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ON THE FINITE GENERATION OF THE MONOID OF EFFECTIVE DIVISOR CLASSES ON RATIONAL SURFACES OF TYPE (n, m)

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ABSTRACT. We prove that every projective rational surface of type (n, m) has only a finite number of (-1)-curves and only a finite number of (-2)-curves, where n and m are nonnegative integers satisfying the inequality $mn - m - 4n < 0$. As a consequence of this result, it follows the finite generation of the monoid of effective divisor classes on such surface. Thus giving in particular, a uniform proof of the classical results stating that the monoid of effective divisor classes on the surface obtained by blowing up the projective plane at points which are either all collinear or which are all on a conic is finitely generated.

1. Introduction

The monoid of effective divisor classes $M(X)$ on a given smooth projective rational surface X defined over an algebraically closed field of arbitrary characteristic may fail to be finitely generated. This is mainly due to the existence of an infinite number of integral curves of strictly negative self-intersection. A classical example of this phenomenon is the one given by Masayoshi Nagata (see [13, Theorem 4a, page 283]). Indeed, he shows that the surface obtained by blowing up nine points in general position has an infinite number of (-1)-curves, see also [5, Exercise 4.15, page 409]. Here, a (-1)-curve on a smooth projective surface Y means that it is a smooth rational curve with self-intersection -1. It may happen also that there exists a surface with an infinite number of (-2)-curves, this phenomenon is illustrated by an example of Brian Harbourne in [2, Example (2.8), page 140], see also [10, Example 5.5, page 111] for some open questions. By a (-2)-curve, we mean a smooth rational curve of self-intersection -2. On the other hand, the monoid of effective divisor classes on a smooth projective rational surface having an effective anticanonical divisor is finitely generated if and only if X has only a finite number of (-1)-curves and only a finite number of (-2)-curves, see [10, Corollary 4.2, page 109]. In this note, we give classes of smooth projective rational surfaces X such that $M(X)$ is finitely generated. These rational surfaces, denoted by $S_{(n,m)}$ and called of type (n, m) where n and m are nonnegative integers, are constructed as follows. Take a degenerate cubic which consists of a line L on the projective plane and a plane (integral) conic C (for example, L and C can be chosen in such a way that they meet each other transversally). Then take n points, say $P_1, ..., P_n$, on L and m points, say $Q_1, ..., Q_m$, on C with the assumption that the set of these $(n + m)$ points should not contain the intersection set of the line L and the conic

C. $S_{(n,m)}$ is then nothing but the surface obtained by blowing-up the projective plane $\mathbb{P}^2$ with center the zero dimensional closed subscheme $\{P_1, ..., P_n, Q_1, ..., Q_m\}$. Classical examples of this kind of surfaces is the ones when either m or n vanishes. For n and m big enough, the complete linear anticanonical system of $S_{(n,m)}$ is reduced to a singleton consisting of a reduced effective divisor having two smooth rational curves of strictly negative self-intersection as irreducible components. In particular, it may happen that a component maybe a (-1)-curve, or a (-2)-curve. Recall that a (-1)-curve (respectively a (-2)-curve) is a smooth rational curve of self-intersection -1 (respectively -2).

The main result of this work is the following:

Theorem 1. *With the notation as above, $S_{(n,m)}$ has only a finite number of (-1)-curves and a finite number of (-2)-curves, where n and m are nonnegative integers satisfying the following inequality: $mn - m - 4n < 0$.*

As a corollary, we have:

Corollary 2. *The monoid of effective divisor classes on $S_{(n,m)}$ is finitely generated by the set of (-1)-curves, by the set of (-2)-curves and eventually by the two irreducible components of the anticanonical divisor of $S_{(n,m)}$, where n and m are nonnegative integers satisfying the two inequalities $mn - m - 4n < 0$ and $n + m \geq 2$.*

Proof. It is a straightforward application of the above Theorem 1 and the Corollary [10, Corollary 4.2, page 109]. □

Hence, it follows by applying the Corollary 2.

Corollary 3. *The monoid of effective divisor classes on $S_{(n,m)}$ is finitely generated by the smooth rational curves of strictly negative self-intersection, where n and m are nonnegative integers satisfying the two inequalities $mn - m - 4n < 0$ and $n + m \geq 2$.*

The following classical result then follows easily:

Corollary 4. *The monoid of effective divisor classes on the surface obtained by the blow up the projective plane either at collinear points or at points all on a conic is finitely generated.*

Proof. Apply the above Corollary 2 to either $S_{(n,0)}$ or $S_{(0,m)}$, where n and m are positive integers which can be chosen larger than or equal to two, otherwise (i.e., if either n or m equals one), the conclusion of the Corollary 4 is trivial. □

Whereas the following nonclassical result follows also easily:

Corollary 5. *The monoid of effective divisor classes on the surface obtained by the blow up the projective plane at a set consisting of four or fewer points in general position and of n general points of a line is finitely generated, where the integer n is larger than or equal to ten.*

Proof. It is a straightforward from the Corollary 2 and the fact that there exists at least one integral conic passing through any four or fewer assigned points in general position, see the Proposition [5, Proposition 4.1, page 396] and its Corollary [5, Corollary 4.2, page 397]. □

Remark 6. *Our choice of the number ten in the Corollary 5 comes from our interest to find new smooth projective rational surfaces X having a canonical divisor K_X of strictly negative self-intersection and for which the monoid of effective divisor classes $M(X)$ is finitely generated. Surfaces X with $K_X^2 \geq 0$ are very well understood by now, see* [13], [14], [11], [2], [12], [3], [4], [6], [8], [7] *and* [9].

The proof of the theorem 1 will be postponed until section three. The structure of the paper is the following: section two gives the background needed and section three gives the proof of the result stated in section one.

2. Preliminaries

Let X be a smooth projective rational surface defined over an algebraically closed field of arbitrary characteristic. A canonical divisor on X, respectively the Picard group of X, will be denoted by K_X and respectively by $Pic(X)$ (see [5, Example 1.4, page 361] and [5, line -6, page 357]. There is an intersection form on $Pic(X)$ induced by the intersection of divisors on X, it will be denoted by a dot, that is, for x and y in $Pic(X)$, $x.y$ is the intersection number of x and y (see [5, Theorem 1.1, page 357]).

The following result known as the Riemann-Roch theorem for smooth projective rational surfaces is stated using the Serre duality.

Theorem 7. *Let D be a divisor on a smooth projective rational surface X having an algebraically closed field of arbitrary characteristic as a ground field. Then the following equality holds:*

$$h^0(X, O_X(D)) - h^1(X, O_X(D)) + h^0(X, O_X(K_X - D)) = 1 + \frac{1}{2}(D^2 - D.K_X).$$

$O_X(D)$ being an invertible sheaf associated to the divisor D and K_X being a canonical divisor on X.

Proof. This is exactly the Theorem [5, Theorem 1.6, page 362] applied to the case when the surface is rational. □

Here we recall some standard results, see [4], [5] and [1]. We begin with the notion of nefness of divisors.

Definition 1. *A divisor D on a smooth projective surface Z is numerically effective, nef in short, if $D.C \geq 0$ for every integral curve C on Z.*

A typical example is given by the following lemma:

Lemma 8. *An integral curve on a smooth projective surface having a nonnegative self-intersection is nef.*

Definition 2. *A divisor class x modulo algebraic equivalence on a smooth projective surface Z is effective respectively numerically effective, nef in short, if an element of x is an effective, respectively numerically effective, divisor on Z.*

Now, we start with some properties which follow from a successive iterations of blowing up closed points of a smooth projective rational surface.

Lemma 9. *Let $\pi^\star : Pic(X) \to Pic(Y)$ be the natural group homomorphism on Picard groups induced by a given birational morphism $\pi : Y \to X$ of smooth projective rational surfaces. Then $\pi^\star$ is an injective intersection-form preserving map of free abelian groups of finite rank. Furthermore, it preserves the dimensions of cohomology groups, the effective divisor classes and the numerically effective divisor classes.*

Proof. See [4, Lemma II.1, page 1193]. □

Lemma 10. *Let x be an element of the Picard group $Pic(X)$ of a smooth projective rational surface X. The effectiveness or the the nefness of x implies the noneffectiveness of $k_X - x$, where k_X denotes the element of $Pic(X)$ which contains a canonical divisor on X. Moreover, the nefness of x implies also that the self-intersection of x is greater than or equal to zero.*

Proof. See [4, Lemma II.2, page 1193]. □

The following result is well known as the adjunction formula:

Lemma 11. *Let Γ be an integral curve on a smooth projective surface Z, then the following equality holds*

$$\Gamma^2 + \Gamma.K_Z = 2p_a(\Gamma) - 2,$$

where K_Z is a canonical divisor on Z and $p_a(\Gamma)$ is the arithmetic genus of the curve Γ.

Proof. See [5, Proposition 1.5, page 361]. □

Using Bézout theorem, one may determine the elements of the anticanonical complete linear system of $S_{(n,m)}$ for $n \geq 4$ and $m \geq 5$.

Lemma 12. *For $n \geq 4$ and $m \geq 5$, the anticanonical complete linear system $|-K_{S_{(n,m)}}|$ on $S_{(n,m)}$ is*

$$|-K_{S_{(n,m)}}| = \{\widetilde{C} + \widetilde{L}\},$$

where $\widetilde{C}$ (respectively, $\widetilde{L}$) is the strict inverse transform of the conic C (respectively, of the line L) by the natural projection from $S_{(n,m)}$ to the projective plane.

From [5, Proposition 3.3, page 387] and the knowledge of a canonical divisor on the projective plane, one may obtain:

Lemma 13. *The self-intersection of a canonical divisor on $S_{(n,m)}$ is equal to $9-(n+m)^2$, where n and m are arbitrary nonnegative integers.*

Remark 14. *The surface in the Corollary 5 has a canonical divisor of self-intersection less than or equal to -1.*

3. Proof of the theorem 1

In this section, we give a proof of the result stated in the theorem of section one. To do so, we need to give explicitly the lattice structure of $Pic(S_{(n,m)})$.
Firstly, the integral basis

$$(\mathcal{E}_0; -\mathcal{E}_1^L, \ldots, -\mathcal{E}_n^L; -\mathcal{E}_1^C, \ldots, -\mathcal{E}_m^C),$$

is defined by:

- $\mathcal{E}_0$ is the class of a line on the projective plane which do not pass through any of the assigned points $P_1, ..., P_n, Q_1, ..., Q_m$ in consideration,
- $\mathcal{E}_i^L$ is the class of the exceptional divisor corresponding to the ith point blown-up P_i for every $i = 1, \ldots, n$,
- $\mathcal{E}_j^C$ is the class of the exceptional divisor corresponding to the jth point blown-up Q_j for every $j = 1, \ldots, m$.

The class of a divisor on $S_{(n,m)}$ will be represented by the $(1 + m + n)$-tuple

$$(a; b_1^L, \ldots, b_n^L; b_1^C, \ldots, b_m^C),$$

Secondly, the intersection form on $Pic(S_{(n,m)})$ is given by:

- $\mathcal{E}_0^2 = 1 = -(\mathcal{E}_i^L)^2 = -(\mathcal{E}_j^C)^2$ for every $i = 1, \ldots, n$ and $j = 1, ..., m$;
- $\mathcal{E}_i^L.\mathcal{E}_{i'}^L = 0$ for every $i, i' = 1, \ldots, n$, with $i \neq i'$;
- $\mathcal{E}_j^C.\mathcal{E}_{j'}^C = 0$ for every $j, j' = 1, \ldots, m$, with $j \neq j'$;
- $\mathcal{E}_i^L.\mathcal{E}_j^C = 0$ for every $i = 1, ..., n, j = 1, \ldots, m$;
- $\mathcal{E}_0.\mathcal{E}_i^L = \mathcal{E}_0.\mathcal{E}_i^C = 0$ for every $i = 1, ..., n, j = 1, \ldots, m$.

Remark 15. *we observe that if the class $(a; b_1^L, ..., b_n^L; b_1^C, ..., b_m^C)$ is effective, then it represents the class of a projective plane curve of degree a and having at least multiplicity $b_1^L, ..., b_n^L$ (respectively, $b_1^C, ..., b_m^C$) at the points $P_1, ..., P_n$ (respectively at the points $Q_1, ..., Q_m$). Also we note by assumption that the classes $\mathcal{E}_0, \mathcal{E}_i^L, \mathcal{E}_j^C$ are all the classes of smooth rational curves on $S_{(n,m)}$.*

We will prove that the set of (-1)-curves and the set of (-2)-curves are both finite. To do so, we first show that the set of (-2)-curves is finite. So let V be a general (-2)-curve on $S_{(n,m)}$. Let π be the natural projection from $S_{(n,m)}$ to $\mathbb{P}^2$ and let $(a; b_1^L, ..., b_n^L; b_1^C, ..., b_m^C)$ be the $(1+n+m)$-tuple representing the class of V in the Picard group $Pic(S_{(n,m)})$ relatively to the integral basis $(\mathcal{E}_0; -\mathcal{E}_1^L, \ldots, -\mathcal{E}_n^L; -\mathcal{E}_1^C, \ldots, -\mathcal{E}_m^C)$. Since $V^2 = -2$, it follows that the degree $a \geq 1$. From the two equalities $V^2 = -2$ and $V.K_{S_{(n,m)}} = 0$, one may obtain the following equalities:

$$(b_1^L)^2 + \ldots + (b_n^L)^2 + (b_1^C)^2 + \cdots + (b_m^C)^2 = a^2 + 2, \tag{1}$$

and

$$b_1^L + \cdots + b_n^L + b_1^C + \cdots + b_m^C = 3a. \tag{2}$$

It follows that if either m or n vanishes, then a also vanishes. Hence there is at most one (-2)-curve, generically there is no (-2)-curve at all. Consequently, we assume that m and n do not vanish. From the equality (2), one may obtain the following two equalities:

$$b_1^L + \cdots + b_n^L = a, \tag{3}$$

and

$$b_1^C + \cdots + b_m^C = 2a. \tag{4}$$

We claim that the integer a is bounded. To see this, we argue as follows. Define x_i^L and y_j^C for every $i = 1, \ldots, n$ and for every $j = 1, \ldots, m$ as follows.

$$x_i^L = \left(b_i^L - \frac{a}{n}\right), \tag{5}$$

and

$$y_j^C = \left(b_j^C - \frac{2a}{m}\right). \tag{6}$$

Then the equations (3) and (4) become respectively:

$$x_1^L + \cdots + x_n^L = 0, \tag{7}$$

$$y_1^C + \cdots + y_n^C = 0. \tag{8}$$

Whereas the equation (1) gives the following equation:

$$(x_1^L)^2 + \ldots + (x_n^L)^2 + (y_1^C)^2 + \cdots + (y_m^C)^2 = 2 + a^2 - \frac{a^2}{n} - \frac{4a^2}{m}, \tag{9}$$

which implies by our assumption that the nonnegative integer a is bounded.

Now we proceed to prove that the set of (-1)-curves on $S_{(n,m)}$ is finite. Indeed, let U be a general (-1)-curve on $S_{(n,m)}$ and let $(a; b_1^L, ..., b_n^L; b_1^C, ..., b_m^C)$ be the $(1+n+m)$-tuple representing the class of U in the Picard group $Pic(S_{(n,m)})$ relatively to the integral basis $(\mathcal{E}_0; -\mathcal{E}_1^L, \ldots, -\mathcal{E}_n^L; -\mathcal{E}_1^C, \ldots, -\mathcal{E}_m^C)$. Since U is general, it follows that the degree $a \geq 1$. From the two equalities $U^2 = -1$ and $U.K_{S_{(n,m)}} = -1$, one may obtain the following two equalities:

$$(b_1^L)^2 + \ldots + (b_n^L)^2 + (b_1^C)^2 + \cdots + (b_m^C)^2 = a^2 + 1, \tag{10}$$

$$(a - b_1^L - \cdots - b_n^L) + (2a - b_1^C - \cdots - b_m^C) = 1, \tag{11}$$

Hence either the following case to whom we refer to as the case 1,

$$(b_1^L)^2 + \ldots + (b_n^L)^2 + (b_1^C)^2 + \cdots + (b_m^C)^2 = a^2 + 1, \tag{12}$$

and

$$b_1^L + \cdots + b_n^L = a - 1, \tag{13}$$

and

$$b_1^C + \cdots + b_m^C = 2a. \tag{14}$$

or the following case to whom we refer to as the case 2

$$(b_1^L)^2 + \ldots + (b_n^L)^2 + (b_1^C)^2 + \cdots + (b_m^C)^2 = a^2 + 1, \tag{15}$$

and

$$b_1^L + \cdots + b_n^L = a, \tag{16}$$

and

$$b_1^C + \cdots + b_m^C = 2a - 1. \tag{17}$$

holds.

It follows that if either m or n vanishes, then a also vanishes. So we may consider the integers m and n to be not equal to zero. Assume that we are in the case 1, and consider the new scalars $(\alpha_i^L)_{i\in\{1,\ldots,n\}}$ and $(\beta_j^C)_{j\in\{1,\ldots,m\}}$ defined by:

$$\alpha_i^L = b_i^L - \frac{a-1}{n}, \quad \textit{for every } i=1,\ldots,n, \tag{18}$$

and

$$\beta_j^C = b_j^C - \frac{2a}{m}, \quad \textit{for every } j=1,\ldots,m. \tag{19}$$

Then the equations (12), (13) and (14) give

$$(\alpha_1^L)^2 + \ldots + (\alpha_n^L)^2 + (\beta_1^C)^2 + \cdots + (\beta_m^C)^2 = 1 + a^2 - \frac{(a-1)^2}{n} - \frac{4a^2}{m}, \tag{20}$$

and

$$\alpha_1^L + \cdots + \alpha_n^L = 0, \tag{21}$$

and

$$\beta_1^C + \cdots + \beta_m^C = 0. \tag{22}$$

It follows then from the equation (20) that a is bounded. With the same strategy, we prove the boundness of a in the case 2.

Remark 16. *Our method of proving the theorem 1 suggests not only to allow that some points to be infinitely near but also to consider other classes of smooth projective rational surfaces other than* $S_{(n,m)}$.

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