DOI:10.1478/C1A0702002

Atti dell'Accademia Peloritana dei Pericolanti
Classe di Scienze Fisiche, Matematiche e Naturali
Vol. LXXXV, C1A0702002 (2007)
Adunanza del 28 novembre 2006

GENERALIZING DOUBLE GRAPHS

MARIA CORINNA MARINO [a] AND NORMA ZAGAGLIA SALVI [b]*

(Nota presentata dal Socio Aggregato Maria Corinna Marino)

ABSTRACT. In this paper we study the graphs which are direct product of a simple graph G with the graphs obtained by the complete graph K_k adding a loop to each vertex; thus these graphs turn out to be a generalization of the double graphs.

1. Introduction

Let G be a finite simple graph, i.e. a graph without loops and multiple edges. In [1] it is introduced and studied the graph, said *double* of G and denoted $D(G)$, obtained by taking two copies of G and joining every vertex v in one component to every vertex w' in the other component corresponding to a vertex w adjacent to v in the first component. The above construction can be generalized in the following way.

As usual $V(G)$ and $E(G)$ denote the set of vertices and edges of G, respectively, and adj denote the adjacency relation of G. For all definitions not given here see [2, 3, 4, 5, 6].

The *direct product* $G \times H$ of two graphs G and H is the graph with $V(G\times H) = V(G) \times V(H)$ and with adjacency defined by (v_1, w_1) adj (v_2, w_2) if and only if v_1 adj v_2 in G and w_1 adj w_2 in H .[5]

The *total graph* T_n on n vertices is the graph associated to the total relation (where every vertex is adjacent to every vertex). It can be obtained by the complete graph K_n adding a loop to every vertex. In [5] it is denoted by K_n^s .

We define the k-*fold* of G as the graph $D^{[k]}(G) = G \times T_k$; clearly for $k = 2$ we obtain the double graphs.

Since the direct product of a simple graph with any graph is always a simple graph, it follows that the k-fold of a simple graph is still a simple graph.

In $D^{[k]}(G)$ we have (v, a) adj (w, b) if and only if v adj w in G. Then, if $V(T_k) = \{0, 1, \dots, k-1\}$, we have that $G_i = \{(v, i) \ : \ v \in V(G)\}$, $0 \le i \le k-1$, are k subgraphs of $D^{[k]}(G)$ isomorphic to G such that $G_0 \cap G_1 \cap \dots \cap G_{k-1} = \varnothing$ and $G_0 \cup G_1 \cup \dots \cup G_{k-1}$ is a spanning subgraph of $D^{[k]}(G)$. Moreover we have an edge between (v, i) and (w, j) and similarly we have an edge between (v, j) and (w, i) , where $0 \le i, j \le k-1$, whenever v adj w in G. We will call $\{G_0, G_1, \dots, G_{k-1}\}$ the *canonical decomposition* of $D^{[k]}(G)$.

The *lexicographic product* (or *composition*) of two graphs G and H is the graph $G \circ H$ with $V(G)\times V(H)$ as vertex set and with adjacency defined by (v_1, w_1) adj (v_2, w_2)

if and only if $v_1 = v_2$ and w_1 adj w_2 in H or v_1 adj v_2 in G. The graph $G \circ H$ can be obtained from G replacing each vertex v of G by a copy H_v of H and joining every vertex of H_v with every vertex of H_w whenever v and w are adjacent in G [5, p. 185].

Lemma 1. *For any graph G we have $G \times T_n \simeq G \circ N_n$, where N_n is the graph on n vertices without edges.*

Proof. For simplicity consider T_n and N_n on the same vertex set. Then the function $f : G \times T_n \to G \circ N_n$, defined by $f(v, k) = (v, k)$ for every $(v, k) \in V(G \times T_n)$, is a graph isomorphism. Indeed, since N_n has no edges, we have that (v, h) adj (w, k) in $G \circ N_n$ if and only if v adj w in G. □

2. Some basic properties of k-fold graphs

In this section we will review some elementary properties of the k-fold graphs. We will write $\mathcal{D}^2[G]$ for the double of the double of G. More generally $D^k(G)$ is obtained by multiplying G by T_2 k times, t.i. $D^k(G) = G \times T_{2^k}$ and $D^{[k]}(G) \neq D^k(G)$

In particular $D^{[k]}(G) \simeq D^m(G)$ when $k = 2^m$. In the following proposition the converse statement is proved.

Proposition 2. *Let k and m positive integers. Then $D^{[k]}(G) \simeq D^m(G)$ if and only if $k = 2^m$.*

Proof. We prove the ”only if” part. Let n be the number of vertices of G. The assumption that $D^{[k]}(G) \simeq D^m(G)$ implies $\mid V(D^{[k]}(G)) \mid=\mid V(D^m(G)) \mid$. Thus, because $\mid V(D^{[k]}(G)) \mid= kn$ and $\mid V(D^m(G)) \mid= 2^m n$, the result follows. □

Recall the following theorem.

Theorem 3 ([5, p. 190])**.** *If $G \circ H \simeq G' \circ H'$ and $|V(H)| = |V(H')|$, then $H \simeq H'$ and $G \simeq G'$.*

An immediate consequence of the theorem is the following

Theorem 4. *Two graphs G_1 and G_2 are isomorphic if and only if $\mathcal{D}^{[k]}(G_1)$ and $\mathcal{D}^{[k]}(G_2)$ are isomorphic.*

Proof. By Lemma 1 $D^{[k]}(G_1) = G_1 \circ N_k$ and $D^{[k]}(G_2) = G_2 \circ N_k$; then the claim holds. □

Proposition 5. *The k-fold graph $\mathcal{D}^{[k]}(G)$ of a graph G on n vertices contains at least $(2^n - 2)(k - 1)k + k$ subgraphs isomorphic to G itself.*

Proof. Let $\{G_0, G_1, \ldots, G_{k-1}\}$ be the canonical decomposition of $\mathcal{D}^{[k]}(G)$. Let S_0 be any subset of $V(G_0)$ and C_0 the complementary set of S_0 with respect to $V(G_0)$; moreover let S_i, where $1 \leq i \leq k-1$, be subsets of $V(G_i)$ and C_i their complementary sets with respect to $V(G_i)$. Then the subsets $S_i \cup C_j$, where $1 \leq i, j \leq k-1$, $i \neq j$, are isomorphic to G. The number of subsets S_0 is $2^n - 2$, because we exclude the cases of $S_0 = \varnothing$ and $S_0 = G_0$; finally we have to add the k subsets $G_0, G_1, \ldots, G_{k-1}$. □

Proposition 6. *For any graph G, G is bipartite if and only if $\mathcal{D}^{[k]}(G)$ is bipartite.*

Proof. Let $\{G_0, G_1, \ldots, G_{k-1}\}$ be the canonical decomposition of $\mathcal{D}^{[k]}(G)$. If G is bipartite then also G_j, where $0 \leq j \leq k-1$, are bipartite. Let $\{V, W\}$ be the partite sets of G and $\{V_j, W_j\}$, be the corresponding partite sets of G_j. Every edge of $\mathcal{D}^{[k]}(G)$ has one extreme in $\cup_{j=0}^{k-1} V_j$, and the other in $\cup_{j=o}^{k-1} W_j$ and hence also $\mathcal{D}^{[k]}(G)$ is bipartite.

Conversely, if $\mathcal{D}^{[k]}(G)$ is bipartite then it does not contain odd cycles. Hence also the subgraph $G_0 \simeq G$ does not contain odd cycles and then it is bipartite. □

A *vertex cut* of a graph G is a subset S of $V(G)$ such that $G \setminus S$ is disconnected. The *connectivity* $\kappa(G)$ of G is the smallest size of a vertex cut of G. A *point of articulation* (resp. *bridge*) is a vertex (resp. edge) whose removal augment the number of connected components. A *block* is a connected graph without articulation points. In the following proposition we present some properties of the k-fold graphs, whose proof is perfectly similar to the proof in the case of the double graphs [1].

Proposition 7. *For any graph $G \neq K_1$ the following properties hold.*

(1) *G is connected if and only if $\mathcal{D}^{[k]}(G)$ is connected.*
(2) *If G is connected, then every pair of vertices of $\mathcal{D}^{[k]}(G)$ belongs to a cycle.*
(3) *Every edge of $\mathcal{D}^{[k]}(G)$ belongs to a 4-cycle.*
(4) *In a k-fold graph there are neither bridges nor articulation points.*
(5) *If G is connected, then $\mathcal{D}^{[k]}(G)$ is a block.*
(6) *The connectivity of $\mathcal{D}^{[k]}(G)$ is $\kappa(\mathcal{D}^{[k]}(G)) = 2^k\ \kappa(G)$.*

A graph G is *Hamiltonian* if it has a spanning cycle.

Proposition 8. *If a graph G is Hamiltonian, then also $\mathcal{D}^{[k]}(G)$ is Hamiltonian.*

Proof. Let $\{G_0, G_1, \ldots, G_{k-1}\}$ be the canonical decomposition of $\mathcal{D}^{[k]}(G)$. Let γ be a spanning cycle of G, vw be an edge of γ and γ' be the path obtained by γ removing the edge vw. Let γ'_i be the corresponding path in G_i, for $i = 0, 1, \ldots, k-1$. Then $\gamma'_0 \cup \{(w, 0), (v, 1)\} \cup \gamma'_1 \cup \{(w, 1), (v, 2)\} \cup \cdots \cup \gamma'_{k-1}\{(w, k-1), (v, 0)\}$ is a spanning cycle of $\mathcal{D}^{[k]}(G)$. □

Proposition 9. *For any graph G_1 and G_2 the following properties hold:*

(1) $\mathcal{D}^{[k]}(G_1 \times G_2) = G_1 \times \mathcal{D}^{[k]}(G_2) = \mathcal{D}^{[k]}(G_1) \times G_2$
(2) $\mathcal{D}^{[k]}(G_1 \circ G_2) = G_1 \circ \mathcal{D}^{[k]}(G_2)$.

Proof. The first identity comes from the definition of k-fold graphs and $(G_1 \times G_2) \times T_k = G_1 \times (G_2 \times T_k)$, while the second one comes from $(G_1 \circ G_2) \circ N_k = G_1 \circ (G_2 \circ N_k)$. □

Let J_k be the matrix of all ones of order k. From the definition it follows immediately that

Proposition 10. *Let A be the adjacency matrix of G. Then the adjacency matrix of $\mathcal{D}^{[k]}(G)$ is*

$$\mathcal{D}^{[k]}[A] = \begin{bmatrix} A & A & \ldots & A \\ A & A & \ldots & A \\ \ldots & & & \\ A & A & \ldots & A \end{bmatrix} = A \otimes J_k.$$

The *rank* $r(G)$ of a graph G is the rank of its adjacency matrix. Then from the above proposition it follows that

Proposition 11. *For any graph G,* $r(\mathcal{D}^{[k]}(G)) = r(G)$.

In the sequel we will use the property that two graphs are isomorphic if and only if their adjacency matrices are similar by means of a permutation matrix.

Let G_1 and G_2 be two graphs. The *sum* $G_1 + G_2$ of G_1 and G_2 is the disjoint union of the two graphs. The *complete sum* $G_1 \boxplus G_2$ of G_1 and G_2 is the graph obtained by $G_1 + G_2$ joining every vertex of G_1 to every vertex of G_2. A graph is *decomposable* if it can be expressed as sums and complete sums of isolated vertices [6, p.183].

Proposition 12. *For any graph* G_1 *and* G_2 *the following properties hold:*

(1) $\mathcal{D}^{[k]}(G_1 + G_2) = \mathcal{D}^{[k]}(G_1) + \mathcal{D}^{[k]}(G_2)$
(2) $\mathcal{D}^{[k]}(G_1 \boxplus G_2) = \mathcal{D}^{[k]}(G_1) \boxplus \mathcal{D}^{[k]}(G_2)$
(3) *The k-fold of a decomposable graph is decomposable.*

Proof. The first two properties can be proved simultaneously as follows. Let A_1 and A_2 be the adjacency matrices of G_1 and G_2, respectively. Then $\begin{bmatrix} A_1 & X \\ X & A_2 \end{bmatrix}$ is the adjacency matrix of $G_1 + G_2$ when $X = O$ and of $G_1 \boxplus G_2$ when X is the matrix J all of whose entries are 1's. Then the adjacency matrix of the k-fold graph is

$$\begin{bmatrix} A_1 & X \\ X & A_2 \end{bmatrix} \otimes J_k.$$

Interchanging first the columns in even positions with those in odd positions and similarly for the rows, we obtain the matrix

$$\begin{bmatrix} A_1 \otimes J_k & X \otimes J_k \\ X \otimes J_k & A_2 \otimes J_k \end{bmatrix}$$

which is the adjacency matrix of $\mathcal{D}^{[k]}(G_1) + \mathcal{D}^{[k]}(G_2)$ when $X = O$ and of $\mathcal{D}[G_1] \boxplus \mathcal{D}[G_2]$ when $X = J$. These properties are also implied by the right-distributive laws of the lexicographic product [5, pp. 185-186]. Finally the third property follows from the fact that $\mathcal{D}^{[k]}$ preserves sums and complete sums and $\mathcal{D}^{[k]}(K_1) = N_k = K_1 + K_1 + \cdots + K_1$. □

Examples

(1) If N_n is the graph on n vertices without edges, then $\mathcal{D}^{[k]}(N_n) = N_{kn}$, while $\mathcal{D}^k(N_n) = N_{2^k . n}$.
(2) Let $K_{m,n}$ be a complete bipartite graph. Then $\mathcal{D}^{[k]}(K_{m,n}) = K_{km,kn}$. Similarly, if $K_{m_1,\ldots,m_n}$ is a complete n-partite graph we have $\mathcal{D}^{[k]}(K_{m_1,\ldots,m_n}) = K_{km_1,\ldots,km_n}$. In particular, if $K_{m(n)}$ is the complete m-partite graph $K_{n,\ldots,n}$, then $\mathcal{D}^{[k]}(K_{m(n)}) = K_{m(kn)}$. Since $K_n = K_{n(1)}$ it follows that the k-fold of the complete graph K_n is the graph $H_n^{[k]} = K_{n(k)}$, which turns out to be a generalization of the hyperoctahedral graph.

(3) For $n \geq 2$, let K_n^- be the graph obtained by the complete graph K_n deleting any edge. Then $K_n^- = N_2 \boxplus K_{n-2}$ and $\mathcal{D}^{[k]}(K_n^-) = \mathcal{D}^{[k]}(N_2) \boxplus \mathcal{D}^{[k]}(K_{n-2}) = N_{2k} \boxplus H_{n-2}^{[k]}$, that is $\mathcal{D}^{[k]}(K_n^-) = K_{2k,k,\ldots,k}$.

A graph G is *circulant* when its adjacency matrix A is circulant, i.e. when every row distinct from the first one, is obtained from the preceding one by shifting every element one position to the right. Let $C(a_1, \ldots, a_n)$ be the circulant graph where $(a_1, \ldots, a_n)$ is the first row of the adjacency matrix (for a suitable ordering of the vertices).

Proposition 13. *A graph G is circulant if and only if $\mathcal{D}^{[k]}(G)$ is circulant. Specifically*

$$\mathcal{D}^{[k]}(C(a_1, \ldots, a_n)) = C(a_1, \ldots, a_n, a_1, \ldots, a_n, \ldots, a_1, \ldots, a_n).$$

Let $\mathcal{R}[G] = G \times K_2$ be the *canonical double covering* of G [7]. In a way similar to the case of double graphs it is possible to prove the following proposition.

Proposition 14. *$\mathcal{D}^{[k]}$ and $\mathcal{R}$ commutes, that is $\mathcal{D}^{[k]}(\mathcal{R}[G]) = \mathcal{R}[\mathcal{D}^{[k]}(G)]$ for every graph G.*

3. Spectral properties of k-fold graphs

The *eigenvalues*, the *characteristic polynomial* and the *spectrum* of a graph are the eigenvalues, the characteristic polynomial and the spectrum of its adjacency matrix [3, p. 12].

Proposition 15. *The characteristic polynomial of the k - fold of a graph G on n vertices is*

$$\varphi(\mathcal{D}^{[k]}(G); \lambda) = (k\lambda^{k-1})^n \, \varphi(G; \lambda/k).$$

In particular the spectrum of $\mathcal{D}^{[k]}(G)$ is $\{0, \ldots, 0, k\lambda_1, \ldots, k\lambda_n\}$, where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of G and 0 is taken $(k-1)n$ times.

Proof. By Proposition 10 it follows that

$$\varphi(\mathcal{D}^{[k]}(G); \lambda) = \begin{vmatrix} \lambda I - A & -A & \ldots & -A \\ -A & \lambda I - A & \ldots & -A \\ \ldots & & & \\ -A & -A & \ldots \lambda I - A & \end{vmatrix} = \begin{vmatrix} \lambda I - kA & -A & \ldots & -A \\ \lambda I - kA & \lambda I - A & \ldots & -A \\ \ldots & & & \\ \lambda I - kA & -A & \ldots & \lambda I - A \end{vmatrix}$$

$$= \begin{vmatrix} \lambda I - kA & -A & \ldots & -A \\ 0 & \lambda I & \ldots & 0 \\ \ldots & & & \\ 0 & 0 & \ldots & \lambda I \end{vmatrix}.$$

□

An *integral graph* is a graph all of whose eigenvalues are integers [3, p. 266].

Proposition 16. *A graph G is integral if and only if $\mathcal{D}^{[k]}(G)$ is an integral graph.*

Proof. Since the characteristic polynomial of a graph is monic with integer coefficients its rational roots are necessarily integers. Then the claim immediately follows from Proposition 15. □

Two graphs are *cospectral* when they are non-isomorphic and have the same spectrum [2], [3]. From Proposition 15 and Theorem 4 we have the following property.

Proposition 17. *Two graphs* G_1 *and* G_2 *are cospectral if and only if* $\mathcal{D}^{[k]}[G_1]$ *and* $\mathcal{D}^{[k]}[G_2]$ *are cospectral.*

Therefore given two cospectral graphs G_1 and G_2, it is always possible to construct an infinite sequence of cospectral graphs. Indeed $\mathcal{D}^{[k]}(G_1)$ and $\mathcal{D}^{[k]}(G_2)$ are cospectral for every $k \in \mathbb{N}$.

The relation between the spectrum of a graph G and its k-fold graph has a consequence for the strongly regular graphs. First recall that a graph G is *d-regular* if every vertex has degree d; then a graph G is d-regular if and only if $\mathcal{D}^{[k]}[G]$ is kd-regular.

A simple graph G is *strongly regular* with parameters (n, d, λ, μ) when it has n vertices, is d-regular, every adjacent pair of vertices has λ common neighbors and every nonadjacent pair has μ common neighbors.[8]

Connected strongly regular graphs, distinct from the complete graph, are characterized [3, p. 103] as the connected regular graphs with exactly three distinct eigenvalues.

Strongly regular graphs with one zero eigenvalue are characterized as follows [3, p. 163]: a regular graph G has eigenvalues k, 0, λ_3 if and only if the complement of G is the sum of $1 - k/\lambda_3$ complete graphs of order $-\lambda_3$. Equivalently, a regular graph has three distinct eigenvalues of which one is zero if and only if it is a multipartite graph $K_{m(n)}$.

We are able now to characterize the strongly regular k-fold graphs in the following proposition proved in a perfectly similar way as in the case of the double graphs.

Proposition 18. *For any graph* G *the following characterizations hold.*

(1) $\mathcal{D}^{[k]}(G)$ *is a connected strongly regular graph if and only if* G *is a complete multipartite graph* $K_{m(kn)}$.
(2) $\mathcal{D}^{[k]}(G)$ *is a disconnected strongly regular graph if and only if* G *is a completely disconnected graph* N_{kn}.

Moreover, since complete bipartite graphs are characterized by their spectrum, we have that

Proposition 19. *Strongly regular k-fold graphs are characterized by their spectrum.*

4. Complexity and Laplacian spectrum

Let $t(G)$ be the *complexity* of the graph G, i.e. the number of its spanning trees. It is well known [9] that

$$t(G) = \frac{1}{n^2} \det(L + J) \tag{1}$$

where n is the number of vertices of G, L is the Laplacian matrix of G and J, as before, is the $n \times n$ matrix all of whose entries are equal to 1.

Theorem 20. *The complexity of the k-fold of a graph* G *on* n *vertices with degrees* $d_1, d_2, \ldots, d_n$ *is*

$$t(\mathcal{D}^{[k]}(G)) = k^{kn-2}\, d_1^{k-1}\, d_2^{k-1} \cdots d_n^{k-1}\, t(G)\,. \tag{2}$$

Proof. Let $v_1, \ldots, v_n$ be the vertices of G and $d_1, \ldots, d_n$ their degrees. As known the Laplacian matrix L of G is equal to $D - A$ where D is the diagonal matrix $\mathrm{diag}(d_1, \ldots, d_n)$ and A is the adjacency matrix of G. Then the Laplacian matrix of $\mathcal{D}^{[k]}(G)$ is

$$\mathcal{D}^{[k]}(L) = \mathcal{D}^{[k]}(D) - \mathcal{D}^{[k]}(A) = \begin{bmatrix} kD & O & \ldots & 0 \\ O & kD & \ldots & 0 \\ \ldots & & & \\ 0 & 0 & \ldots & kD \end{bmatrix} - \begin{bmatrix} A & A & \ldots & A \\ A & A & \ldots & A \\ A & A & \ldots & A \end{bmatrix}, \tag{3}$$

then

$$\mathcal{D}^{[k]}(L) = \begin{bmatrix} kD - A & -A & \ldots & -A \\ -A & kD - A & \ldots & -A \\ \ldots & & & \\ -A & -A & \ldots & kD - A \end{bmatrix}. \tag{4}$$

Hence it follows that

$$t(\mathcal{D}^{[k]}(G)) = \frac{1}{(kn)^2}.det(\mathcal{D}^{[k]}(L)+J) = \frac{1}{(kn)^2}.det \begin{bmatrix} kD - A + J & -A + J & \ldots & -A + J \\ -A + J & kD - A + J & \ldots & -A + J \\ \ldots & & & \\ -A + J & -A + J & \ldots & kD - A + J \end{bmatrix}. \tag{5}$$

Summing to the first the remaining columns, we have

$$t(\mathcal{D}^{[k]}(G)) = \frac{1}{(kn)^2} \begin{bmatrix} kD - kA + kJ & -A + J & \ldots & -A + J \\ kD - kA + kJ & kD - A + J & \ldots & -A + J \\ \ldots & & & \\ kD - kA + kJ & -A + J & \ldots & kD - A + J \end{bmatrix} \tag{6}$$

$$= \frac{1}{(kn)^2} \begin{bmatrix} kD - kA + kJ & -A + J & \ldots & -A + J \\ 0 & kD & \ldots & 0 \\ \ldots & & & \\ 0 & 0 & \ldots & kD \end{bmatrix}. \tag{7}$$

Then

$$t(\mathcal{D}^{[k]}(G)) = \frac{1}{(kn)^2}\, |kD - kA + kJ| \cdot |kD|^{k-1} = k^{nk-2}\, t(G).(d_1)^{k-1}.(d_2)^{k-1} \ldots (d_n)^{k-1}$$

and the theorem follows. □

As an immediate consequence we have the following

Theorem 21. *The complexity of the double of a d-regular graph G on n vertices is*

$$t(\mathcal{D}^{[k]}(G)) = k^{nk-2}.t(G).d^{n(k-1)}\,. \tag{8}$$

Finally, from (5), it can be proved the following

Proposition 22. *Let* G *be a graph on* n *vertices with degrees* $d_1, d_2, \ldots, d_n$ *and let* $\{\lambda_1, \ldots, \lambda_n\}$ *be its Laplacian spectrum. Then the Laplacian spectrum of* $\mathcal{D}^{[k]}(G)$ *is* $\{kd_1, \cdots, kd_n, k\lambda_1, \ldots, k\lambda_n\}$. *In particular,* G *has an integral Laplacian spectrum if and only if the same hold for* $\mathcal{D}^{[k]}(G)$.

5. Independent sets

An *independent set* of vertices of a graph G is a set of vertices in which no pair of vertices is adjacent. Let $\mathcal{I}_h[G]$ be the set of all independent subsets of size h of G and let $i_h(G)$ be its size. The *independence polynomial* of G is defined as

$$I(G;x) = \sum_{h\geq 0} \sum_{S\in\mathcal{I}_h[G]} x^{|S|} = \sum_{h\geq 0} i_h(G)\, x^h.$$

Proposition 23. *For any graph* G *we have* $\mathcal{I}_h[\mathcal{D}^{[k]}(G)] \simeq \mathcal{I}_h[G] \times \mathbf{k}^h$, *where* $\mathbf{k} = \{0, 1, \cdots, k-1\}$. *In particular* $i_h(\mathcal{D}^{[k]}(G)) = k^h i_h(G)$ *and* $I(\mathcal{D}^{[k]}(G); x) = I(G, kx)$.

Proof. Let the vertices of G be linearly ordered in some way. Let $S = \{(v_1, w_1), \ldots, (v_h, w_h)\}$ be an independent set of $\mathcal{D}^{[k]}(G) = G \times T_k$. Since T_k is a total graph, it follows that $\pi_1(S) = \{v_1, \ldots, v_h\}$ is an arbitrary independent subset of G and $\pi_2(S)$ is equivalent to an arbitrary sequence $(w_1, \ldots, w_h)$ of length h (where the order is established by the order of $\pi_1(S)$ induced by the order of $V(G)$). The claim follows. □

The (vertex) *independence number* $\alpha(G)$ of a graph G is the maximum size of the independent sets of vertices of G. Equivalently, $\alpha(G)$ is the degree of the polynomial $I(G, x)$. Then Proposition 23 implies the following

Proposition 24. *For any graph* G *we have that* $\alpha(\mathcal{D}^{[k]}(G)) = k\alpha(G)$.

6. Morphisms

A *morphism* $f : G \to H$ between two graphs G and H is a function from the vertices of G to the vertices of H which preserves adjacency (i.e. v adj w implies $f(v)$ adj $f(w)$, for every $v, w \in V(G)$) [10, 11]. An *isomorphism* between two graphs is a bijective morphism whose inverse function is also a morphism.

Let $\mathbf{Hom}(G, H)$ be the set of all morphisms between G and H and let $\mathbf{k}^{V[G]}$ be the set of all functions from $V(G)$ to $\mathbf{k} = \{0, 1, \cdots, k-1\}$.

Lemma 25. *For every graph* G *and* H, $\mathbf{Hom}(G, \mathcal{D}^{[k]}(H)) = \mathbf{Hom}(G, H) \times \mathbf{k}^{V[G]}$.

Proof. From the universal property of the direct product (in the categorical sense [12]) we have $\mathbf{Hom}(G, G_1 \times G_2) = \mathbf{Hom}(G, G_1) \times \mathbf{Hom}(G, G_2)$. Since $\mathcal{D}^{[k]}(G) = G \times T_k$ and $\mathbf{Hom}(G, T_k) = \mathbf{k}^{V[G]}$, the lemma follows. □

We now extend $\mathcal{D}^{[k]}$ to morphisms in the following way: for any graph morphism $f : G \to H$ let $\mathcal{D}^{[k]}[f] : \mathcal{D}^{[k]}(G) \to \mathcal{D}^{[k]}(H)$ be the morphism defined by $\mathcal{D}^{[k]}[f](v, k) = (f(v), k)$ for every $(v, k) \in \mathcal{D}[G]$. In this way $\mathcal{D}^{[k]}$ is an endofunctor of the category of finite simple graphs and graph morphisms.

A morphism $r : G \to H$ between two graphs G and H is a *retraction* if there exists a morphism $s : H \to G$ such that $r \circ s = 1_H$. If there exists a retraction $r : G \to H$ then H is a *retract* of G. Since $\mathcal{D}^{[k]}$ is a functor it preserves retractions and retracts.

Proposition 26. *Every graph G is a retract of $\mathcal{D}^{[k]}(G)$. More generally every retract of G is also a retract of $\mathcal{D}^{[k]}(G)$.*

Proof. Consider the morphisms $r : \mathcal{D}^{[k]}(G) \to G$ and $s : G \to \mathcal{D}^{[k]}(G)$ defined by $r(v,k) = v$ for every $(v,k) \in V(\mathcal{D}^{[k]}(G))$ and $s(v) = (v,0)$ for every $v \in V(G)$. Then r, which is the projection of $G \times T_k$ on G, is a retraction. The second part of the proposition follows from the fact that $\mathcal{D}^{[k]}$ is a functor and the composition of retractions is a retraction. □

References

[1] E. Munarini, A. Scagliola, C. Perelli Cippo, N. Zagaglia Salvi, "Double graphs", *Discrete Math.* (2007), to appear.
[2] N. Biggs, *Algebraic Graph Theory* (Cambridge Univ. Press, Cambridge, 1993)
[3] D. M. Cvetković, M. Doob, H. Sachs, *Spectra of graphs. Theory and applications* (Johann Ambrosius Barth, Heidelberg, 1995)
[4] F. Harary, *Graph theory* (Addison-Wesley, Reading, 1969)
[5] W. Imrich, S. Klavžar, *Product graphs* (Wiley-Interscience, New York, 2000)
[6] R. Merris, *Graph Theory* (Wiley-Interscience, New York, 2001)
[7] L. Porcu, "Sul raddoppio di un grafo", *Istituto Lombardo (Rend. Sc.) A* **110**, 453 (1976)
[8] P. Cameron, "Strongly regular graphs," in *Topics in Algebraic Graph Theory*, edited by L. W. Beineke and R. J. Wilson (Cambridge Univ. Press., Cambridge, 2004)
[9] R. A. Brualdi, H. J. Ryser, *Combinatorial Matrix Theory* (Cambridge Univ. Press, Cambridge, 1991)
[10] G. Hahn, C. Tardif, "Graph homomorphisms: structure and symmetry, in *Graph Symmetry*, edited by G. Hahn and G. Sabidussi (Kluver, 1997)
[11] P. Hell, J. Nesetril, *Graphs and Homomorphisms* (Oxford University Press, Oxford, 2004)
[12] T. S. Blyth, *Categories* (Longman, Harlow, 1986)

[a] Maria Corinna Marino
Università degli Studi di Messina
Dipartimento di Matematica
Contrada Papardo, Salita Sperone 31
98166 Messina, Italy

[b] Norma Zagaglia Salvi
Politecnico di Milano
Dipartimento di Matematica
Piazza Leonardo da Vinci, 32
20133 Milano, Italy.
* **E-mail**: norma.zagaglia@polimi.it

Presented: November 28, 2006
Published on line: October 1, 2007