

## IMPULSIVE WENDROFF'S TYPE INEQUALITIES AND THEIR APPLICATIONS

ANNA MARIA PICCIRILLO,<sup>a\*</sup> MICHELE CIARLETTA,<sup>b</sup> AND SERGIY D. BORYSENKO<sup>c</sup>

**ABSTRACT.** Integro-sum inequalities for functions of two independent variables are investigated. New analogies of Wendroff's type inequalities for discontinuous functions are obtained. As applications, some hyperbolic equations with impulse influence are considered. New conditions of boundedness of solutions are obtained.

### 1. Introduction

For differential models described by systems of ordinary differential equations with impulsive influence the method of "integro-sum" inequalities plays an important role to investigate the properties of existence, uniqueness, boundness, and stability of solutions for perturbed systems [1-19]. The generalization of this method and its applications to estimating the solutions of impulsive hyperbolic differential equations have not been sufficiently investigated before.

In the present article we found new analogies of Wendroff's inequality for discontinuous functions with finite jumps on some curves  $\Gamma_j \subset \mathbb{R}_+^2$  and discontinuities of Lipschitz and non-Lipschitz type. New conditions of boundedness for solutions of impulsive nonlinear hyperbolic equations are obtained.

A first generalization of Wendroff's result for discontinuous functions was obtained in [19]; the most important results in the theory of integral inequalities for discontinuous functions of several variables were described in [8] (see also [7]).

Our paper is devoted to a generalization of results obtained in [1], [7], [10-12] and is based on new analogies of the Wendroff-type inequality.

We consider some set  $D^* \subset \mathbb{R}^2$ :

$$D^* = D \setminus \Gamma, D = \bigcup_j D_j, j = 1, 2, \dots; \Gamma = \bigcup_j \Gamma_j, \Gamma_j = \{(x, y) : \varphi_j(x, y) = 0, \\ j = 1, 2, \dots\}, \Gamma_k \cap \Gamma_{k+1} = \emptyset, k = 1, 2, \dots,$$

where

$\varphi_j(x, y)$  are real-valued continuously differentiable functions such that  $\text{grad } \varphi_j(x, y) > 0$ , for all  $j = 1, 2, \dots$ ;

$$D_1 = \{(x, y) : x \geq 0, y \geq 0, \varphi_1(x, y) < 0\};$$

$D_k = \{(x, y) : x \geq 0, y \geq 0, \varphi_{k-1}(x, y) > 0, \varphi_k(x, y) < 0, \forall k > 2, k \in \mathbb{N}\};$   
 $G_p = \{(u, v) : (x, y) \in D_p, 0 \leq u \leq x, 0 \leq v \leq y, p \in \mathbb{N}\};$   
 $\mu_{\varphi_n}$  is the Lebesgue-Stiljtes measure concentrated on the curves  $\{\Gamma_n\}$ .

Let us consider a real-valued nonnegative, discontinuous, nondecreasing function  $u(x, y)$  in  $D^*$ , which has finite jumps on the curves  $\{\Gamma_j\}$ .

Let  $g(x, y)$  be a positive nondecreasing continuous function in  $\mathbb{R}_+^2$ , and let us assume that  $u(x, y)$  satisfies the following integro-sum inequality in  $D^*$ :

$$u(x, y) \leq g(x, y) + \int \int_{G_n} \Phi(\tau, s, u(\tau, s)) d\tau ds + \sum_{j=1}^{n-1} \int_{\Gamma_j \cap G_n} W(x, y, u(x, y)) d\mu_{\varphi_j} \quad (1)$$

where  $\Phi, W$ , defined in  $D^*$ , are nonnegative, nondecreasing functions for 3-d argument, with fixed first and second arguments.

Let the integro-sum equation of the following form be set

$$\sigma(x, y) = g(x, y) + \int \int_{G_n} \Phi(\tau, s, \sigma(\tau, s)) d\tau ds + \sum_{j=1}^{n-1} \int_{\Gamma_j \cap G_n} W(x, y, \sigma(x, y)) d\mu_{\varphi_j} \quad (2)$$

where  $\sigma(x, y)$  is a nonnegative discontinuous function, which has finite jumps on the curves  $\{\Gamma_j\}$ . Then, for  $x \geq 0, y \geq 0$  the estimate

$$u(x, y) \leq \sigma_g(x, y) \quad (3)$$

is valid, where  $\sigma_g(x, y)$  is some solution of the integro-sum equation (2), continuous in domain  $D^*$ , and  $u(x, y)$  satisfies the inequality (1). The results of §2 are based on the estimate (3).

## 2. Previous results

**2.1. Lipschitz-type discontinuities.** Let us assume that  $W = \beta_j(x, y)u(x, y)$ . Then the next statement takes place:

**Proposition 1.** In the following let us assume

$$F_i(x, y) \stackrel{\text{def}}{=} \int_0^x \int_0^y f_i(\tau, s) d\tau ds, \quad i = 1, 2, 3, 5 \quad (4)$$

$$\prod \beta_j(x, y) \stackrel{\text{def}}{=} \prod_{j=1}^{n-1} \left( 1 + \int_{\Gamma_j \cap G_n} \beta_j(x, y) d\mu_{\varphi_j} \right). \quad (5)$$

**A:** *The estimate*

$$u(x, y) \leq g(x, y) \exp[F_1(x, y)] \prod (\beta_j(x, y)), \quad (6)$$

if  $\Phi = f_1(x, y)u(x, y)$ ,  $f_1 \geq 0 : f_1 \in C(\mathbb{R}_+^2)$ ,  $\beta_j \in C(\mathbb{R}_+^2)$ ,  $\forall j = 1, 2, \dots$ , holds for all  $(x, y) \in D^*$ .

**B:** *The estimate*

$$u(x, y) \leq g(x, y) \exp[F_2(x, y)] \prod (\beta_j(x, y)) \cdot \left(1 + \int_0^x \int_0^y f_3(\tau, s) g^{-1}(\tau, s) \exp[-F_2(\tau, s)] d\tau ds\right) \quad (7)$$

holds, if  $\Phi = f_2(x, y)u(x, y) + f_3(x, y)$ ,  $f_i \geq 0$ ,  $f_i \in C(\mathbb{R}_+^2)$ ,  $i = 2, 3$ .

**C:** *The following assertions hold:*

**i:** *The estimate*

$$u(x, y) \leq g(x, y) \prod (\beta_j(x, y)) \left[1 + (1 - \alpha) \int_0^x \int_0^y f_4(\tau, s) \cdot g^{\alpha-1}(\tau, s) d\tau ds\right]^{1/(1-\alpha)} \quad (8)$$

is true, if  $0 < \alpha < 1$ ,  $\Phi = f_4(x, y)u^\alpha(x, y)$ .

**ii:** *The estimate*

$$u(x, y) \leq g(x, y) \prod (\beta_j(x, y)) \left[1 + (1 - \alpha) \prod^{\alpha-1} (\beta_j(x, y)) \cdot \int_0^x \int_0^y f_4(\tau, s) g^{\alpha-1}(\tau, s) d\tau ds\right]^{-1/(1-\alpha)}$$

holds for  $\alpha > 1$  and for an arbitrary  $(x, y) \in D^*$  such that

$$\int_0^x \int_0^y f_4(\tau, s) g^{\alpha-1}(\tau, s) d\tau ds < \left[(\alpha - 1) \prod^{\alpha-1} (\beta_j(x, y))\right]^{-1}.$$

**D:** *The following assertions hold:*

**i:** *The estimate*

$$u(x, y) \leq g(x, y) \prod (\beta_j(x, y)) \exp[F_5(x, y)] \cdot \left[1 + (1 - \alpha) \int_0^x \int_0^y f_6(\tau, s) g^{\alpha-1}(\tau, s) \cdot \exp[(\alpha - 1)F_5(\tau, s)] d\tau ds\right]^{1/(1-\alpha)} \quad (9)$$

holds for  $0 < \alpha < 1$ ,  $\Phi = f_5(x, y)u(x, y) + f_6(x, y)u^\alpha(x, y)$ .

**ii:** *The estimate*

$$u(x, y) \leq g(x, y) \prod (\beta_j(x, y)) \exp[F_5(x, y)] \cdot \left[1 + (1 - \alpha) \prod^{\alpha-1} (\beta_j(x, y)) \int_0^x \int_0^y f_6(\tau, s) g^{\alpha-1}(\tau, s) \cdot \exp[(\alpha - 1)F_5(\tau, s)] d\tau ds\right]^{-1/(\alpha-1)} \quad (10)$$

is true for  $\alpha > 1$  and arbitrary  $(x, y) \in D^*$  such that

$$\int_0^x \int_0^y f_6 g^{\alpha-1} \exp[(\alpha - 1)F_5] d\tau ds < \left[(\alpha - 1) \prod^{\alpha-1} (\beta_j(x, y))\right]^{-1}.$$

**Remark.** The proofs of Propositions 1-5 are similar to the proof of proposition 6, so we omitted these.

**Proposition 2.** *Let us suppose that function  $u(x_1, x_2)$  satisfies following integro-sum inequality in  $D^*$  :*

$$u(x_1, x_2) \leq q(x_1, x_2) + g(x_1, x_2) \iint_{G_n} \bar{\psi}(\tau, s) W[u(\tau, s)] d\tau ds + \sum_{j=1}^{n-1} \int_{\Gamma_j \cap G_n} \beta_j(x, y) u(x_1, x_2) d\mu_{\varphi_j}, \quad (11)$$

where  $q(x_1, x_2)$  is positive and nondecreasing,  $g(x_1, x_2) \geq 1$ ,  $\beta_j(x_1, x_2) \geq 0$ ,  $\bar{\psi}(\tau, s) \geq 0$ ; the function  $W$  belongs to the class  $\Phi_1$  of functions such that:

1.  $W(\sigma_1 \sigma_2) \leq W(\sigma_1) W(\sigma_2), \forall \sigma_1, \sigma_2 > 0$ ;
2.  $W : [0, \infty[ \rightarrow [0, \infty[, W(0) = 0$ ;
3.  $W$  is nondecreasing.

Moreover  $u(x_1, x_2)$  is a nonnegative discontinuous function, which has finite jumps on the curves  $\{\Gamma_j\}, j = 1, 2, \dots$

Then for arbitrary  $\{0 < x_1 < \infty, 0 < x_2 < \infty\}$  the following estimate is fulfilled:

$$u(x_1, x_2) \leq q(x_1, x_2) g(x_1, x_2) \Psi_i^{-1} \left\{ \iint_{D_i} \frac{\bar{\psi}(\tau, s)}{q(\tau, s)} W[q(\tau, s) g(\tau, s)] d\tau ds \right\} \quad (12)$$

$$\forall x \in D_i : \iint_{D_i} \frac{\bar{\psi}(\tau, s)}{q(\tau, s)} W[q(\tau, s) g(\tau, s)] d\tau ds \in \text{Dom}(\Psi_i^{-1}),$$

$$\Psi_0(V) \stackrel{\text{def}}{=} \int_1^V \frac{d\sigma}{W(\sigma)}, \quad \Psi_i(V) \stackrel{\text{def}}{=} \int_{C_i}^V \frac{d\sigma}{W(\sigma)}, \quad i = 1, 2$$

where  $V = (V_1, V_2), \sigma = (\sigma_1, \sigma_2)$  and

$$C_i = \left( 1 + \int_{\Gamma_i \cap G_n} \beta_j(x_1, x_2) g(x_1, x_2) du_{\varphi_i} \right) \times \Psi_i^{-1} \left\{ \iint_{G_{i+1} \setminus G_i} \frac{\bar{\psi}(\tau, s)}{q(\tau, s)} W[q(\tau, s) g(\tau, s)] d\tau ds \right\}.$$

**Proposition 3.** *Let us suppose that nonnegative discontinuous function  $u(x_1, x_2)$ , which has finite jumps on the curves  $\{\Gamma_j\}$ , satisfies inequality (11), where the function  $W$  belongs to the class  $\Phi_1$  of function such that:*

1.  $W : [0, \infty[ \rightarrow [0, \infty[$  is continuous and nondecreasing;
2.  $\forall t > 0, u \geq 0, t^{-1}(W(u)) \leq W(t^{-1}u)$ ;
3.  $W(0) = 0$ .

If all functions  $q, g, \bar{\psi}, \beta_j$  satisfy the conditions of proposition 2 and  $q(x_1, x_2) \geq 1$ , then for arbitrary  $\{0 \leq x_1 \leq x_1^*, 0 \leq x_2 \leq x_2^*\}$  the following inequality is justified:

$$u(x_1, x_2) \leq q(x_1, x_2)g(x_1, x_2)\Psi_i^{-1} \left\{ \iint_{D_i} \bar{\psi}(\tau, s)g(\tau, s)d\tau ds \right\}, \quad i = 0, 1, \dots$$

where

$$\begin{aligned} \bar{\Psi}_0(V) &= \int_1^V \frac{d\sigma}{W(\sigma)}, \quad \bar{\Psi}_i(V) = \int_{C_i}^V \frac{d\sigma}{W(\sigma)}, \quad i = 1, 2, \dots \\ C_i &= \left( 1 + \int_{\Gamma_i \cap G_n} \beta_j(x_1, x_2)g(x_1, x_2)d\mu_{\varphi_i} \right) \bar{\Psi}_{i-1}^{-1} \left\{ \iint_{D_i} \bar{\psi}(\tau, s)g(\tau, s)d\tau ds \right\}, \\ (x_1^*, x_2^*) &= x^* = \sup_x \left\{ x : \iint_{G_{i+1} \setminus G_i} \bar{\psi}(\tau, s)g(\tau, s)d\tau ds \in \text{Dom}(\bar{\Psi}_i^{-1}(V)), \quad i = 1, 2, \dots \right\}. \end{aligned}$$

**2.2. Non-Lipschitz-type discontinuities.** Let us consider the following inequality:

$$\begin{aligned} u(x_1, x_2) &\leq \varphi(x_1, x_2) + \iint_{G_n} f(\sigma_1, \sigma_2) u^\alpha(\sigma_1, \sigma_2) d\sigma_1 d\sigma_2 + \\ &+ \sum_{j=1}^{n-1} \int_{\Gamma_j \cap G_n} \beta_j(x_1, x_2) u^m(x_1, x_2) d\mu_{\varphi_j}. \end{aligned} \quad (13)$$

**Proposition 4.**

If the function  $u(x_1, x_2)$  satisfies inequality (13) with  $f \geq 0$ ,  $\beta_j \geq 0$ ,  $\alpha = 1$ ,  $m > 0$ , then the following estimates are valid:

$$\begin{aligned} u(x_1, x_2) &\leq \varphi(x_1, x_2) \prod_{j=1}^{\infty} \int_{\Gamma_j \cap G_{j+1}} \varphi^{m-1}(x_1, x_2) \beta_j(x_1, x_2) d\mu_{\varphi_j} \cdot \\ &\cdot \exp \left[ \int_0^{x_1} \int_0^{x_2} f(\sigma_1, \sigma_2) d\sigma_1 d\sigma_2 \right], \quad \text{if } 0 < m \leq 1, \end{aligned} \quad (14)$$

$$\begin{aligned} u(x_1, x_2) &\leq \varphi(x_1, x_2) \prod_{j=1}^{\infty} \left( 1 + \int_{\Gamma_j \cap G_{j+1}} \varphi^{m-1}(x_1, x_2) \beta_j(x_1, x_2) d\mu_{\varphi_j} \right) \cdot \\ &\cdot \exp \left[ m \int_0^{x_1} \int_0^{x_2} f(\sigma_1, \sigma_2) d\sigma_1 d\sigma_2 \right], \quad \text{if } m \geq 1. \end{aligned} \quad (15)$$

**Proposition 5.**

If the function  $u(x_1, x_2)$  satisfies inequality (13) with  $\alpha = m > 0$ ,  $m \neq 1$  and the conditions of above Theorem are valid, then the following estimates hold:

$$u(x_1, x_2) \leq \varphi(x_1, x_2) \prod_{j=1}^{\infty} \left( 1 + \int_{\Gamma_j \cap G_{j+1}} \beta_j(x_1, x_2) \varphi^{m-1}(x_1, x_2) d\mu_{\varphi_j} \right) \cdot \left[ 1 + (1-m) \int_0^{x_1} \int_0^{x_2} \varphi^{m-1}(\sigma_1, \sigma_2) f(\sigma_1, \sigma_2) d\sigma_1 d\sigma_2 \right]^{\frac{1}{1-m}}, \text{ for } 0 < m < 1; \quad (16)$$

$$u(x_1, x_2) \leq \varphi(x_1, x_2) \prod_{j=1}^{\infty} \left( 1 + m \int_{\Gamma_j \cap G_{j+1}} \beta_j(x_1, x_2) \varphi^{m-1}(x_1, x_2) d\mu_{\varphi_j} \right) \cdot \left[ 1 - (m-1) \left[ \prod_{j=1}^{\infty} \left( 1 + m \int_{\Gamma_j \cap G_{j+1}} \beta_j(x_1, x_2) \varphi^{m-1}(x_1, x_2) d\mu_{\varphi_j} \right) \right]^{m-1} \cdot \int_0^{x_1} \int_0^{x_2} \varphi^{m-1}(\sigma_1, \sigma_2) f(\sigma_1, \sigma_2) d\sigma_1 d\sigma_2 \right]^{\frac{1}{1-m}}, \text{ for } m > 1 \quad (17)$$

$$\text{such that } \int_0^{x_1} \int_0^{x_2} \varphi^{m-1}(\sigma_1, \sigma_2) f(\sigma_1, \sigma_2) d\sigma_1 d\sigma_2 \leq \frac{1}{m},$$

$$\prod_{j=1}^{\infty} \left( 1 + m \int_{\Gamma_j \cap G_{j+1}} \beta_j(x_1, x_2) \varphi^{m-1}(x_1, x_2) d\mu_{\varphi_j} \right) < \left( 1 + \frac{1}{m-1} \right)^{\frac{1}{1-m}}. \quad (18)$$

**Proposition 6.**

Let us suppose that function  $u(x_1, x_2)$  satisfies inequality

$$u(x_1, x_2) \leq \bar{\varphi}(x_1, x_2) + g(x_1, x_2) \iint_{G_n} \bar{\psi}(\tau, s) u^m(\tau, s) d\tau ds + \sum_{j=1}^{n-1} \int_{\Gamma_j \cap G_n} \beta_j(x_1, x_2) u^n(x_1, x_2) d\mu_{\varphi_j}, \quad (19)$$

where  $\bar{\varphi}$  is a positive and nondecreasing function,  $g(x_1, x_2) \geq 1$ ,  $\beta_j(x_1, x_2) \geq 0$ ,  $\bar{\psi}(\tau, s) \geq 0$ ; the function  $u(x_1, x_2)$  is nonnegative and has finite jumps on the curves  $\Gamma_j$ ,  $j = 1, 2, \dots; m, n > 0$ . With these conditions the following estimates take place:

$$u(x_1, x_2) \leq \bar{\varphi}(x_1, x_2) g(x_1, x_2) \left( \Psi_i^{-1} \iint_{D_i} \bar{\psi}(\tau, s) \bar{\varphi}^{m-1}(\tau, s) g^m(\tau, s) d\tau ds \right) \quad (20)$$

$$\forall (x_1, x_2) \in D_i : \iint_{D_i} \bar{\psi}(\tau, s) g(\tau, s) d\tau ds \in (\Psi_i^{-1}),$$

$$\Psi_i(V) = \int_{C_i}^V \sigma^{-m} d\sigma, \quad C_0 = 1,$$

$$C_i = \left( 1 + \int_{\Gamma_i \cap G_{i+1}} \beta_i(x_1, x_2) g^n(x_1, x_2) \bar{\varphi}^{n-1}(x_1, x_2) d\mu_{\varphi_i} \right) \times \\ \times \Psi_{i-1}^{-1} \left( \iint_{G_{i+1} \setminus G_i} \bar{\psi}(\tau, s) \bar{\varphi}^{m-1}(\tau, s) g^m(\tau, s) d\tau ds \right), \quad i = 1, 2, \dots,$$

where  $m = 1$

$$\Psi_i^{-1}(V) = C_i \exp V, \quad i = 1, 2, \dots;$$

if  $0 < m < 1 \forall (x_1, x_2) \in D_i :$

$$\Psi_i^{-1}(V) = (C_i + (1 - m)V)^{\frac{1}{1-m}}, \quad i = 1, 2, \dots;$$

if  $m > 1$ ,

$$\Psi_i^{-1}(V) = [C_i - (m - 1)V]^{-\frac{1}{m-1}}, \quad i = 1, 2, \dots,$$

$$\forall (x_1, x_2) \in D_i : \iint_{D_i} \bar{\psi}(\tau, s) \bar{\varphi}^{m-1}(\tau, s) g^m(\tau, s) d\tau ds < \frac{C_i}{m-1}.$$

*Proof.* Obviously we have

$$\frac{u(x_1, x_2)}{\bar{\varphi}(x_1, x_2)} \leq 1 + g(x_1, x_2) \iint_{G_n} \frac{\bar{\psi}(\tau, s)}{\bar{\varphi}(\tau, s)} u^m(\tau, s) d\tau ds + \\ + \sum_{j=1}^{n-1} \int_{\Gamma_j \cap G_n} \frac{\beta_j(y_1, y_2)}{\bar{\varphi}(x_1, x_2)} u^n(y_1, y_2) d\mu_{\varphi_j} \leq g(x_1, x_2) (1 + \\ + \iint_{G_n} \frac{\bar{\psi}(\tau, s)}{\bar{\varphi}(\tau, s)} u^m(\tau, s) d\tau ds + \\ + \sum_{j=1}^{n-1} \int_{\Gamma_j \cap G_n} \frac{\beta_j(y_1, y_2)}{\bar{\varphi}(x_1, x_2)} u^n(y_1, y_2) d\mu_{\varphi_j})$$

Denoting by:

$$u^*(x_1, x_2) := \left( 1 + \iint_{G_n} \frac{\bar{\psi}(\tau, s)}{\bar{\varphi}(\tau, s)} u^m(\tau, s) d\tau ds + \right. \\ \left. + \sum_{j=1}^{n-1} \int_{\Gamma_j \cap G_n} \frac{\beta_j(y_1, y_2)}{\bar{\varphi}(x_1, x_2)} u^n(y_1, y_2) d\mu_{\varphi_j} \right),$$

then for  $k > 0$  we get

$$\begin{aligned}
 u^k(x_1, x_2) &\leq \bar{\varphi}^k(x_1, x_2) g^k(x_1, x_2) u^{*k}(x_1, x_2), \Rightarrow \\
 u^*(x_1, x_2) &\leq 1 + \iint_{G_n} \frac{\bar{\psi}(\tau, s)}{\bar{\varphi}(\tau, s)} u^m(\tau, s) d\tau ds + \\
 &+ \sum_{j=1}^{n-1} \int_{\Gamma_j \cap G_n} \frac{\beta_j(y_1, y_2)}{\bar{\varphi}(y_1, y_2)} u^n(y_1, y_2) d\mu_{\varphi_j} \leq \\
 &\leq 1 + \iint_{G_n} \bar{\psi}(\tau, s) \bar{\varphi}^{m-1}(\tau, s) g^m(\tau, s) u^{*m}(\tau, s) d\tau ds + \\
 &+ \sum_{j=1}^{n-1} \int_{\Gamma_j \cap G_n} \beta_j(y_1, y_2) u^{*n}(y_1, y_2) g^n(y_1, y_2) \bar{\varphi}^{n-1}(y_1, y_2) d\mu_{\varphi_j}. \quad (21)
 \end{aligned}$$

If  $(x_1, x_2) \in D_1$  from (20) it follows

$$u^*(x_1, x_2) \leq 1 + \iint_{G_1} \bar{\psi}(\tau, s) \bar{\varphi}^{m-1}(\tau, s) g^m(\tau, s) u^{*m}(\tau, s) d\tau ds.$$

Then  $\forall (x_1, x_2) \in D_1$

$$\begin{aligned}
 u^*(x_1, x_2) &\leq \Psi_0^{-1} \left( \iint_{G_1} \bar{\psi}(\tau, s) \bar{\varphi}^{m-1}(\tau, s) g^m(\tau, s) d\tau ds \right), \\
 \Psi_0(V) &= \int_1^V \sigma^{-m} d\sigma,
 \end{aligned}$$

where

$$a) \quad \Psi_0^{-1}(V) = \exp V, \quad i = 1, 2, \dots, \text{ if } m = 1;$$

$$b) \quad \Psi_0^{-1}(V) = (1 + (1 - m)V)^{\frac{1}{1-m}}, \quad i = 1, 2, \dots \text{ if } 0 < m < 1 \quad \forall (x_1, x_2) \in D_1;$$

$$c) \quad \Psi_0^{-1}(V) = [1 - (m - 1)V]^{\frac{-1}{m-1}}, \quad i = 1, 2, \dots \text{ if } m > 1$$

$\forall (x_1, x_2) \in D_1 :$

$$\iint_{G_1} \bar{\psi}(\tau, s) \bar{\varphi}^{m-1}(\tau, s) g^m(\tau, s) d\tau ds < \frac{1}{m-1}.$$



Now if we consider  $D_2$ , then  $G_2 = G_2^1 \cup G_2^2$ ,  $G_2^2 = D_2 \cap G_2$ . Obviously  $\forall (x_1, x_2) \in D_2$  it results

$$\begin{aligned} u^*(x_1, x_2) &\leq 1 + \iint_{G_2^1} \bar{\psi}(\tau, s) \bar{\varphi}^{m-1}(\tau, s) g^m(\tau, s) u^{*m}(\tau, s) d\tau ds + \\ &+ \int_{\Gamma_1 \cap G_2} \beta_1(y_1, y_2) u^{*n}(y_1, y_2) g^n(y_1, y_2) \bar{\varphi}^{n-1}(y_1, y_2) d\mu_{\varphi_j} + \\ &+ \iint_{G_2^2} \bar{\psi}(\tau, s) \bar{\varphi}^{m-1}(\tau, s) g^m(\tau, s) u^{*m}(\tau, s) d\tau ds. \end{aligned}$$

Take points  $A_i(x_{i1}^1, x_{i2}^1) \in \Gamma_1 \cap G_2$ ,  $i = 0, \dots, n-1$  and consider inequality

$$\begin{aligned} u^*(x_1, x_2) &\leq \sum_{i=0}^{n-1} \left( 1 + \int_0^{x_{i1}^1} \int_0^{x_{i2}^1} \bar{\psi}(\tau, s) \bar{\varphi}^{m-1}(\tau, s) g^m(\tau, s) u^{*m}(\tau, s) d\tau ds + \right. \\ &+ \beta_1(x_{i1}^1, x_{i2}^1) u^{*n}(x_{i1}^1, x_{i2}^1) g^n(x_{i1}^1, x_{i2}^1) \bar{\varphi}^{n-1}(x_{i1}^1, x_{i2}^1) \Delta\mu_{\varphi_1}^i + \\ &\left. + \int_{x_{i1}^1}^{x_1} \int_{x_{i2}^1}^{x_2} \bar{\psi}(\tau, s) \bar{\varphi}^{m-1}(\tau, s) g^m(\tau, s) u^{*m}(\tau, s) d\tau ds \right), \end{aligned}$$

where  $\Delta\mu_{\varphi_1}^i$  is variation of the measure  $\varphi_1$  on segment  $A_i A_{i+1}$ . Then, it follows

$$\begin{aligned} u^*(x_1, x_2) &\leq \sum_{i=0}^{n-1} \left( \Psi_0^{-1} \left( \int_0^{x_{i1}^1} \int_0^{x_{i2}^1} \bar{\psi}(\tau, s) \bar{\varphi}^{m-1}(\tau, s) g^m(\tau, s) d\tau ds \right) + \right. \\ &+ \beta_1(x_{i1}^1, x_{i2}^1) g^n(x_{i1}^1, x_{i2}^1) \bar{\varphi}^{n-1}(x_{i1}^1, x_{i2}^1) \Delta\mu_{\varphi_1}^i \times \\ &\times \Psi_0^{-1} \left( \left( \int_0^{x_{i1}^1} \int_0^{x_{i2}^1} \bar{\psi}(\tau, s) \bar{\varphi}^{m-1}(\tau, s) g^m(\tau, s) d\tau ds \right) \right) + \\ &\left. + \int_{x_{i1}^1}^{x_1} \int_{x_{i2}^1}^{x_2} \bar{\psi}(\tau, s) \bar{\varphi}^{m-1}(\tau, s) g^m(\tau, s) u^{*m}(\tau, s) d\tau ds \right) \leq \\ &\leq \sum_{i=0}^{n-1} \left( (1 + \beta_1(x_{i1}^1, x_{i2}^1) g^n(x_{i1}^1, x_{i2}^1) \bar{\varphi}^{n-1}(x_{i1}^1, x_{i2}^1) \Delta\mu_{\varphi_1}^i) \times \right. \\ &\times \left( \Psi_0^{-1} \left( \int_0^{x_{i1}^1} \int_0^{x_{i2}^1} \bar{\psi}(\tau, s) \bar{\varphi}^{m-1}(\tau, s) g^m(\tau, s) d\tau ds \right) \right) + \\ &\left. + \int_{x_{i1}^1}^{x_1} \int_{x_{i2}^1}^{x_2} \bar{\psi}(\tau, s) \bar{\varphi}^{m-1}(\tau, s) g^m(\tau, s) u^{*m}(\tau, s) d\tau ds \right). \end{aligned}$$

If  $0 \leq i \leq n-1$  then in  $D_2$  we get:

$$u^*(x_1, x_2) \leq \Psi_1^{-1} \left( \iint_{D_1} \bar{\psi}(\tau, s) \bar{\varphi}^{m-1}(\tau, s) g^m(\tau, s) d\tau ds \right), \quad (22)$$

with

$$\begin{aligned}\Psi_1(V) &= \int_{C_1}^V \sigma^{-m} d\sigma \\ C_1 &= \left(1 + \int_{\Gamma_1 \cap G_2} \beta_1(x_1, x_2) g^n(x_1, x_2) \bar{\varphi}^{n-1}(x_1, x_2) d\mu_{\varphi_i}\right) \times \\ &\quad \times \Psi_0^{-1} \left( \iint_{G_2 \setminus G_1} \bar{\psi}(\tau, s) \bar{\varphi}^{m-1}(\tau, s) g^m(\tau, s) d\tau ds \right),\end{aligned}$$

where

$$a) \quad \Psi_1^{-1}(V) = C_1 \exp V, \quad i = 1, 2, \dots, \quad m = 1$$

$$b) \quad \Psi_1^{-1}(V) = (C_1 + (1 - m)V)^{\frac{1}{1-m}}, \quad i = 1, 2, \dots, \quad 0 < m < 1 \quad \forall (x_1, x_2) \in D_2$$

$$c) \quad \Psi_1^{-1}(V) = [C_1 - (m - 1)V]^{\frac{-1}{m-1}}, \quad i = 1, 2, \dots, \quad m > 1$$

$\forall (x_1, x_2) \in D_2$  :

$$\iint_{G_1} \bar{\psi}(\tau, s) \bar{\varphi}^{m-1}(\tau, s) g^m(\tau, s) d\tau ds < \frac{C_1}{m-1}.$$

From Eq. (22) and the inequality

$$u(x_1, x_2) \leq \bar{\varphi}(x_1, x_2) g(x_1, x_2) u^*(x_1, x_2)$$

it follows that the estimate (20) holds in  $D_2$ .

Let us suppose  $(x_1, x_2) \in D_k$ ,  $G = G_{k+1}^1 \cup G_{k+2}^2$ ,  $G_{k+1}^1 = D_{k+1} \cap G$ ,  $G_{k+1}^2 = G \setminus G_{k+1}^1$  and consider the domain  $D_{k+1}$ :

$$\begin{aligned}u^*(x_1, x_2) &\leq 1 + \iint_{G_{k+1}^1} \bar{\psi}(\tau, s) \bar{\varphi}^{m-1}(\tau, s) g^m(\tau, s) u^{*m}(\tau, s) d\tau ds + \\ &\quad + \int_{\Gamma_k \cap G_2} \beta_k(y_1, y_2) u^{*n}(y_1, y_2) g^n(y_1, y_2) \bar{\varphi}^{n-1}(x_1, x_2) d\mu_{\varphi_j} + \\ &\quad + \iint_{G_{k+1}^2} \bar{\psi}(\tau, s) \bar{\varphi}^{m-1}(\tau, s) g^m(\tau, s) u^{*m}(\tau, s) d\tau ds.\end{aligned}\tag{23}$$

By using the inductive method, if we suppose that (20) is fulfilled in  $D_k$ , then from Eq. (23) we obtain the corresponding estimate in the domain  $D_{k+1}$ . So, the estimate (20) holds for arbitrary  $(0 \leq x_1 < \infty, 0 \leq x_2 < \infty)$ .  $\square$

### 3. Applications

**3.1. Lipschitz-type discontinuities.** Let us suppose that the evolution of some real processes may be described by hyperbolic partial differential equations with impulse perturbations concentrated on the surfaces

$$\frac{\partial^2 u(x_1, x_2)}{\partial x_1 \partial x_2} = H(x, u(x)), \quad (x_1, x_2) \in \Gamma_i,$$

$$\begin{aligned} u(x_1, 0) &= \phi_1(x_1), \\ u(0, x_2) &= \phi_2(x_2), \\ \phi_1(0) &= \phi_2(0), \end{aligned} \quad (24)$$

$$\Delta u|_{(x_1, x_2) \in \Gamma_i} = \int_{\Gamma_i \cap G_n} \beta_i(x_1, x_2) u(x_1, x_2) d\mu_{\phi_i}.$$

Here  $\Delta u|_{(x_1, x_2) \in \Gamma_i}$  are the characterised values of finite jumps  $u(x)(x = (x_1, x_2))$ , when the solution of (24) meet with hypersurfaces  $\Gamma_i : u(x) \cap \Gamma_i$ . We investigate Eq. (24) in the domain  $D^* \subset \mathbb{R}_+^2$ , which was described in the Introduction.

Denoting by  $\phi(x_1, x_2)$  the boundary conditions in Eq. (24), every solution of Eq. (24), satisfying the boundary conditions, is also a solution of the Volterra integro-sum equation:

$$\begin{aligned} u(x_1, x_2) &= \phi(x_1, x_2) + \iint_{G_n} H(\tau, s, u(\tau, s)) d\tau ds + \\ &+ \sum_{j=1}^{n-1} \int_{\Gamma_j \cap G_n} \beta_j(x_1, x_2) u(x_1, x_2) d\mu_{\phi_j} \end{aligned} \quad (25)$$

Let us suppose that

$$|H(\tau, s, u(\tau, s))| \leq \psi(\tau, s) W[|u(\tau, s)|], \quad (26)$$

where  $\psi(\tau, s) \geq 0$ ,  $W(\sigma) \in \Phi_1$ .

By using the result of proposition 2, we obtain following statement:

**Proposition 7.** *If  $H(x, u(x))$  in (24) satisfies condition (26), then for all solutions of Eq. (24) the following inequality is valid for all  $x_1 > 0, x_2 > 0$*

$$|u(x_1, x_2)| \leq |\phi(x_1, x_2)| \Psi_i^{-1} \left\{ \iint_{D_i} \frac{\psi(\tau, s)}{|\phi(\tau, s)|} W[|\phi(\tau, s)|] d\tau ds \right\} \quad (27)$$

$\forall x \in D_i :$

$$\iint_{D_i} \frac{\psi(\tau, s)}{|\phi(\tau, s)|} W[|\phi(\tau, s)|] d\tau ds \in \text{Dom}(\Psi_i^{-1}),$$

where

$$\Psi_0(V_1) = \int_1^{V_1} \frac{d\sigma}{W(\sigma)}, \quad \Psi_i(V_1) = \int_{C_i}^{V_1} \frac{d\sigma_1}{W(\sigma_1)} \quad i = 1, 2, \dots$$

$$C_i = \left( 1 + \int_{\Gamma_i \cap G_{i+1}} |\beta_i(x_1, x_2)| d\mu_{\phi_i} \right) \Psi_{i-1}^{-1} \left( \iint_{G_{i+1} \setminus G_i} \frac{\psi(\tau, s)}{|\phi(\tau, s)|} W[|\phi(\tau, s)|] d\tau ds \right).$$

By using the result of proposition 3 we obtain:

**Proposition 8.**

If the function  $H$  satisfies (26), where  $W$  belongs to the class of functions  $\bar{\Phi}_1 : W \in \bar{\Phi}_1$ , then all solutions of Eq. (24) satisfy such estimate:

$$|u(x_1, x_2)| \leq |\phi(x_1, x_2)| \bar{\Psi}_j^{-1} \left( \iint_{D_j} \psi(\tau, s) d\tau ds \right), \quad \forall j = 0, 1, \dots$$

where

$$\bar{\Psi}_0(V) = \int_1^V \frac{d\sigma}{W(\sigma)}, \quad \bar{\Psi}_i(V) = \int_{C_1}^V \frac{d\sigma}{W(\sigma)}, \quad i = 1, 2, \dots$$

$$C_i = \left( 1 + \int_{\Gamma_i \cap G_{i+1}} |\beta_i(x_1, x_2)| d\mu_{\phi_i} \right) \bar{\Psi}_{i-1}^{-1} \left( \iint_{G_{i+1} \setminus G_i} \psi(\tau, s) d\tau ds \right),$$

$\forall x : 0 < x < x^* :$

$$x^* = \sup_x \left\{ x : \iint_{G_{i+1} \setminus G_i} \bar{\psi}(\tau, s) d\tau ds \in \text{Dom}(\bar{\Psi}_i^{-1}), \quad i = 1, 2, \dots \right\}.$$

From Proposition 4 the next result follows:

**Proposition 9.**

Let us suppose that the following conditions hold:

- A):**  $|H(x_1, x_2, u(x_1, x_2))| \leq f(x_1, x_2) |u(x_1, x_2)|^\alpha$ ,  $\alpha = \text{const} > 0$ , where  $f$  is continuous nonnegative function in  $\mathbb{R}_+^2$ .  
**B):**  $\exists M = \text{const} > 0 : |\phi(x_1, x_2)| \leq M$ .

Then for solutions of Eqs. (24), the following estimates take place:

- (1)  $|u(x_1, x_2)| \leq M \prod_{j=1}^{\infty} \left( 1 + \int_{\Gamma_j \cap G_{j+1}} |\beta_j(x_1, x_2)| d\mu_{\phi_j} \right) \cdot \exp \left[ \int_0^{x_1} \int_0^{x_2} f(\sigma_1, \sigma_2) d\sigma_1 d\sigma_2 \right]$ , if  $\alpha = 1$ ;  
 (2)  $|u(x_1, x_2)| \leq M \prod_{j=1}^{\infty} \left( 1 + \int_{\Gamma_j \cap G_{j+1}} |\beta_j(x_1, x_2)| d\mu_{\phi_j} \right) \times \times \left[ 1 + (1 - \alpha) M^{\alpha-1} \int_0^{x_1} \int_0^{x_2} f(\sigma_1, \sigma_2) d\sigma_1 d\sigma_2 \right]^{\frac{1}{1-\alpha}}$ ,  
 if  $0 < \alpha < 1$ ;

$$(3) \quad |u(x_1, x_2)| \leq M \prod_{j=1}^{\infty} \left( 1 + \int_{\Gamma_i \cap G_{j+1}} |\beta_j(x_1, x_2)| d\mu_{\phi_i} \right) \times \\ \times \left\{ 1 + (\alpha - 1) M^{\alpha-1} \left[ \prod_{j=1}^{\infty} \left( 1 + \int_{\Gamma_i \cap G_{j+1}} |\beta_j(x_1, x_2)| d\mu_{\phi_i} \right) \right]^{\alpha-1} \right. \\ \left. \times \int_0^{x_1} \int_0^{x_2} f(\sigma_1, \sigma_2) d\sigma_1 d\sigma_2 \right\}^{-\frac{1}{\alpha-1}}$$

for  $\alpha > 1$  and arbitrary  $(x_1, x_2) \in D^*$  such that

$$\int_0^{x_1} \int_0^{x_2} f(\sigma_1, \sigma_2) d\sigma_1 d\sigma_2 < \left\{ (\alpha - 1) M^{\alpha-1} \left[ \prod_{j=1}^{\infty} \left( 1 + \int_{\Gamma_i \cap G_{j+1}} |\beta_j(x_1, x_2)| d\mu_{\phi_i} \right) \right]^{\alpha-1} \right\}^{-1}.$$

From Proposition 9 we obtain the following statement:

**Proposition 10.** *Let us consider Eq. (24) under the following conditions:*

- (1)  $|H(x_1, x_2, u(x_1, x_2))| \leq \psi(x_1, x_2) |u(x_1, x_2)|^\alpha$ ;
- (2)  $\exists M = \text{const} > 0 : |\varphi(x_1, x_2)| \leq M$ ;
- (3)  $\exists \xi, \eta$  :

$$\prod_{j=1}^{\infty} \left( 1 + \int_{\Gamma_i \cap G_{j+1}} |\beta_j(x_1, x_2)| d\mu_{\varphi_i} \right) \leq \xi < \infty; \\ \int_0^{x_1} \int_0^{x_2} \psi(\sigma_1, \sigma_2) d\sigma_1 d\sigma_2 \leq \eta < \infty.$$

Then all solutions  $u(x_1, x_2)$  of Eq. (24) are bounded for  $0 < \alpha \leq 1$ .

If additionally

$$\prod_{j=1}^{\infty} \left( 1 + \int_{\Gamma_i \cap G_{j+1}} |\beta_j(x_1, x_2)| d\mu_{\varphi_i} \right) < \frac{M^{1-\alpha}}{(\alpha - 1)\eta}$$

all solutions of Eq. (24) are bounded also for  $\alpha > 1$ .

**3.2. Hölder type discontinuities.** Let us consider the following problem:

$$\frac{\partial^2 u(x_1, x_2)}{\partial x_1 \partial x_2} = F(x, u(x)), \quad x = (x_1, x_2) \in \Gamma_i$$

$$u(x_1, 0) = \psi_1(x_1)$$

$$u(0, x_2) = \psi_2(x_2) \quad (28)$$

$$\psi_1(0) = \psi_2(0)$$

$$\Delta u|_{x \in \Gamma_i} = \int_{\Gamma_i \cap G_n} \beta_i(x) u^m(x) d\mu_{\phi_i}, \quad m > 0,$$

where  $\Delta u|_{x \in \Gamma_i}$  characterizes the values of discontinuities of solution of (28) when solution of (28) meets curves  $\Gamma_i : u(x) \cap \Gamma_i$ . In (28) we suppose that boundary conditions  $\psi_i(x)$  are bounded, i.e.

$$|\psi(x_1, x_2)| \leq M = \text{const.} < \infty$$

and  $F(x, u)$  satisfies the estimate:

$$|F(x, u)| \leq f(x_1, x_2) |u(x_1, x_2)|^\alpha, \quad (29)$$

with  $f \geq 0$ ,  $\alpha = \text{const.} > 0$ .

In the particular case  $m = 1$ , the equation of problem (28) was investigated in [7], [8], [16]. By using Propositions 4 and estimates A)–D) we obtain the following statement:

**Proposition 11.** *Let us suppose that for problem (28) the assumptions in Introduction about curves  $\Gamma_i$ , domains  $B_k$ ,  $G_k$  and functions  $\varphi_k$  are valid. Moreover, let  $F$  satisfy inequality (29).*

*I. Then the following estimates take place:*

$$\begin{aligned} A') \quad \alpha = 1, m \leq 1 \Rightarrow |u(x_1, x_2)| &\leq M \prod_{j=1}^{\infty} \left( 1 + M^{m-1} \int_{\Gamma_j \cap G_{j+1}} |\beta_j^*(x_1, x_2)| d\mu_{\varphi_j} \right) \cdot \\ &\cdot \exp \left[ \int_0^{x_1} \int_0^{x_2} f(\sigma_1, \sigma_2) d\sigma_1 d\sigma_2 \right], \end{aligned}$$

$$\begin{aligned} B') \quad \alpha = 1, m \geq 1 \Rightarrow |u(x_1, x_2)| &\leq M \prod_{j=1}^{\infty} \left( 1 + M^{m-1} \int_{\Gamma_j \cap G_{j+1}} |\beta_j^*(x_1, x_2)| d\mu_{\varphi_j} \right) \cdot \\ &\cdot \exp \left[ m \int_0^{x_1} \int_0^{x_2} f(\sigma_1, \sigma_2) d\sigma_1 d\sigma_2 \right], \end{aligned}$$

$$C') \quad 0 < \alpha = m < 1 \Rightarrow |u(x_1, x_2)| \leq M \prod_{j=1}^{\infty} \left( 1 + M^{m-1} \int_{\Gamma_j \cap G_{j+1}} |\beta_j^*(x_1, x_2)| d\mu_{\varphi_j} \right) \cdot$$

$$\cdot \left[ 1 + (1 - m)M^{m-1} \int_0^{x_1} \int_0^{x_2} f(\sigma_1, \sigma_2) d\sigma_1 d\sigma_2 \right]^{\frac{1}{1-m}},$$

$$D') \quad \alpha = m > 1 \Rightarrow |u(x_1, x_2)| \leq M \prod_{j=1}^{\infty} \left( 1 + mM^{m-1} \int_{\Gamma_j \cap G_{j+1}} |\beta_j^*(x_1, x_2)| d\mu_{\varphi_j} \right).$$

$$\cdot \left[ 1 - (m - 1)M^{m-1} \left[ \prod_{j=1}^{\infty} \left( 1 + mM^{m-1} \int_{\Gamma_j \cap G_{j+1}} |\beta_j^*(x_1, x_2)| d\mu_{\varphi_j} \right) \right]^{m-1} \cdot \right. \\ \left. \cdot \int_0^{x_1} \int_0^{x_2} f(\sigma_1, \sigma_2) d\sigma_1 d\sigma_2 \right]^{-\frac{1}{m-1}},$$

for all  $x_1, x_2 > 0$  :

$$\int_0^{x_1} \int_0^{x_2} f(\sigma_1, \sigma_2) d\sigma_1 d\sigma_2 \leq \frac{1}{mM^{m-1}}, \quad (30)$$

$$\prod_{j=1}^{\infty} \left( 1 + mM^{m-1} \int_{\Gamma_j \cap G_{j+1}} |\beta_j^*(x_1, x_2)| d\mu_{\varphi_j} \right) < \left( 1 + \frac{1}{m-1} \right)^{\frac{1}{1-m}}. \quad (31)$$

II. All solutions  $u(x_1, x_2)$  of (28) are bounded in the cases  $A') - C')$  only if the values

$\prod_{j=1}^{\infty} \left( 1 + \int_{\Gamma_j \cap G_{j+1}} |\beta_j^*| d\mu_{\varphi_j} \right), \int_0^{x_1} \int_0^{x_2} f(\sigma_1, \sigma_2) d\sigma_1 d\sigma_2$  are bounded. Referring to the case  $D')$ , (31), (32) guarantee conditions of boundedness for all solutions of (28).

## References

- [1] S. D. Borysenko and A. M. Piccirillo, *Impulsive integral inequalities: Applications* (Lunaset Editrice, Caserta, 2011).
- [2] S. Borysenko, "On some generalizations Bellman-Bihari result for integro-functional inequalities for discontinuous functions and their applications", in *The Nonlinear Analysis and Applications 2009: Materials of the International scientific conference (April 02-04th 2009, Kyiv)*, Kyiv: NTUU "KPI" (2009), p. 83.
- [3] S. D. Borysenko and A. M. Piccirillo, *Impulsive differential models: Stability* (Lunaset Editrice, Caserta, 2012), to appear.
- [4] A. Gallo and A. M. Piccirillo, "New Wendroff's type inequalities for discontinuous functions and its applications", *Nonlinear Studies* **19**(1), 1-11 (2012).
- [5] S. D. Borysenko, M. Ciarletta, and G. Iovane, "Integro-sum inequalities and motion stability of systems with impulse perturbations", *Nonlinear Analysis* **62**, 417-428 (2005).

- [6] S. Borysenko and G. Iovane, *Integro-sum inequalities and qualitative analysis dynamical systems with perturbations* (Tipogr. Elda - Mercato, S. Severino, 2006).
- [7] S. D. Borysenko and G. Iovane, "About some integral inequalities of Wendroff type for discontinuous functions", *Nonlinear Analysis* **66**, 2190-2203 (2007).
- [8] S. Borysenko and G. Iovane, *Differential models: Dynamical systems with perturbations* (Aracne, Rome, 2009).
- [9] S. D. Borysenko, G. Iovane, and P. Giordano, "Investigations of the properties motion for essential nonlinear systems perturbed by impulses on some hypersurfaces", *Nonlinear Analysis* **62**, 345-363 (2005).
- [10] A. Gallo and A. M. Piccirillo, "About new analogies of Gronwall-Bellman-Bihari type inequalities for discontinuous functions and estimated solutions for impulsive differential systems", *Nonlinear Analysis* **67**(5), 1550-1559 (2007).
- [11] A. Gallo and A. M. Piccirillo, "On some generalizations Bellman-Bihari result for integro-functional inequalities for discontinuous functions and their applications", *Boundary Value Problems*, Vol. 2009, ID 634624 (2009) [14 pages].
- [12] G. Iovane, "Some new integral inequalities of Bellman-Bihari type with delay for discontinuous functions", *Nonlinear Analysis* **66**, 498-508 (2007).
- [13] G. Iovane and S. D. Borysenko, "Boundedness, stability, practical stability of motion impulsive systems", *Proc. DE&CAS* (Brest, 2005), pp. 15-21.
- [14] V. Lakshmikantham, D. D. Bainov, and P. S. Simeonov, *Theory of impulsive differential equations* (World Scientific Publ., Singapore, 1989).
- [15] A. M. Piccirillo, S. S. Borysenko, and S. D. Borysenko, "Qualitative analysis behaviour of the solutions of impulsive differential systems", *Atti. Accad. Pelor. Pericol. Cl. Sc. Fis. Mat. Nat.*, **89**(2), C1A8902002 (2011) [12 pages]; DOI: [10.1478/C1A8902002](https://doi.org/10.1478/C1A8902002).
- [16] Yu. A. Mitropolsky, G. Iovane, and S. D. Borysenko, "About a generalization of Bellman-Bihari type inequalities for discontinuous functions and their applications", *Nonlinear Analysis* **66**, 2140-2165 (2007).
- [17] J. J. Nieto, "Impulsive periodic problems of first order", *Applied Mathematics Letters*, **15**(4), 489-493 (2002).
- [18] J. J. Nieto, "Periodic boundary value problems for first-order impulsive ordinary differential equations", *Nonlinear Analysis*, **51**(7), 1223-1232 (2002).
- [19] A. M. Samoilenko, S. D. Borysenko, C. Cattani, G. Matarazzo, and V. Yasinsky, *Differential models: Stability* (Naukova Dumka, Kyiv, 2001).

---

<sup>a</sup> Seconda Università degli Studi di Napoli  
Dipartimento di Ingegneria Civile  
Via Roma 21, 81031 Aversa (CE), Italy

<sup>b</sup> Università degli Studi di Salerno  
Dipartimento di Ingegneria Elettronica ed Ingegneria Informatica  
Via Ponte don Melillo, 84084 Fisciano (SA), Italy

<sup>c</sup> National Technical University of Ukraine  
Department of Differential Equations  
37 Prospect Peremogy, Kyev 03056, Ukraine

\* To whom correspondence should be addressed | Email: [annamaria.piccirillo@unina2.it](mailto:annamaria.piccirillo@unina2.it)