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A GENERALIZATION OF MINIMAL DISTRIBUTIONS

CORNELIA LIVIA BEJAN [a] AND MASSIMILIANO FERRARA *[b]

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ABSTRACT. We connect here various harmonic geometric objects (including a generalization of minimal distributions that we define here), we relate them with the theory of foliations and we construct some examples.

1. Introduction

Recently, an interesting study on the interrelations between foliations and several geometric structures on manifolds was published in [1, 2]. Since the structure we deal with is the almost product one, for our purpose we refer to [3].

By taking into account the increasing number of harmonic and minimal properties introduced on geometric objects, we study here non - holonomic manifolds, we generalize the notion of minimal distribution and apply it in the case of foliations.

All manifolds throughout will be assumed paracompact and of class C^∞ and all maps will be taken of C^∞ - class. Einstein convention on repeated indices is applied.

2. Connections on manifolds

Let (M, P, G) be a semi-Riemannian almost product manifold, that is a manifold M endowed with a non-degenerate symmetric $(0, 2)$ - tensor field G and a $(1, 1)$ - tensor field $P \neq \pm Id$, $P^2 = Id$, which is compatible with G:

$$G\left(PX, PY\right) = G\left(X, Y\right), \ \forall X, Y \in \Gamma\left(TM\right). \tag{1.1}$$

Corresponding to the eigenvalues 1 and -1 of P, we denote respectively with V and H the eigendistributions and with v and h the projectors from TM onto them:

$$TM = V \perp H = v\left(TM\right) \perp h\left(TM\right), \tag{1.2}$$

where the decomposition into a direct sum is orthogonal.

Conversely any non - degenerate distribution V on a semi - Riemannian manifold (M, G) defines an almost product structure P whose eigendistributions corresponding to the eigenvalues 1 and -1 are V and its orthogonal complement H, respectively.

As an example, we recall from [4] that a connection Γ on a manifold M is a $(1,1)$ - tensor field Γ on the tangent bundle $\pi : TM \to M$ such that $t\Gamma = -\Gamma t = t$, where t denotes the canonical almost tangent structure on TM, defined by $t : T(TM) \to T(TM)$, $t\left(y^i \frac{\partial}{\partial x^i} + z^k \frac{\partial}{\partial y^k}\right) = y^i \frac{\partial}{\partial y^i}$ in local coordinates. Obviously, $t^2 = 0$. It is easily seen that Γ defines an almost product structure Γ on TM such that at each point $u \in TM$, the eigenspace corresponding to the eigenvalue -1 of Γ of is $\ker \pi_{*,u}$. Particular connections Γ are the linear ones.

The almost product structure Γ is expressed in local coordinates by:

$$\Gamma\left(y^i \frac{\partial}{\partial x^i} + z^k \frac{\partial}{\partial y^k}\right) = y^i \frac{\partial}{\partial x^i} - \left(2y^i \omega_i^k + z^k\right) \frac{\partial}{\partial y^k}, \tag{1.3}$$

where ω_i^k are the connection components of Γ.

However, (TM, Γ, G^c) is not a semi - Riemannian almost product manifold, since the almost product structure Γ is not compatible with the semi - Riemannian structure G^c given by the complete lift of the metric G on M.

3. Harmonicity

We recall that a map $\varphi : (F, g) \to (M, G)$ between semi - Riemannian manifolds is harmonic, provided it satisfies the Euler - Lagrange equations, that is its tension field $\tau(\varphi)$ expressed in local coordinates $\left(x^i\right)$ on M by:

$$\tau(\varphi) = g^{ij}\left[\nabla^G_{\partial/\partial x^i} d\varphi\left(\frac{\partial}{\partial x^j}\right) - d\varphi\left(\nabla^g_{\partial/\partial x^i} \frac{\partial}{\partial x^j}\right)\right] = 0, \tag{2.1}$$

where ∇^g and ∇^G are respectively the Levi - Civita connections of g and G.

In particular, when the map φ is an isometric immersion, then φ is harmonic if and only if the immersed submanifold $\varphi(F)$ is minimal in M, that is its mean curvature vector field (defined as the trace of its second fundamental form) vanishes identically.

Also, a foliation is called a minimal or a harmonic foliation if its mean curvature vector field vanishes identically.

This notion of minimality extends to a distribution V (integrable or not) of a semi-Riemannian manifold (M, g) as follows: V is minimal if

$$\mathrm{trace}\{(X, Y) \in V \times V \to \nabla^g_X Y\} \in V.$$

From [5], a connection Γ on the semi - Riemannian manifold (M, G) is harmonic if it is a harmonic almost product structure on (TM, G^c) where G^c is the complete lift of the metric G.

Equivalently, in local coordinates a connection Γ on (M, G) is harmonic if and only if

$$G^{ij}\left({}^G\Gamma^k_{ij} - \frac{\partial}{\partial y^i}\omega^k_j\right)\frac{\partial}{\partial y^k} = 0, \tag{2.2}$$

where ω_j^k are the connection components of Γ and ${}^G\Gamma$ is the Levi - Civita connection of G.

As it is pointed out in [6], p. 21, a map $\varphi : (F, g) \to \left(M, \nabla^M\right)$ from a semi - Riemannian manifold to an affine manifold is harmonic, provided it satisfies the Euler - Lagrange equations $\tau(\varphi) = 0$, where the tension field $\tau(\varphi)$ is expressed in local coordinates by:

$$\tau(\varphi) = g^{ij}\left[\nabla^M_{\partial/\partial x^i} d\varphi\left(\frac{\partial}{\partial x^j}\right) - d\varphi\left(\nabla^g_{\partial/\partial x^i}\frac{\partial}{\partial x^j}\right)\right], \tag{2.3}$$

where ∇^g is the Levi - Civita connection of g. Obviously, this definition generalizes the notion of harmonic maps between Riemannian manifolds. In particular, if F is a leaf of a distribution V on M, then we call (F, g) harmonic immersed in (M, ∇^M).

If (F, g) is a submanifold, harmonic immersed in (M, ∇^M), that is the inclusion map $\varphi : (F, g) \to (M, \nabla^M)$ is harmonic and moreover we assume g on F is the restriction of a semi - Riemannian structure (say g) inherited from M, whose Levi - Civita connection ∇^g on M coincides with ∇^M, then F is called minimal submanifold of M. If V is an integrable distribution, which defines a minimal foliation, then in this particular case, $\tau(\varphi) = 0$ is equivalent with:

$$h\nabla^g_{e_i} e_i = 0, \tag{2.4}$$

for any orthonormal frame $\{e_i\}$ of V, where h is the projector from TM to H (the orthogonal complement of V with respect to g).

It is natural to extend the notion of minimal distribution when ∇^g and ∇^M do not necessarily coincide.

Definition 2.1. Let (M, ∇^M) be an affine manifold endowed with a semi - Riemannian structure g and a non - degenerate distribution V. We say that V is ∇^M - minimal with respect to g if

$$\nabla^M_{e_i} e_i - v\nabla^g_{e_i} e_i = 0, \tag{2.5}$$

for any orthonormal frame $\{e_i\}$ of V, where v is the projector of TM on V and ∇^g is the Levi - Civita connection of g.

In particular, if ∇^M is the Levi - Civita connection of a semi - Riemannian structure G on M, then V is G - minimal with respect to g if (2.5) is satisfied. Obviously, when g and G coincide, we get the classical notion of minimality.

Remark that Definition 2.1 does not depend on the orthonormal frame $\{e_i\}$ chosen in (2.5).

4. φ - Morphisms

Let $\varphi : (F, g) \to (M, G)$ be a map between Riemannian manifolds. From vector bundles category theory $\Phi : TF \to TM$ is a φ - morphism, provided the fibre restriction $\Phi_p : T_pF \to T_{\varphi(p)}M$ is linear at any $p \in F$. Thus Φ determines a 1 - form $\Phi \in A^1(\varphi^{-1}TM)$ with values in the pull - back bundle $\varphi^{-1}TM$. A linear connection D of a vector bundle $E \to F$ defines the exterior derivative d and the coderivative δ of any bundle valued - form $\omega \in A^1(E)$ respectively by

$$d\omega(X, Y) = D\omega(X, Y) - D\omega(Y, X),$$

where $D\omega(X, Y) = (D_X\omega)Y, \forall X, Y \in \Gamma(TF)$ and $\delta\omega = -div\omega = -\mathrm{trace}D\omega$.

We say that ω is closed (resp. coclosed) if $d\omega = 0$ (resp. $\delta\omega = 0$). Obviously, d and δ depend on D.

Let $E=\varphi^{-1}TM$ and $\nabla^{\varphi^{-1}TM}$ be respectively the pull - back bundle and the unique (see [6]) linear connection on $\varphi^{-1}TM$, which satisfies

$$\nabla_X^{\varphi^{-1}TM}\varphi^* U=\varphi^*\nabla_{d\varphi X}^M U,\ \forall X\in\Gamma\left(TF\right),\ U\in\Gamma\left(TM\right), \tag{3.1}$$

where $\varphi^*U\in\Gamma\left(\varphi^{-1}TM\right),\left(\varphi^*U\right)_p=U_{\varphi(p)},\ \forall p\in F.$

In local coordinates, we have:

$$\Phi\left(x,y\right)=\left(\varphi^\alpha\left(x\right),\Phi_i^\alpha\left(x\right)y^i\right) \tag{3.2}$$

and

$$\delta\Phi=g^{ij}\left(\nabla^{\varphi^{-1}TM}\Phi\right)\frac{\partial}{\partial x^j}=g^{ij}\left(\Phi_j^\alpha\frac{\partial\varphi^\beta}{\partial x^i}{}^G\Gamma_{\alpha\beta}^\gamma-{}^g\Gamma_{ij}^k\Phi_k^\alpha\right)\frac{\partial}{\partial u^\gamma}, \tag{3.3}$$

where ${}^G\Gamma$ and ${}^g\Gamma$ are respectively the Levi - Civita connections of G and g.

This relation gives the coderivative of the φ - morphism Φ, [7].

Proposition 3.1. A linear connection ∇^M on a semi - Riemannian manifold (M,G) is harmonic if and only if the identity $I:TM\to TM$ is coclosed.

Proof. As the connection ∇^M is linear, then its connection components ω_j^k satisfy:

$$\omega_j^k\left(x^i,y^i\right)=y^{iG}\Gamma_{ij}^k. \tag{3.4}$$

From (3.3) we complete the proof.

Remark. From (2.2) and (3.4), it follows that a linear connection ∇^M is harmonic on (M,G) if and only if

$$\mathrm{trace}\{(X,Y)\in\Gamma\left(TM\right)\times\Gamma\left(TM\right)\to\left(\nabla_X^G Y-\nabla_X^M Y\right)\}=0,$$

where ∇^G is the Levi-Civita connection of G.

Therefore, we may say that ∇^M is harmonic on a distribution V on (M,G) if

$$\mathrm{trace}\{(X,Y)\in V\times V\to\left(\nabla_X^G Y-\nabla_X^M Y\right)\}=0. \tag{3.5}$$

As a conclusion at the end of this section, we obtain

Theorem 3.1. *Let $\left(M,\nabla^M\right)$ be an affine manifold endowed with a semi - Riemannian structure G and a non - degenerate distribution V. Then any two of the following assertions imply the third one:*

(a) The distribution V is ∇^M - minimal with respect to G;

(b) The distribution V on (M,G) is minimal (in the classical sense);

(c) The connection ∇^M is harmonic on V.

Proof. As the connection ∇^M is linear, then its connection components ω_j^k satisfy (3.4) and we may use the above remark.

Let $\{e_i\}$ denote on orthonormal frame of V with respect to G. By applying (3.5), it follows that ∇^M is harmonic if and only if

$$\nabla_{e_i}^G e_i-\nabla_{e_i}e_i=0, \tag{3.6}$$

where ∇^G is the Levi - Civita connection of G.

From (2.4) and (1.2) we have

$$\begin{aligned}\nabla_{e_i}^M e_i-v\nabla_{e_i}^G e_i&=\left(\nabla_{e_i}^M e_i-\nabla_{e_i}^G e_i\right)+\left(\nabla_{e_i}^G e_i-v\nabla_{e_i}^G e_i\right)\\&=\left(\nabla_{e_i}^M e_i-\nabla_{e_i}^G e_i\right)+h\nabla_{e_i}^G e_i\end{aligned}$$

From (2.5), (3.6) and (2.4) we complete the proof.

As a consequence, we obtain:

Corollary 3.2. *Let $\left(M, \nabla^M\right)$ be an affine manifold endowed with a semi - Riemannian structure G and an integrable non - degenerate distribution V.*

Then for any integral submanifold F of V, any two of the following assertions imply the third one.

(a) The inclusion map $i : (F, G) \to \left(M, \nabla^M\right)$ is harmonic;

(b) The submanifold (F, G) of (M, G) is minimal;

(c) On the manifold (F, G), the connection ∇ defined as the restriction of ∇^M to the submanifold F is harmonic.

5. Examples

As examples, on a semi - Riemannian almost product manifold (M, P, G), beside the Levi - Civita connection ∇^G one has the Schouten - Van Kampen connection ∇^0 and the Vrănceanu connection ∇^*:

(4.1)
$$\nabla^0_X Y = v\nabla^G_X vY + h\nabla^G_X hY,$$

$$\nabla^*_X Y = v\nabla^G_{vX} vY + v\left[hX, vY\right] + h\nabla^G_{hX} hY + h\left[vX, hY\right], \forall X, Y \in \Gamma\left(TM\right).$$

Several properties and relations among them are given in [1].

Proposition 4.1. Let (M, P, G) be a semi - Riemannian almost product manifold and ∇^G its Levi - Civita connection. Then the Schouten - Van Kampen connection ∇^0 (resp. the Vrănceanu connection) is harmonic if and only if both the eigendistributions V and H corresponding to the eigenvalues 1 and -1 of P are minimal (in the classical sense).

Proof. Let $\{e_i\}$, $\{e_a\}$ be an orthonormal frame of V and H, respectively. We denote $\{e_\alpha\} = \{e_i\} \cup \{e_a\}$ an orthonormal frame on M.

From (4. 1) we have:

$$\nabla^G_{e_\alpha} e_\alpha - \nabla^0_{e_\alpha} e_\alpha = \nabla^G_{e_i} e_i + \nabla^G_{e_a} e_a - \nabla^0_{e_i} e_i - \nabla^0_{e_a} e_a = \nabla^G_{e_i} e_i + \nabla^G_{e_a} e_a - v\nabla^0_{e_i} e_i - h\nabla^0_{e_a} e_a$$
$$= h\nabla^G_{e_i} e_i + v\nabla^G_{e_a} e_a.$$

From (2.4), the statement is valid for ∇^0 and analogous for ∇^*, which complete the proof.

From (4.1) and Definition 2.1, it follows:

Proposition 4.2. Let (M, P, G) be a semi - Riemannian almost product manifold and ∇^G its Levi - Civita connection. Then both the distributions V and H corresponding to the eigenvalues 1 and -1 of P are ∇^0-minimal as well as ∇^*-minimal, where ∇^0 and ∇^* are Schouten - Van Kampen and Vrănceanu connections, respectively.

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[a] Cornelia Livia Bejan
Mathematical Seminary,
"Al. I. Cuza" University,
Blvd. Carol I, 11
700506 Iaşi, Romania

[b] Massimiliano Ferrara
Department "V. Pareto",
University of Messina,
Via Dei Verdi, 75
98100 Messina, Italy
* **E-mail**: mferrara@unime.it

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