$$

3. Multiscale representation of functions

Let us consider the class of (real or complex) functions $f(x)$, such that the following integrals

$$\left\{ \begin{array}{lcl} \alpha_k &=& \langle f(x), \varphi_k^0(x)\rangle = \displaystyle\int_{-\infty}^{\infty} f(x)\overline{\varphi}_k^0(x)\mathrm{d}x \\ \alpha_k^* &=& \langle f(x), \overline{\varphi}_k^0(x)\rangle = \displaystyle\int_{-\infty}^{\infty} f(x)\varphi_k^0(x)\mathrm{d}x \\ \beta_k^n &=& \langle f(x), \psi_k^n(x)\rangle = \displaystyle\int_{-\infty}^{\infty} f(x)\overline{\psi}_k^n(x)\mathrm{d}x \\ \beta{^*}_k^n &=& \langle f(x), \overline{\psi}_k^n(x)\rangle = \displaystyle\int_{-\infty}^{\infty} f(x)\psi_k^n(x)\mathrm{d}x \end{array} \right. \tag{12}$$

exist and are finite.

From the orthogonality conditions (11), the function $f(x)$ can be reconstructed in terms of harmonic wavelets as (see e.g. [7])

$$\begin{array}{rcl} f(x) &=& \displaystyle\sum_{k=-\infty}^{\infty} \alpha_k\varphi_k^0(x) + \sum_{n=0}^{\infty}\sum_{k=-\infty}^{\infty} \beta_k^n\psi_k^n(x) \\ &+& \displaystyle\sum_{k=-\infty}^{\infty} \alpha_k^*\overline{\varphi}_k^0(x) + \sum_{n=0}^{\infty}\sum_{k=-\infty}^{\infty} \beta{^*}_k^n\overline{\psi}_k^n(x) \end{array} \tag{13}$$

which involve the basis and (for a complex function) its conjugate basis. For a real function $f(x) = \overline{f}(x)$ it is

$$\alpha_k^*(x) = \alpha_k(x) \quad , \qquad \beta{^*}_k^n(x) = \beta_k^n(x) \; .$$

According to (8),(9), in the Fourier domain, it is

$$\left\{ \begin{array}{lcl} \alpha_k &=& 2\pi\langle \widehat{f(x)}, \widehat{\varphi_k^0(x)}\rangle = \displaystyle\int_{-\infty}^{\infty} \widehat{f}(\omega)\overline{\widehat{\varphi}_k^0(\omega)}\mathrm{d}\omega = \int_0^{2\pi} \widehat{f}(\omega)e^{i\omega k}\mathrm{d}\omega \\ \alpha_k^* &=& 2\pi\langle \widehat{f(x)}, \widehat{\overline{\varphi}_k^0(x)}\rangle = \ldots = \displaystyle\int_0^{2\pi} \widehat{f}(\omega)e^{-i\omega k}\mathrm{d}\omega \\ \beta_k^n &=& 2\pi\langle \widehat{f(x)}, \widehat{\psi_k^n(x)}\rangle = \ldots = 2^{-n/2}\displaystyle\int_{2^{n+1}\pi}^{2^{n+2}\pi} \widehat{f}(\omega)e^{i\omega k/2^n}\mathrm{d}\omega \\ \beta{^*}_k^n &=& \langle \widehat{f(x)}, \widehat{\overline{\psi}_k^n(x)}\rangle = \ldots = 2^{-n/2}\displaystyle\int_{2^{n+1}\pi}^{2^{n+2}\pi} \widehat{f}(\omega)e^{-i\omega k/2^n}\mathrm{d}\omega \; , \end{array} \right. \tag{14}$$

being $\widehat{\overline{f(x)}} = \overline{\widehat{f}(-\omega)}$.

The approximation of (13) up to the scale $N \leq \infty$ and to a finite translation $M \leq \infty$ it is

$$
\begin{aligned}
f(x) \cong \Pi^{N,M} f(x) &= \sum_{k=0}^{M} \alpha_k \varphi_k^0(x) + \sum_{n=0}^{N} \sum_{k=-M}^{M} \beta_k^n \psi_k^n(x) \\
&+ \sum_{k=0}^{M} \alpha_k^* \overline{\varphi}_k^0(x) + \sum_{n=0}^{N} \sum_{k=-M}^{M} \beta^{*n}_{\ k} \overline{\psi}_k^n(x) \; .
\end{aligned} \tag{15}
$$

Since wavelets are localized functions, they can capture with a few terms the main features of functions defined in a short range interval. It should be noticed, however, that, for a non trivial function $f(x) \neq 0$ the corresponding wavelet coefficients (14), in general, vanish when either

$$\widehat{f}(\omega) = 0 \; , \; \forall k \qquad \text{or} \qquad \widehat{f}(\omega) = Cnst. \; , \; k \neq 0 \; .$$

In particular, it can be seen that the wavelet coefficients (14) trivially vanish when

$$
\left\{
\begin{array}{lcll}
f(x) &=& \sin(2k\pi x) & , \quad k \in \mathbb{Z} \\
 & & & \\
f(x) &=& \cos(2k\pi x) & , \quad k \in \mathbb{Z} \quad (k \neq 0) \; .
\end{array}
\right. \tag{16}
$$

For instance from $(12)_1$, for the functions $\cos(2k\pi x)$ it is

$$
\begin{aligned}
\alpha_k &= \int_{-\infty}^{\infty} \cos(2k\pi x) \overline{\varphi}_k^0(x) \mathrm{d}x = \frac{1}{2} \int_{-\infty}^{\infty} \left(e^{-2ih\pi x} + e^{2ih\pi x} \right) \overline{\varphi}_k^0(x) \mathrm{d}x \\
&= \frac{1}{2} \left[\int_{-\infty}^{\infty} e^{-2ih\pi x} \overline{\varphi}_k^0(x) \mathrm{d}x + \int_{-\infty}^{\infty} e^{2ih\pi x} \overline{\varphi}_k^0(x) \mathrm{d}x \right]
\end{aligned}
$$

from where by the change of variable $2\pi x = \xi$ and taking into account (7) there follows

$$\alpha_k = \frac{1}{2} \left[\widehat{\overline{\varphi}_k^0(x)} + \widehat{\varphi}_k^0(x) \right]_{x=2\pi h} \; .$$

According to (8) it is

$$\widehat{\varphi}_k^0(2\pi h) = \frac{1}{2} e^{-i2\pi hk} \chi(2\pi + 2\pi h) \overset{(2)}{=} \frac{1}{2} \chi(2\pi + 2\pi h)$$

and, because of (4)

$$\chi(2\pi + 2\pi h) = 1 \quad , \quad 0 < h < 1$$

so that

$$\widehat{\varphi}_k^0(2\pi h) = 0 \quad , \quad \forall h \neq 0 \quad .$$

There follows that $\alpha_h = 0$, as well as the remaining wavelet coefficients of $\cos(2k\pi x)$ (with $k \in \mathbb{Z}$ and $k \neq 0$). Analogously, it can be shown that all wavelet coefficients of $\sin(2k\pi x)$ ($\forall k \in \mathbb{Z}$) are zero.

As a consequence, a given function $f(x)$, for which the coefficients (12) are defined, admits the same wavelet coefficients of

$$f(x) + \sum_{h=0}^{\infty} \left[A_h \sin(2h\pi x) + B_h \cos(2h\pi x) \right] - B_0 \; , \tag{17}$$

or (by a simple transformation) in terms of complex exponentials,

$$f(x) - C_0 + \sum_{h=-\infty}^{\infty} C_h e^{2ih\pi x} \ , \tag{18}$$

so that the wavelet coefficients of $f(x)$ are defined unless an additional trigonometric series (the coefficients A_h , B_h , C_h being constant) as in (17).

4. Connection Coefficients for Harmonic Wavelets

Equation (9) describes the basic structure of the functional space defined on the basis functions (6). The investigation of the differential properties of the basis leads us to the computation of their derivatives. Moreover, in the application of the Frobenius method, it is assumed that a certain unknown function (with its derivatives) can be expressed in terms of the basis (and its derivatives). For this reason, as a first step, we need to compute the derivatives of the wavelet basis (see e.g. [2,3,6,10–12,16,18,19]), through the connection coefficients [2,3,13,16–18].

The differential properties of wavelets are based on the knowledge of the following inner products:

$$\left\{ \begin{array}{l} \lambda^{(\ell)}{}_{kh} \stackrel{\text{def}}{=} \left\langle \dfrac{\mathrm{d}^\ell}{\mathrm{d}x^\ell} \varphi^0_k(x) \, , \varphi^0_h(x) \right\rangle \quad , \quad \Lambda^{(\ell)}{}^{m}_{kh} \stackrel{\text{def}}{=} \left\langle \dfrac{\mathrm{d}^\ell}{\mathrm{d}x^\ell} \varphi^0_k(x) \, , \psi^m_h(x) \right\rangle \\ \\ \gamma^{(\ell)}{}^{nm}_{kh} \stackrel{\text{def}}{=} \left\langle \dfrac{\mathrm{d}^\ell}{\mathrm{d}x^\ell} \psi^n_k(x) \, , \psi^m_h(x) \right\rangle \quad , \quad \zeta^{(\ell)}{}^{n}_{kh} \stackrel{\text{def}}{=} \left\langle \dfrac{\mathrm{d}^\ell}{\mathrm{d}x^\ell} \psi^n_k(x) \, , \varphi^0_h(x) \right\rangle \ , \end{array} \right. \tag{19}$$

and the corresponding inner products with conjugate functions:

$$\left\{ \begin{array}{l} \overline{\lambda}^{(\ell)}{}_{kh} \stackrel{\text{def}}{=} \left\langle \dfrac{\mathrm{d}^\ell}{\mathrm{d}x^\ell} \overline{\varphi}^0_k(x) \, , \overline{\varphi}^0_h(x) \right\rangle \quad , \quad \overline{\Lambda}^{(\ell)}{}^{m}_{kh} \stackrel{\text{def}}{=} \left\langle \dfrac{\mathrm{d}^\ell}{\mathrm{d}x^\ell} \overline{\varphi}^0_k(x) \, , \overline{\psi}^m_h(x) \right\rangle \\ \\ \overline{\gamma}^{(\ell)}{}^{nm}_{kh} \stackrel{\text{def}}{=} \left\langle \dfrac{\mathrm{d}^\ell}{\mathrm{d}x^\ell} \overline{\psi}^n_k(x) \, , \overline{\psi}^m_h(x) \right\rangle \quad , \quad \overline{\zeta}^{(\ell)}{}^{n}_{kh} \stackrel{\text{def}}{=} \left\langle \dfrac{\mathrm{d}^\ell}{\mathrm{d}x^\ell} \overline{\psi}^n_k(x) \, , \overline{\varphi}^0_h(x) \right\rangle \ . \end{array} \right. \tag{20}$$

The coefficients (19),(20) can be easily computed in the Fourier domain(see for a proof [2]) (for the first and second order connection coefficients of *periodic* harmonic wavelets see also [6,8,9]) and it can be shown that

Theorem 1. *The only non trivial connection coefficients* $\lambda^{(\ell)}{}_{kh}$, $\gamma^{(\ell)}{}^{nm}_{kh}$ *are given by*

$$
(21)\quad \left\{
\begin{array}{rcl}
\lambda^{(\ell)}{}_{kh} &=& \dfrac{1}{2\pi}\left[i^{\ell}(1-|\mu(h-k)|)\dfrac{(2\pi)^{\ell+1}}{\ell+1} \right.\\
&-& \left. i\mu(h-k)\displaystyle\sum_{s=1}^{\ell}(-1)^{[1+\mu(h-k)](2\ell-s+1)/2}\,\dfrac{\ell!\,(2i\,\pi)^{\ell-s+1}}{(\ell-s+1)!|h-k|^s}\right]\\
\gamma^{(\ell)}{}^{nm}_{kh} &=& \Big\{(1-|\mu(h-k)|)\dfrac{i^{\ell}\pi^{\ell+1}2^{[1+(n+m)/2](\ell+1)}(2^{\ell+1}-1)}{\ell+1}\\
&-& i\,\mu(h-k)\displaystyle\sum_{s=1}^{\ell+1}(-1)^{[1+\mu(h-k)](2\ell-s+1)/2}\,\dfrac{\ell!\,(i\,\pi)^{\ell-s+1}}{(\ell-s+1)!|h-k|^s}\\
&\times& [2^{[1+(n+m)/2](\ell+1)-2s}(2^{\ell+1}-2^s)]\Big\}\dfrac{2^{-(n+m)/2}}{2\pi}\delta^{nm}
\end{array}
\right.
$$

for $\ell \geq 1$, *and* $\lambda^{(\ell)}{}_{kh} = \delta_{kh}$ *when* $\ell = 0$.

Theorem 2. *The connection coefficients* $(20)_1$ $\overline{\lambda}^{(\ell)}{}_{kh}$ *are given by*

$$(22)\qquad \overline{\lambda}^{(\ell)}{}_{kh} = \lambda^{(\ell)}{}_{hk}$$

for $\ell \geq 1$, *and* $\overline{\lambda}^{(\ell)}{}_{kh} = \delta_{kh}$ *when* $\ell = 0$.

It should be noticed, from their definition, that the coefficients (21) are not symmetric in the sense that $\lambda^{(\ell)}{}_{kh} \neq \lambda^{(\ell)}{}_{hk}$. Analogously, it is

$$(23)\qquad \overline{\gamma}^{(\ell)}{}^{nm}_{kh} = \gamma^{(\ell)}{}^{nm}_{hk}$$

for $\ell \geq 1$, and $\overline{\gamma}^{(\ell)}{}^{nm}_{kh} = \delta_{kh}\delta^{nm}$ when $\ell = 0$.

The mixed connection coefficients $(19)_{2,4}$ are trivially vanishing:

$$(24)\qquad \Lambda^{(\ell)}{}^{m}_{kh} = 0 \quad , \quad \zeta^{(\ell)}{}^{n}_{kh} = 0 \quad , \quad \overline{\Lambda}^{(\ell)}{}^{m}_{kh} = 0 \quad , \quad \overline{\zeta}^{(\ell)}{}^{n}_{kh} = 0\,.$$

There follows that the fundamental connection coefficients are (21),(22),(23).

These coefficients enable us to characterize any order derivative of the basis. In fact, according to (19) it is

$$(25)\qquad \frac{\mathrm{d}^{\ell}\varphi^{0}_{k}(x)}{\mathrm{d}x^{\ell}} = \sum_{m=0}^{\infty}\sum_{h=-\infty}^{\infty}\lambda^{(\ell)}{}^{m}_{kh}\,\varphi^{m}_{h}(x)\,.$$

A good approximation is obtained by fixing a finite value of M

$$(26)\qquad \boxed{\frac{\mathrm{d}^{\ell}\varphi^{0}_{k}(x)}{\mathrm{d}x^{\ell}} \cong \sum_{h=0}^{M}\lambda^{(\ell)}{}_{kh}\varphi^{0}_{h}(x)}\,.$$

Analogously we have,

$$(27)\qquad \frac{\mathrm{d}^{\ell}\psi^{n}_{k}(x)}{\mathrm{d}x^{\ell}} = \sum_{m=0}^{\infty}\sum_{k,h=-\infty}^{\infty}\gamma^{(\ell)}{}^{nm}_{kh}\,\psi^{m}_{h}(x)$$

and a good approximation, which depends both on the dilation N and translational parameter M, it is

$$\boxed{\frac{d^\ell \psi_k^n(x)}{dx^\ell} \cong \sum_{m=0}^{N} \sum_{h=-M}^{M} \gamma^{(\ell)}{}_{kh}^{nm}\, \psi_h^m(x)} \tag{28}$$

with $n \leq N$. Of course, since the harmonic wavelets are oscillating functions the approximation improves by increasing the translational parameters, however the approximation can be considered sufficiently good for a very low value of M.

For the corresponding conjugate functions we have

$$\frac{d^\ell \overline{\varphi}_k^0(x)}{dx^\ell} = \sum_{m=0}^{\infty} \sum_{h=-\infty}^{\infty} -\lambda^{(\ell)}{}_{kh}^{m}\, \overline{\varphi}_h^m(x) \quad , \quad \frac{d^\ell \overline{\varphi}_k^0(x)}{dx^\ell} \cong \sum_{h=0}^{M} \lambda^{(\ell)}{}_{kh} \overline{\varphi}_h^0(x)$$

and

$$\frac{d^\ell \overline{\psi}_k^n(x)}{dx^\ell} = \sum_{m=0}^{\infty} \sum_{k,h=-\infty}^{\infty} -\gamma^{(\ell)}{}_{kh}^{nm}\, \overline{\psi}_h^m(x)\ , \ \frac{d^\ell \overline{\psi}_k^n(x)}{dx^\ell} \cong \sum_{m=0}^{N} \sum_{h=-M}^{M} \overline{\gamma}^{(\ell)}{}_{kh}^{nm}\, \psi_h^m(x)\ .$$

Thanks to Eqs. (25),(27), and to the orthonormality conditions (11) the remaining mixed coefficients are trivially vanishing.

5. Poisson's problem with source

Let us consider the Poisson problem

$$\begin{cases} \dfrac{d^2 u}{dx^2} + \mu^2 u = f(x) \\ \\ u(x_0) = a \quad , \quad u(x_1) = b\ , \end{cases} \tag{29}$$

with x_0 , x_1 , a , b given numbers. By using the Frobenius method, the solution is searched in the form of harmonic wavelet series up to a finite scale of approximation (15), so that the projection into a finite dimensional wavelet space is:

$$\begin{cases} \left\langle \Pi^{N,M} \dfrac{d^2}{dx^2} u(x), \varphi_k^0 \right\rangle + \left\langle \Pi^{N,M} u(x), \varphi_k^0 \right\rangle & = \left\langle \Pi^{N,M} f(x), \varphi_k^0 \right\rangle\ , \\ \left\langle \Pi^{N,M} \dfrac{d^2}{dx^2} u(x), \overline{\varphi}_k^0 \right\rangle + \left\langle \Pi^{N,M} u(x), \overline{\varphi}_k^0 \right\rangle & = \left\langle \Pi^{N,M} f(x), \overline{\varphi}_k^0 \right\rangle\ , \\ \left\langle \Pi^{N,M} \dfrac{d^2}{dx^2} u(x), \psi_k^n(x) \right\rangle + \left\langle \Pi^{N,M} u(x), \psi_k^n(x) \right\rangle & = \left\langle \Pi^{N,M} u(x), \psi_k^n(x) \right\rangle\ , \\ \left\langle \Pi^{N,M} \dfrac{d^2}{dx^2} u(x), \overline{\psi}_k^n(x) \right\rangle + \left\langle \Pi^{N,M} u(x), \overline{\psi}_k^n(x) \right\rangle & = \left\langle \Pi^{N,M} u(x), \overline{\psi}_k^n(x) \right\rangle\ , \\ & \quad (k = 0, 1, \ldots, M\ ; n = 0, \ldots, N)\ , \end{cases}$$

where we assume that $u(x)$ is in the form:

$$
\begin{aligned}
u(x) \quad &\cong \quad \Pi^{N,M} u(x) = \left[\sum_{k=0}^{M} \alpha_k \varphi_k^0(x) + \sum_{n=0}^{N} \sum_{k=-M}^{M} \beta_k^n \psi_k^n(x) \right] \\
&+ \quad \left[\sum_{k=0}^{M} \alpha_k^* \overline{\varphi}_k^0(x) + \sum_{n=0}^{N} \sum_{k=-M}^{M} \beta^{*n}_{\ k} \overline{\psi}_k^n(x) \right] \\
&- \quad B_0 + \sum_{k=0}^{\infty} A_k \sin 2k\pi x + B_k \cos 2k\pi x \ .
\end{aligned} \tag{30}
$$

The second derivative, on account of (22),(23),(26),(28), is

$$
\begin{aligned}
\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} \cong &\left[\sum_{k=0}^{M} \alpha_k \sum_{h=0}^{M} \lambda^{(\ell)}{}_{kh} \varphi_h^0(x) + \sum_{n=0}^{N} \sum_{k=-M}^{M} \beta_k^n \sum_{m=0}^{N} \sum_{h=-M}^{M} \gamma^{(\ell)}{}_{kh}^{nm} \psi_h^m(x) \right] \\
+ &\left[\sum_{k=0}^{M} \alpha_k^* \sum_{h=0}^{M} \lambda^{(\ell)}{}_{hk} \overline{\varphi}_h^0(x) + \sum_{n=0}^{N} \sum_{k=-M}^{M} \beta^{*n}_{\ k} \sum_{m=0}^{N} \sum_{h=-M}^{M} \gamma^{(\ell)}{}_{hk}^{nm} \overline{\psi}_h^m(x) \right] \\
&- \sum_{k=0}^{\infty} 4k^2\pi^2 \left(A_k \sin 2k\pi x + B_k \cos 2k\pi x \right) \ .
\end{aligned} \tag{31}
$$

On the given function $f(x)$ it is assumed that it can be represented in the form (30), that is

$$
\begin{aligned}
f(x) \quad &\cong \quad \left[\sum_{k=0}^{M} a_k \varphi_k^0(x) + \sum_{n=0}^{N} \sum_{k=-M}^{M} b_k^n \psi_k^n(x) \right] \\
&+ \quad \left[\sum_{k=0}^{M} a_k^* \overline{\varphi}_k^0(x) + \sum_{n=0}^{N} \sum_{k=-M}^{M} b^{*n}_{\ k} \overline{\psi}_k^n(x) \right] \\
&- \quad d_0 + \sum_{k=0}^{\infty} c_k \sin 2k\pi x + d_k \cos 2k\pi x \ .
\end{aligned} \tag{32}
$$

By putting (30),(31),(32) into (29) and by the scalar product with the basis functions φ_i^0 ,$\overline{\varphi}_i^0$, $\psi_i^r(x)$, $\overline{\psi}_i^r(x)$ we obtain the algebraic system:

(33)
$$\left\{\begin{array}{rcll}
\displaystyle\sum_{k=0}^{M}\alpha_k\lambda^{(2)}{}_{ki}+\mu^2\alpha_i &=& a_i & ,\quad (i=0,\dots,M)\\
\displaystyle\sum_{k=0}^{M}\alpha_k^*\lambda^{(2)}{}_{ik}+\mu^2\alpha_i^* &=& a_i^* & ,\quad (i=0,\dots,M)\\
\displaystyle\sum_{n=0}^{N}\sum_{k=-M}^{M}\beta_k^n\gamma^{(2)}{}_{ki}^{nr}+\mu^2\beta_i^r &=& b_i^r & ,\quad (r=0,\dots,N; i=-M,\dots,M)\\
\displaystyle\sum_{n=0}^{N}\sum_{k=-M}^{M}\beta^*{}_k^n\gamma^{(2)}{}_{ki}^{nr}+\mu^2\beta^*{}_i^r &=& b_i^r & ,\quad (r=0,\dots,N; i=-M,\dots,M)\\
-4k^2\pi^2A_k+\mu^2A_k &=& c_k & ,\quad (k=0,\dots,\infty)\\
-4k^2\pi^2B_k+\mu^2B_k &=& d_k & ,\quad (k=1,\dots,\infty)\,.
\end{array}\right.$$

The boundary conditions $(29)_{2,3}$ give two additional equations:

$$(34)\qquad \left\{\begin{array}{l}
\displaystyle\sum_{k=0}^{M}\alpha_k\varphi_k^0(x_0)+\sum_{n=0}^{N}\sum_{k=-M}^{M}\beta_k^n\psi_k^n(x_0)+\sum_{k=0}^{M}\alpha_k^*\overline{\varphi}_k^0(x_0)\\
\displaystyle+\sum_{n=0}^{N}\sum_{k=-M}^{M}\beta^*{}_k^n\overline{\psi}_k^n(x_0)-B_0+\sum_{k=0}^{\infty}A_k\sin 2k\pi x_0+B_k\cos 2k\pi x_0=a\,,\\
\displaystyle\sum_{k=0}^{M}\alpha_k\varphi_k^0(x_1)+\sum_{n=0}^{N}\sum_{k=-M}^{M}\beta_k^n\psi_k^n(x_1)+\sum_{k=0}^{M}\alpha_k^*\overline{\varphi}_k^0(x_1)\\
\displaystyle+\sum_{n=0}^{N}\sum_{k=-M}^{M}\beta^*{}_k^n\overline{\psi}_k^n(x_1)-B_0+\sum_{k=0}^{\infty}A_k\sin 2k\pi x_1+B_k\cos 2k\pi x_1=b\,.
\end{array}\right.$$

5.1. Localized Source. As an example, let us consider the following problem:

$$(35)\qquad \left\{\begin{array}{l}
\dfrac{\mathrm{d}^2u}{\mathrm{d}x^2}+\mu^2u=\dfrac{1}{2}\left(\psi_0^0(x)+\overline{\psi}_0^0(x)\right)\\[2ex]
u(-1)=0\quad,\quad u(1)=0\,.
\end{array}\right.$$

A good approximation of the second derivative of the functions $\psi^0_0(x)$, $\overline{\psi}^0_0(x)$, can be obtained at the lowest level $N = 0,\ M = 0$ so that system (33) becomes

$$
(36)\qquad \left\{
\begin{array}{rcl}
\alpha_0 \lambda^{(2)}{}_{00} + \mu^2 \alpha_0 &=& 0\ , \\[2ex]
\alpha^*_0 \lambda^{(2)}{}_{00} + \mu^2 \alpha^*_0 &=& 0\ , \\[2ex]
\beta^0_0 \gamma^{(2)}{}^{00}_{00} + \mu^2 \beta^0_0 &=& \dfrac{1}{2}\ , \\[2ex]
\beta^*{}^0_0 \gamma^{(2)}{}^{00}_{00} + \mu^2 \beta^*{}^0_0 &=& \dfrac{1}{2}\ , \\[2ex]
-4k^2\pi^2 A_k + \mu^2 A_k &=& 0 \quad , \quad (k = 0, \dots, \infty) \\[2ex]
-4k^2\pi^2 B_k + \mu^2 B_k &=& 0 \quad , \quad (k = 1, \dots, \infty)\ .
\end{array}
\right.
$$

When $\mu \neq \pm\sqrt{-\gamma^{(2)}{}^{00}_{00}}$, i.e. $\mu \neq \pm\pi\sqrt{28/3}$, the above algebraic system is solved, for the wavelet coefficients, by

$$
\beta^0_0 = \frac{1}{2}\frac{1}{\gamma^{(2)}{}^{00}_{00} + \mu^2}\ ,\ \beta^*{}^0_0 = \frac{1}{2}\frac{1}{\gamma^{(2)}{}^{00}_{00} + \mu^2}\ ,\ \alpha_k = 0\ , \alpha^*{}_k = 0
$$

and from the last 2 equations

$$
(37)\qquad \left\{
\begin{array}{l}
k = 0 \Rightarrow A_0 = 0\ , \\[2ex]
k \neq 0 \left\{ \begin{array}{l} k = k^* = \mu/(2\pi) \Rightarrow A_k = A_{k^*}\ ,\ B_k = B_{k^*} \\ k \neq \mu/(2\pi) \Rightarrow A_k = 0\ ,\ B_k = 0\ ,\ \forall\, k \neq k^*\ . \end{array} \right.
\end{array}
\right.
$$

However, since k^* must be an integer the nontrivial term A_{k^*} exists only when $\mu = 2\pi h$ with $h \in \mathbb{N}$.

The boundary conditions $(35)_{2,3}$, together with the equations (34), give two additional equations

$$
(38)\qquad
\begin{array}{l}
\alpha_0 \varphi^0_0(-1) + \beta^0_0 \psi^0_0(-1) + \alpha^*_0 \overline{\varphi}^0_0(-1) + \beta^*{}^0_0 \overline{\psi}^0_0(-1) - B_0 + \displaystyle\sum_{k=0}^{\infty} B_k = 0\ , \\[3ex]
\alpha_0 \varphi^0_0(1) + \beta^0_0 \psi^0_0(1) + \alpha^*_0 \overline{\varphi}^0_0(1) + \beta^*{}^0_0 \overline{\psi}^0_0(1) - B_0 + \displaystyle\sum_{k=0}^{\infty} B_k = 0
\end{array}
$$

Taking into account the definitions (6) it is

$$
(39)\qquad \left\{
\begin{array}{l}
\varphi^0_0(-1) = \overline{\varphi}^0_0(-1) = 0 \quad , \quad \varphi^0_0(1) = \overline{\varphi}^0_0(1) = 0 \\[2ex]
\psi^0_0(-1) = \overline{\psi}^0_0(-1) = 0 \quad , \quad \psi^0_0(1) = \overline{\psi}^0_0(1) = 0\ ,
\end{array}
\right.
$$

so that the boundary conditions (38) are fulfilled by

$$
B_k = 0\ ,\ (k = 0, \dots, \infty)\ .
$$

There follows that the only nontrivial contribution to the trigonometric series is $A_{k^*} \sin k^* x = \sin \mu x$

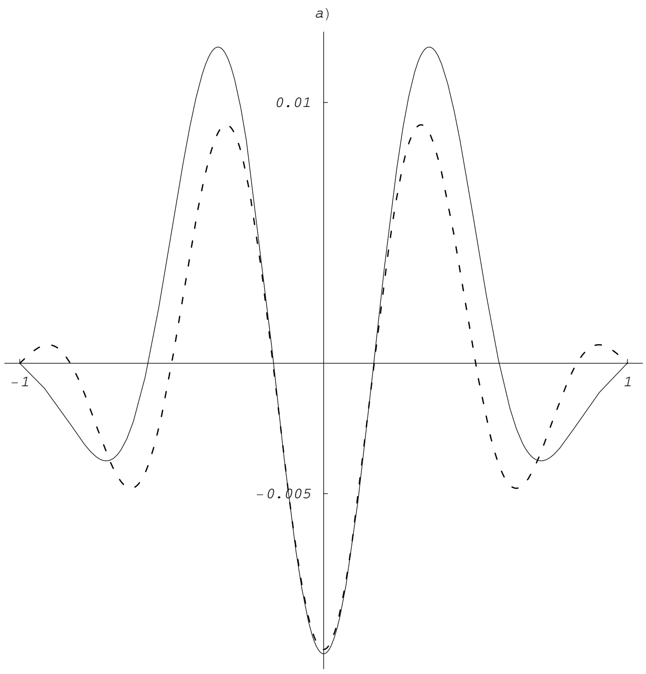

FIGURE 1. *Numerical and wavelet (dashed line) solution, at the lowest scale approximation $M = N = 0$ of the problem* (35)*, with $\mu \neq 2\pi h$.*

If we put these coefficients in the Eq. (30) we obtain the wavelet solution (Fig. 1)

$$u(x) = \begin{cases} A_{k*} \sin 2\pi h x + \dfrac{3}{6 - 56\pi^2} \left(\psi_0^0(x) + \overline{\psi}_0^0(x) \right) \ , \ (\mu = 2\pi h, \ h \in \mathbb{N}) \\ \dfrac{3}{6 - 56\pi^2} \left(\psi_0^0(x) + \overline{\psi}_0^0(x) \right) \ , \ (\mu \neq 2\pi h) \ , \end{cases}$$

which, as can be easily seen, fulfills also the boundary conditions $(29)_{2,3}$.

The main advantage of this method is that, in presence of a localized function $f(x)$, the wavelet method is sufficiently accurate nearby the origin and simple, comparing with any other numerical algorithm, which contain complicate exponential integral functions like

$$Ei(z) = \int_{-z}^{\infty} \frac{e^{-t}}{t} \mathrm{d}t \ .$$

However, it should be noticed that the approximation depends on the approximation of the derivative of the wavelet basis (26),(28). For instance we have a good approximation of the derivative

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2} \psi_0^0(x) \cong \gamma^{(2)}{}^{00}_{00} \psi_0^0(x)$$

with only one function and this implies a simple system (33) to be solved. Instead the derivative,

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\varphi^0_0(x) \cong \sum_{k=-4}^{4} \lambda^{(2)}{}_{0h}\varphi^0_h(x)$$

it is sufficiently good with at least 9 functions $\varphi^0_h(x)$, and this obviously increases the complexity of the solution of (33).

Conclusion

Complex harmonic wavelets were used for computing the approximate wavelet solution of a Poisson problem with a localized source. Wavelet methods, based on the Daubechies wavelets, were know for the solution of sourceless Poisson's problem. In this paper it has been shown that, for a localized source, the harmonic wavelets can give a very good approximation of the solution by using only a few instances of the wavelet family thus showing the importance of the localization property in this kind of problems.

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