Atti dell'Accademia Peloritana dei Pericolanti
Classe I di Scienze Fis. Mat. e Nat.
Vol. LXXXI-LXXXII, C1A0401005 (2003-04)
Adunanza del 29 aprile 2004

ON SOME PROPERTIES OF RIEMANNIAN MANIFOLDS WITH LOCALLY CONFORMAL ALMOST COSYMPLECTIC STRUCTURES

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ABSTRACT. Let M be a $2m+1$-dimensional Riemannian manifold and let ∇ be the Levi-Civita connection and ξ be the Reeb vector field, η the Reeb covector field and X be the structure vector field satisfying a certain property on M. In this paper the following properties are proved:

- (i) ξ and X define a 3-covariant vanishing structure;
- (ii) the Jacobi bracket corresponding to ξ vanishes;
- (iii) the harmonic operator acting on $X^\flat$ gives
$$\triangle X^\flat = f\|X\|^2 X^\flat,$$
which proves that $X^\flat$ is an eigenfunction of $\triangle$, having $f\|X\|^2$ as eigenvalue;
- (iv) the 2-form Ω and the Reeb covector η define a Pfaffian transformation, i.e.
$$\mathcal{L}_X \Omega = 0,$$
$$\mathcal{L}_X \eta = 0;$$
- (v) $\nabla^2 X$ defines the Ricci tensor;
- (vi) one has
$$\nabla_X X = fX, \quad f = \text{scalar},$$
which show that X is an affine geodesic vector field;
- (vii) the triple $(X, \xi, \phi X)$ is an involutive 3-distribution on M, in the sense of Cartan.

1. Introduction

Let $M(g, \Omega, \phi, \eta, \xi)$ be an $2m+1$-dimensional Riemannian manifold with metric tensor g and associated Levi-Civita connection ∇. The quadruple $(\Omega, \phi, \eta, \xi)$ consists of a structure 2-form Ω of rank $2m$, and endomorphism ϕ ot the tangent bundle, the Reeb vector field ξ, and its corresponding Reeb covector field η, respectively. We assume that the 2-form Ω satisfies the relation

$$d\Omega = \lambda\, \eta \wedge \Omega, \tag{1}$$

where λ is constant, and that the 1-form η is given by

$$\eta = \lambda df, \tag{2}$$

for some scalar function f on M. We may therefore notice that a locally conformal almost cosymplectic structure [1], [3], [4] is defined on the manifold M. In addition, we assume that the field ϕ of endomorphism of the tangent spaces defines a quasi-Sasakian structure, thus realizing in particular the identity

$$\phi^2 = -Id + \eta \otimes \xi.$$

Moreover, we will assume the presence on M of a structure vector field X satisfying the property

$$\nabla X = f dp + \lambda \nabla \xi, \tag{3}$$

where dp is a canonical vector valued 1-form on M.

In the present paper various properties involving the above mentioned objects are studied. In particular, for the Lie differential of Ω and π with respect to X, one has

$$\mathcal{L}_X \eta = 0,$$

$$\mathcal{L}_X \Omega = 0,$$

which shows that η and Ω define Pfaffian transformations [1].

2. Preliminaries

Let (M, g) be an n-dimensional Riemannian manifold and let ∇ be the covariant differential operator defined by the metric tensor. We assume in the sequel that M is oriented and that the connection ∇ is symmetric. Let $\Gamma TM = \Xi(M)$ be the set of sections of the tangent bundle TM, and

$$\flat : TM \xrightarrow{\flat} T^*M \quad \text{and} \quad \natural : TM \xleftarrow{\natural} T^*M$$

the classical isomorphism defined by the metric tensor g (i.e. $\flat$ is the index-lowering operator, and $\natural$ is the index-raising operator). Following [1], we denote by

$$A^q(M, TM) = \Gamma Hom(\Lambda^q TM, TM),$$

the set of vector valued q-forms ($q < dimM$), and we write for

$$d^\nabla : A^q(M, TM) \to A^{q+1}(M, TM). \tag{4}$$

the covariant derivative operator with respect to ∇. It should be noticed that in general ${d^\nabla}^2 = d^\nabla \circ d^\nabla \neq 0$,unlike $d^2 = d \circ d = 0$. Furthermore, we denote by $dp \in A^1(M, TM)$ the canonical vector valued 1-form of M, which is also called the soldering form of M [3]; since ∇ is assumed to be symmetric, we recall that the identity $d^\nabla(dp) = 0$ is valid. The operator

$$d^\omega = d + e(\omega),$$

acting on ΛM is called the cohomology operator [3]. Here, $e(\omega)$ means the exterior product by the closed 1-form ω, i.e.

$$d^\omega u = du + \omega \wedge u,$$

with $u \in \Lambda M$. A form $u \in \Lambda M$ such that

$$d^{\omega} u = 0,$$

is said to be d^{ω}-closed, and ω is called the cohomology form. A vector field $X \in \Xi(M)$ which satisfies

$$d^{\nabla}(\nabla X) = \nabla^2 X = \pi \wedge dp \in A^2(M, TM), \qquad \pi \in \Lambda^1 M, \tag{5}$$

and where π is conformal to $X^{\flat}$, is defined to be an exterior concurrent vector field [1]. In this case, if $\mathcal{R}$ denotes the Ricci tensor field of ∇, one has

$$\mathcal{R}(X, Z) = -2m\lambda^3(\kappa + \eta) \wedge dp, \qquad Z \in \Xi(M).$$

3. Results

In terms of a local field of adapted vectorial frames $\mathcal{O} = \text{vect}\{e_A | A = 0, \cdots 2m\}$ and its associated coframe $\mathcal{O}^* = \text{covect}\{\omega^A | A = 0, \cdots 2m\}$, the soldering form dp can be expressed as

$$dp = \sum_{A=0}^{2m} \omega^A \otimes e_A.$$

We recall that E.Cartan's structure equations can be written as

$$\nabla e_A = \sum_{B=0}^{2m} \theta_A^B \otimes e_B, \tag{6}$$

$$d\omega^A = -\sum_{B=0}^{2m} \theta_B^A \wedge \omega^B, \tag{7}$$

$$d\theta_B^A = -\sum_{C=0}^{2m} \theta_B^C \wedge \theta_C^A + \Theta_B^A. \tag{8}$$

In the above equation θ (respectively Θ) are the local connection forms in the tangent bundle TM (respectively the curvature 2-forms on M). In terms of the frame fields $\mathcal{O}$ and $\mathcal{O}^*$ with $e_0 = \xi$ and $\omega^0 = \eta$, the structure vector field X and the 2-form Ω can be expressed as

$$X = \sum_{a=1}^{2m} X^a e_a, \tag{9}$$

$$\Omega = \sum_{i=1}^{m} \omega^i \wedge \omega^{i^*}, \qquad i^* = i + m. \tag{10}$$

Taking the Lie differential of Ω and η with respect to X, one calculates

$$\mathcal{L}_X \eta = 0, \tag{11}$$

$$\mathcal{L}_X \Omega = 0. \tag{12}$$

According to [2] the above equations prove that η and Ω define a Pfaffian transformation [3]. Next, by (2) one gets that

$$\theta_0^a = \lambda \omega^a. \tag{13}$$

Since we also assume that

$$\nabla X = f dp + \lambda \nabla \xi, \tag{14}$$

we futher also derive that

$$\nabla \xi = \lambda(dp - \eta \otimes \xi). \tag{15}$$

Since the q-th covariant differential $\nabla^q Z$ of a vector field $Z \in \Xi(M)$ is defined inductively, i.e.

$$\nabla^q Z = d^{\nabla}(\nabla^{q-1} Z),$$

this yields

$$\nabla^2 \xi = \lambda^2 \eta \otimes dp, \tag{16}$$

$$\nabla^3 \xi = 0. \tag{17}$$

Hence, one may say that the 3-covariant Reeb vector field ξ is vanishing. Next, by (13), one derives that

$$\nabla^2 X = \lambda^3 (df + \eta) \wedge dp = \frac{1+\lambda}{\lambda} \eta \wedge dp, \tag{18}$$

and consecutively one gets that

$$\nabla^3 X = 0. \tag{19}$$

This shows that both vector fields ξ and X together define a 3-vanishing structure. Moreover, by reference to [3], it follows from (18) that one may write that

$$\nabla^2 X = -\frac{1}{2m} \mathrm{Ric}(X) - X^\flat \wedge dp, \tag{20}$$

where Ric is the Ricci tensor. Reminding that by the definition of the operator ϕ

$$\phi e_i = e_{i^*} \qquad i \in \{1, \cdots m\},$$
$$\phi e_{i^*} = -e_i \qquad i^* = i + m,$$

one can check that indeed $\phi^2 = -Id$. Acting with ϕ on the vector field X, one obtains in a first step that

$$\phi X = \sum_{i=1}^{m} X^i e_{i^*} - X^{i^*} e_i \qquad i^* = i + m. \tag{21}$$

Calculating the Lie derivative of ϕ w.r.t. ξ, one gets

$$(\mathcal{L}_\xi \phi) X = [\xi, \phi X] - \phi[\xi, X]. \tag{22}$$

Since clearly

$$[\xi, \phi X] = 0, \tag{23}$$

there follows that

$$(\mathcal{L}_\xi \phi) X = 0. \tag{24}$$

Hence, the Jacobi bracket corresponding to the Reeb vector field ξ vanishes. By reference to the definition of the divergence

$$\mathrm{div} Z = \sum_{A=0}^{2m} \omega^A (\nabla_{e_A} Z)$$

one obtains in the case under consideration that

$$\mathrm{div} X = 2m(\lambda + f^2), \tag{25}$$

and

$$\mathrm{div}\phi X = 0. \tag{26}$$

Calculating the differential of the dual form $X^\flat$ of X, one gets

$$dX^\flat = \sum_{a=1}^{2m} \left(dX^a + \sum_{b=1}^{2m} X^b \theta_b^a \right) \wedge \omega^a. \tag{27}$$

Since

$$dX^a + \sum_{b=1}^{2m} X^b \theta_b^a = \lambda \omega^a, \tag{28}$$

one has that

$$dX^\flat = 0, \tag{29}$$

which means that the Pfaffian $X^\flat$ is closed. This implies that $X^\flat$ is an eigenfunction of the Laplacian $\triangle$, and one can write that

$$\triangle X^\flat = f\|X\|^2 X^\flat.$$

If we set

$$2l = \|X\|^2, \tag{30}$$

one also derives by (28) that

$$dl = \lambda X^\flat. \tag{31}$$

Returning to the operator ϕ, one calculates that

$$\nabla(\phi X) = \lambda \phi dp - \sum_{i=1}^{m} \left(\sum_{a=1}^{2m} (X^a \theta_a^i) \otimes e_{i^*} + \sum_{a=1}^{2m} (X^a \theta_a^{i^*}) \otimes e_i \right). \tag{32}$$

Hence there follows that

$$[\xi, X] = \rho\xi - \phi C, \tag{33}$$

$$[\xi, \phi X] = ((C^0)^2 + C^0(1-\lambda))\xi, \tag{34}$$

$$[X, \phi X] = \nabla_\xi \phi C = C^0 \xi - C \tag{35}$$

which shows that the triple $\{X, \xi, \phi X\}$ defines a 3-distribution on M. It is also interesting to draw the attention on the fact that X possesses the following property. From (14) and (15) one derives that

$$\nabla_X X = fX, \tag{36}$$

which means that X is an affine geodesic vector field. Finally, if we denote by Σ the exterior differential system which defines X, it follows by Cartan's test [1] that the caracteristic numbers are

$$r = 3, \qquad s_0 = 1, \qquad s_1 = 2.$$

Since $r = s_0 + s_1$, it follows that Σ is an involution and the existence of X depends on the arbitrary function of 1 argument. Summarizing, we can organize our results into the following

Theorem 3.1. *Let M be a $2m+1$-dimensional Riemannian manifold and let ∇ be the Levi-Civita connection and ξ be the Reeb vector field, η the Reeb covector field and X be the structure vector field satisfying the property (3) on M. One has the following properties:*

- (i) *ξ and X define a 3-covariant vanishing structure;*

- (ii) *the Jacobi bracket corresponding to* ξ *vanishes;*
- (iii) *the harmonic operator acting on* $X^\flat$ *gives*

$$\triangle X^\flat = f||X||^2 X^\flat,$$

which proves that $X^\flat$ *is an eigenfunction of* $\triangle$*, having* $f||X||^2$ *as eigenvalue;*
- (iv) *the 2-form* Ω *and the Reeb covector* η *define a Pfaffian transformation, i.e.*

$$\mathcal{L}_X\Omega = 0,$$
$$\mathcal{L}_X\eta = 0;$$

- (v) $\nabla^2 X$ *defines the Ricci tensor;*
- (vi) *one has*

$$\nabla_X X = fX, \quad f = \text{scalar},$$

which show that X *is an affine geodesic vector field;*
- (vii) *the triple* $(X, \xi, \phi X)$ *is an involutive 3-distribution on* M*, in the sense of Cartan.*

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