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FEYNMAN'S TRANSITION AMPLITUDES IN THE SPACE $\mathcal{S}'_n$

DAVID CARFÌ

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ABSTRACT. In this paper we propose a rigorous formulation for Feynman's propagator of Quantum Mechanics; the space in which we build up the propagator is the space of tempered distributions $\mathcal{S}'_n$.

The goals of the paper are the following ones:

1) a rigorous and operative definition of Feynman's propagator in $\mathcal{S}'_n$;

2) the basic indications for a straightforward and not-ambiguous calculus for the propagators, easy to teach and use;

3) a rigorous formulation and proof of the famous Feynman's Transition Amplitudes Theorem, in the space of tempered distributions; and of one its original generalization.

1. Introduction

The paper deals with a well known Feynman theorem: the Transition Amplitudes Theorem for an abstract dynamical system (physical, biologic, economic, ...). This Theorem gives the probability that the considered system pass from a state to another.

Of this theorem there is no a rigorous proof, neither in the context of Hilbert spaces, but this is not the worse problem; the very critical point is that there is not a clear, univocal and unambiguous statement of the result; this affect badly on the use of this indubitably good result, firstly because it is not clear what is the precise means of the symbols and operations that Feynman presents, and secondly because the context proposed (the Hilbert spaces) appear, at a deep view, inadequate indeed for its application. Nevertheless, its efficiency in the applications, thanks to the good intuitions of physicists, made it a basic instrument in many questions of experimental, computational and theoretical analysis of dynamical systems.

The paper presents a good-founded and rigorously proved version of Feynman theorem, in a form much close to the original one, but in a context quite different from the standard setting of Hilbert spaces: the theorem is stated in the context of tempered distributions.

In the Laurent Schwartz' spaces we have to redefine all the objects appearing in the Feynman Theorem (not all clearly defined neither in Hilbert spaces), and we shall use technics very far from those of Hilbert spaces.

We have to note that the a tempered distribution takes the place of the transition amplitudes. To better understand the situation we need a confrontation. In Hilbert spaces the transition amplitude from a condition (t, u) to another (t', u') is the scalar product $(u|u')$:

then a complex number. In the space of tempered distributions the situation is more complicated, firstly because we have not a scalar product. Secondly, the system can pass from a state u (a pure state) to one of the infinite states of a continuous family w, namely the family of eigenstates of an $\mathcal{S}$-observable; this is the very case of Feynman Theorem. Now we have to use the $\mathcal{S}$-Linear Algebra which, since w is a regular family being the eigenbasis of an $\mathcal{S}$-observable, is able to compute the product of u with the family w, the result is a tempered distribution but in general not a regular one.

The surprise is that after few definitions, the Feynman theorem, in the form desired by Feynman, becomes at last a correct statement readily provable, showing, an unexpected and strong connection with the change of basis Theorem of Linear Algebra, as we prove in the paper.

2. The Feynman propagators

In this paper the $\mathcal{S}$-Linear Algebra in the space $\mathcal{S}'_n$ is systematically used, for it can be seen [1, 2, 3, 4, 5].

Definition (of Feynman propagator). *We call a function*

$$G : \mathbb{R}^2 \to \mathcal{S}\left(\mathbb{R}^n, \mathcal{S}'_n\right),$$

associating to each pair of times an $\mathcal{S}$-family in $\mathcal{S}'_n$, $\mathcal{S}$-propagator if, for every real t, $G(t,t) = \delta$, where δ is the Dirac family in $\mathcal{S}'_n$. Moreover, G is said a Feynmann propagator if

1) for every real t, $G(t,t) = \delta$;

2) for every pair of reals t_0 and t, the family $G(t_0,t)$ is invertible and

$$G(t_0,t) = G(t,t_0)^{-1};$$

3) for every triple of times t_0, t_1 and t_2, we have

$$G(t_0,t_2) = G(t_0,t_1) \cdot G(t_1,t_2).$$

Remark. A propagator G is then defined as an $\mathcal{S}$-family-valued function; so we have

$$G(t_0,t) = (G(t_0,t)_y)_{y\in\mathbb{R}^n},$$

for every pair of times (t_0,t).

Remark. Note that, in the above definition, 2 derives from 1 and 3. In fact, from 3 we have, for $t_2 = t_0$,

$$G(t_0,t_0) = G(t_0,t_1) \cdot G(t_1,t_0),$$

and then, applying 1 ($G(t_0,t_0) = \delta$), we obtain 2.

Definition (propagator of a process). *Let $u : \mathbb{R} \to \mathcal{S}'_n$ be a process in the space $\mathcal{S}'_n$. We say that a propagator*

$$G : \mathbb{R}^2 \to \mathcal{S}\left(\mathbb{R}^n, \mathcal{S}'_n\right)$$

is a Green function, or a propagator, for u if

$$u(t) = \int_{\mathbb{R}^n} u(t_0)G(t_0,t),$$

for every t_0 and t in T.

Remark. Hence $G(t_0, t)$ is a family such that the state of u at the time t is the superposition of the family $G(t_0, t)$ with respect to the system of coefficients coinciding with the state of u at t_0.

Remark. If a process u admits a Feynman propagator, then it is a *strongly causal* and *reversible* process. In fact, by definition, the state of the process at every time t_0, determines the state of the process at every other time t. Moreover, u is reversible, since, if

$$u(t) = \int_{\mathbb{R}^n} u(t_0)G(t_0, t),$$

then

$$u(t_0) = \int_{\mathbb{R}^n} u(t)G(t_0, t)^{-1}.$$

3. The evolution operators

Definition (evolution operator). *An evolution operator in $\mathcal{S}'_n$ is a function*

$$E : \mathbb{R} \times \mathcal{S}'_n \to C^0 (\mathbb{R}, \mathcal{S}'_n),$$

i.e., an operator sending every initial condition (t_0, u_0) belonging to $\mathbb{R}\times\mathcal{S}'_n$ into a process

$$E_{(t_0,u_0)} : \mathbb{R} \to \mathcal{S}'_n,$$

such that

1) $E_{(t_0,u_0)}(t_0) = u_0$ for every initial condition (t_0, u_0);
2) if $E_{(t,u)}(t_0) = u_0$ then $E_{(t_0,u_0)}(t) = u$ for every (t_0, u_0) and (t, u);
3) if $E_{(t_0,u_0)}(t_1) = u_1$ and $E_{(t_1,u_1)}(t_2) = u_2$ then $E_{(t_0,u_0)}(t_2) = u_2$.

Remark. In other terms, a mapping

$$E : \mathbb{R} \times \mathcal{S}'_n \to C^0 (\mathbb{R}, \mathcal{S}'_n),$$

is an evolution operator if and only if the binary relation $=_E$ on the time-states space $\mathbb{R}\times\mathcal{S}'_n$ defined by

$$(t_0, u_0) =_E (t, u) \ \ if\ and\ only\ if \ \ E_{(t_0,u_0)}(t) = u,$$

is an equivalence relation.

Definition (the propagator of an evolution). *Let E be an evolution operator. We say that a function*

$$G : \mathbb{R}^2 \to \mathcal{S} (\mathbb{R}^n, \mathcal{S}'_n)$$

is a Green function, or propagator, for E if

$$E_{(t_0,u_0)}(t) = \int_{\mathbb{R}^n} u_0G(t_0, t),$$

for every u_0 in $\mathcal{S}'_n$ and for every t_0 and t in $\mathbb{R}$.

Theorem. *Let*

$$E : \mathbb{R} \times \mathcal{S}'_n \to C^0 (\mathbb{R}, \mathcal{S}'_n),$$

be an operator that sends every initial condition (t_0, u_0) *belonging to* $\mathbb{R}\times\mathcal{S}'_n$ *into a process*

$$E_{(t_0,u_0)} : \mathbb{R} \to \mathcal{S}'_n;$$

and let

$$G : \mathbb{R}^2 \to \mathcal{S}\left(\mathbb{R}^n, \mathcal{S}'_n\right),$$

be a family-valued function such that

$$E_{(t_0,u_0)}(t) = \int_{\mathbb{R}^n} u_0 G(t_0, t),$$

for every u_0 *in* $\mathcal{S}'_n$ *and for every* t_0 *and* t *in* $\mathbb{R}$*.*

Then, E *is an evolution operator if and only if* G *is a Feynman propagator.*

Proof. Assume E be an evolution operator, we must verify the properties of the Feynman propagator.

1) Let t be a real; for every state u, we have

$$u = E_{(t,u)}(t) = \int_{\mathbb{R}^n} uG(t, t);$$

thus $G(t, t) = \delta$.

2) Consider two instant of time t_0 and t. For every state u_0 in $\mathcal{S}'_n$ set

$$u := E_{(t_0,u_0)}(t) = \int_{\mathbb{R}^n} u_0 G(t_0, t);$$

by axiom 2, we have

$$u_0 = E_{(t,u)}(t_0) = \int_{\mathbb{R}^n} uG(t, t_0),$$

then, consequently

$$u_0 = \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} u_0 G(t_0, t) \right) G(t, t_0) = \int_{\mathbb{R}^n} u_0 \int_{\mathbb{R}^n} G(t_0, t)G(t, t_0).$$

This is equivalent to

$$\int_{\mathbb{R}^n} G(t_0, t)G(t, t_0) = \delta,$$

and then

$$G(t_0, t) = G(t, t_0)^{-1}.$$

3) Consider three instants of time t_0, t_1 and t_2, we have

$$G(t_0, t_2) = G(t_0, t_1) \cdot G(t_1, t_2).$$

In fact, if $E_{(t_0,u_0)}(t_1) = u_1$ and $E_{(t_1,u_1)}(t_2) = u_2$ then $E_{(t_0,u_0)}(t_2) = u_2$. Now $E_{(t_0,u_0)}(t_1) = u_1$ is equivalent to

$$\int_{\mathbb{R}^n} u_0 G(t_0, t_1) = u_1,$$

and $E_{(t_1,u_1)}(t_2) = u_2$ is equivalent to

$$\int_{\mathbb{R}^n} u_1 G(t_1, t_2) = u_2.$$

Hence

$$
\begin{aligned}
u_2 &= \int_{\mathbb{R}^n} u_1 G(t_1,t_2) = \\
&= \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} u_0 G(t_0,t_1) \right) G(t_1,t_2) = \\
&= \int_{\mathbb{R}^n} u_0 \int_{\mathbb{R}^n} G(t_0,t_1) G(t_1,t_2).
\end{aligned}
$$

On the other hand, we have

$$u_2 = E_{(t_0,u_0)}(t_2) = \int_{\mathbb{R}^n} u_0 G(t_0,t_2),$$

and so

$$\int_{\mathbb{R}^n} u_0 G(t_0,t_2) = \int_{\mathbb{R}^n} u_0 \left(\int_{\mathbb{R}^n} G(t_0,t_1) G(t_1,t_2) \right),$$

thus

$$G(t_0,t_2) = G(t_0,t_1) \cdot G(t_1,t_2).$$

The viceversa is a simple calculation. ■

4. Operator-valued propagators

Definition (operator-valued propagator). *An operator-valued propagator is a function*

$$S : \mathbb{R}^2 \to {}^{\mathcal{S}}\mathrm{End}(\mathcal{S}'_{\mathrm{n}}),$$

i.e. it is an operator-valued function, verifying the following properties:

1) $S(t_0,t_0) = (\cdot)_{\mathcal{S}'_n}$*;*
2) $S(t_0,t)^{-1} = S(t,t_0)$*;*
3) $S(t_0,t_1) \circ S(t_1,t_2) = S(t_0,t_2)$*.*

The following evident theorem shows the relation among operator-valued and Feynman propagators.

Theorem. *Let* $S : \mathbb{R}^2 \to {}^{\mathcal{S}}\mathrm{End}(\mathcal{S}'_{\mathrm{n}})$ *be an operator-valued function, and let*

$$G : \mathbb{R}^2 \to \mathcal{S}\left(\mathbb{R}^n, \mathcal{S}'_n\right),$$

be a family-valued function such that, for every $y \in \mathbb{R}^n$*,*

$$G(t_0,t)_y = S(t_0,t)\delta_y.$$

Then, S *is an operator-valued propagator if and only if* G *is a Feynman propagator.*

Definition. *Let* $u : \mathbb{R} \to \mathcal{S}'_n$ *be a process is* $\mathcal{S}'_n$*. The process* u *is said generated by an operator-valued function*

$$S : \mathbb{R}^2 \to {}^{\mathcal{S}}\mathrm{End}(\mathcal{S}'_{\mathrm{n}}),$$

if, for every pair of times t *and* t_0*, we have*

$$u(t) = S(t_0,t)u(t_0).$$

Theorem. *Let* $S : \mathbb{R}^2 \to {}^{\mathcal{S}}\mathrm{End}(\mathcal{S}'_{\mathrm{n}})$ *be an operator-valued propagator, and let* $u : \mathbb{R} \to \mathcal{S}'_n$ *be a process generated by* S. *Then, the function*

$$G : \mathbb{R}^2 \to \mathcal{S}\left(\mathbb{R}^n, \mathcal{S}'_n\right),$$

defined by

$$G(t_0, t)_y = S(t_0, t)\delta_y,$$

for every y *in* $\mathbb{R}^n$, *i.e., by* $G(t_0, t) = S(t_0, t)\delta$, *is a Green function for* u.

Proof. In fact, by the $\mathcal{S}$-linearity of $S(t_0, t)$ we have

$$u(t) = S(t_0, t) \int_{\mathbb{R}^n} u(t_0)\delta = \int_{\mathbb{R}^n} u(t_0) S(t_0, t)\delta. \blacksquare$$

Theorem. *Let the operator valued function*

$$S : \mathbb{R}^2 \to {}^{\mathcal{S}}\mathrm{End}\left(\mathcal{S}'_{\mathrm{n}}\right)$$

be of the form

$$S(t_0, t) = \exp(-i(t - t_0)H),$$

for some $\mathcal{S}$*-diagonalizable operator* H. *Then* S *is an operator-valued propagator.*

Proof. It's enough to prove that the Green function of S is a Feynman propagator. For every times t_0 and t, we have

$$\begin{aligned}
G(t_0, t) &= S(t_0, t)\delta \\
&\quad \text{(by definition of Green function)} \\
&= \exp(-i(t - t_0)H)\delta = \\
&\quad \text{(by assumption)} \\
&= \exp(-i(t - t_1 + t_1 - t_0)H)\delta = \\
&= \exp(-i(t - t_1)H) \circ \exp(-i(t_1 - t_0)H)(\delta) = \\
&= \exp(-i(t - t_1)H)(G(t_0, t_1)) = \\
&= \exp(-i(t - t_1)H)(\int_{\mathbb{R}^n} G(t_0, t_1)\delta) = \\
&\quad \text{(expanding in the Dirac basis)} \\
&= \int_{\mathbb{R}^n} G(t_0, t_1) \exp(-i(t - t_1)H)\delta = \\
&\quad \text{(by } \mathcal{S}\text{-linearity of } \exp(-i(t - t_1)H)) \\
&= \int_{\mathbb{R}^n} G(t_0, t_1) G(t_1, t) = \\
&= G(t_0, t_1) \cdot G(t_1, t). \blacksquare
\end{aligned}$$

5. The Feynman's propagator of a free particle

Let us evaluate the Green function of the evolution of a free particle. The operator-valued propagator, in this case, is of the form

$$S(t_0,t) = \exp(-\frac{i}{\hbar}(t-t_0)H),$$

where H is the $\mathcal{S}$-linear operator defined by

$$H = \frac{1}{2m}P^2,$$

where

$$P : \mathcal{S}'_1 \to \mathcal{S}'_1 : u \mapsto -i\hbar u'$$

is the momentum operator on $\mathcal{S}'_1$. Moreover, let φ be the Dirac-orthonormal standard eigenbasis of P, that is the following family of regular tempered distributions

$$\varphi = \left(\frac{1}{\sqrt{2\pi\hbar}}\left[e^{-\frac{i(p|\cdot)}{\hbar}}\right]\right)_{p\in\mathbb{R}}.$$

It's obvious that $P\varphi_p = p\varphi_p$, for every real p. We have, for every real q,

$$\begin{aligned}
G(0,t)_q &= \exp(-itH)(\delta_q) = \\
&= \exp(-itH)(\int_{\mathbb{R}^n}(\delta_q \mid \varphi)\varphi) = \\
&= \int_{\mathbb{R}^n}(\delta_q \mid \varphi)\exp(-itH)(\varphi) = \\
&= \int_{\mathbb{R}^n}[\delta_q \mid \varphi]\exp(-it\frac{(\cdot)^2}{2m})\varphi = \\
&= \int_{\mathbb{R}^n}\exp(-it\frac{(\cdot)^2}{2m})\left(\int_{\mathbb{R}^n}\delta_q\varphi^{-1}\right)\varphi = \\
&= \int_{\mathbb{R}^n}\exp(-it\frac{(\cdot)^2}{2m})\left(\int_{\mathbb{R}^n}\delta_q\overline{\varphi}\right)\varphi = \\
&= \int_{\mathbb{R}^n}\exp(-it\frac{(\cdot)^2}{2m})\overline{\varphi_q}\varphi.
\end{aligned}$$

The family φ is a regular family, and denoted by f_q the $\mathcal{S}$-function generating the regular tempered distribution φ_q, the function

$$\exp(-it\frac{(\cdot)^2}{2m})\overline{f_q}$$

is an $\mathcal{S}$-function (it the product of a bounded function by an $\mathcal{S}$-function). Hence the superposition

$$\int_{\mathbb{R}^n}\exp(-it\frac{(\cdot)^2}{2m})\overline{\varphi_q}\varphi$$

is a regular distribution of class $\mathcal{S}$; say g_q the generating $\mathcal{S}$-function, it's simple to see that

$$g_q(q') = \int_{\mathbb{R}^n}\exp(-it\frac{p^2}{2m})\overline{f_q}(p)f_p(q^{'})dp,$$

in fact, the superposition

$$[g_q] = \int_{\mathbb{R}^n} \exp(-it\frac{(\cdot)^2}{2m})\overline{\varphi_q}\varphi$$

is the Fourier transform of the tempered distribution

$$\exp(-it\frac{(\cdot)^2}{2m})\overline{\varphi_q},$$

and so g_p is the Fourier transform of the function

$$p \mapsto \exp(-it\frac{p^2}{2m})\overline{f_q}(p).$$

Substituting the expression of f, we have

$$\begin{aligned} g_q(q') &= \int_{\mathbb{R}^n} \exp(-it\frac{p^2}{2m})\overline{f_q}(p)f_p(q')dp = \\ &= \frac{1}{2\pi\hbar}\int_{\mathbb{R}^n} \exp(-it\frac{p^2}{2m})\exp(\frac{ipq}{\hbar})\exp(-\frac{ipq'}{\hbar})dp = \\ &= \frac{1}{2\pi\hbar}\int_{\mathbb{R}^n} \exp(-it\frac{p^2}{2m} + \frac{ipq}{\hbar} - \frac{ipq'}{\hbar})dp. \end{aligned}$$

The last integral is classic, after the standard calculation, we, at last, conclude

$$G(0,t)_q = [g_q] = \left[\left(\frac{m}{2\pi it}\right)^{1/2} e^{im((\cdot)-q)^2/2t}\right].$$

6. Feynman's theorem on transition amplitudes and one generalization

Definition (of transition amplitude). *Let S be the operator-valued propagator in $\mathcal{S}'_n$ of a process, let u_0 be a state (tempered distribution) of the process and t, t_0 two times. Assume W be an $\mathcal{S}$-observable of the system with a regular $\mathcal{S}$-eigenbasis w. The transition amplitude from condition (t_0, u_0) to the multicondition (t, w) is, by definition, the tempered distribution*

$$\langle w \mid S(t_0, t)u\rangle .$$

Interpretation. The Dirac scalar product $\langle w \mid S(t_0, t)u\rangle$ is the probability-distribution of transition amplitudes from the time-state (t_0, u) to each time state (t, w_i).

Theorem (the fundamental Feynman's relation for transition amplitudes). *Let V and W be two $\mathcal{S}$-observables (of a system) with $\mathcal{S}$-eigenbases v and w respectively. Let S be an operatorial propagator (of the system), and assume that S is with unitary values, i.e., assume that*

$$\langle S(t_1, t_0)w \mid u\rangle = \langle w \mid S(t_0, t_1)u\rangle ,$$

for every t_0, t_1 in T, and for every tempered distribution u.

Then, for every triple of times t_0, t_1, t_2 and for every state u, we have

$$\langle w \mid S(t_0, t)u\rangle = \int_{\mathbb{R}^n} \langle S(t, t_1)w \mid v\rangle \langle S(t_1, t_0)v \mid u\rangle .$$

Proof. We have

$$\begin{aligned}
\langle w \mid S(t_0,t)u\rangle &= \langle S(t,t_0)w \mid u\rangle = \\
&= \langle S(t,t_1)S(t_1,t_0)w \mid u\rangle = \\
&= \langle S(t,t_1)w \mid S(t_0,t_1)u\rangle = \\
&= \int_{\mathbb{R}^n} \langle S(t,t_1)w \mid v\rangle \langle v \mid S(t_0,t_1)u\rangle = \\
&= \int_{\mathbb{R}^n} \langle S(t,t_1)w \mid v\rangle \langle S(t_1,t_0)v \mid u\rangle \,. \blacksquare
\end{aligned}$$

Now we pass to the generalization. We first need a theorem.

Theorem. *Let $w \in \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$ and let $A : \mathcal{S}'_n \to \mathcal{S}'_n$ be an invertible $\mathcal{S}$-linear operator. Then, the following assertions hold true*

1) w is $\mathcal{S}$-linearly independent if and only if the family Aw is $\mathcal{S}$-linearly independent;

2) $^{\mathcal{S}}\mathrm{span(Aw)} = \mathrm{A}\left(^{\mathcal{S}}\mathrm{span(w)}\right)$;

3) if w is $\mathcal{S}$-linearly independent, for each $u \in A\left(^{\mathcal{S}}\mathrm{span(w)}\right)$, we have

$$[u \mid Aw] = [A^{-1}u \mid w].$$

Proof. 1) Let w be $\mathcal{S}$-linearly independent and let a belong to $\mathcal{S}'_m$ such that

$$\int_{\mathbb{R}^m} aA(w) = 0_{\mathcal{S}'_n}.$$

Applying A^{-1}, we obtain

$$0_{\mathcal{S}'_n} = A^{-1}0_{\mathcal{S}'_n} = A^{-1}\int_{\mathbb{R}^m} aA(w) = \int_{\mathbb{R}^m} aA^{-1}A(w) = \int_{\mathbb{R}^m} aw.$$

Since w is $\mathcal{S}$-linearly independent we deduce $a = 0_{\mathcal{S}'_n}$, and then Aw is $\mathcal{S}$-linearly independent too.

2) Let $u \in A\left(^{\mathcal{S}}\mathrm{span(w)}\right)$. Then, there exists an $a \in \mathcal{S}'_m$ such that $u = A\int_{\mathbb{R}^m} aw$. Thus, we have

$$u = \int_{\mathbb{R}^m} aAw,$$

so $u \in {}^{\mathcal{S}}\mathrm{span(Aw)}$, and hence $A\left(^{\mathcal{S}}\mathrm{span(w)}\right) \subseteq {}^{\mathcal{S}}\mathrm{span(Aw)}$.

Viceversa, let $u \in {}^{\mathcal{S}}\mathrm{span(Aw)}$. Then, there exists an $a \in \mathcal{S}'_m$ such that

$$u = \int_{\mathbb{R}^m} aAw,$$

and hence,

$$u = A\int_{\mathbb{R}^m} aw,$$

and hence $u \in A\left(^{\mathcal{S}}\mathrm{span(w)}\right)$, hence $^{\mathcal{S}}\mathrm{span(Aw)} \subseteq \mathrm{A}\left(^{\mathcal{S}}\mathrm{span(w)}\right)$. Concluding

$$^{\mathcal{S}}\mathrm{span(Aw)} = \mathrm{A}\left(^{\mathcal{S}}\mathrm{span(w)}\right).$$

3) For every $u \in {}^{\mathcal{S}}\mathrm{span(A^{-1}w)}$ e have

$$u = \int_{\mathbb{R}^m} [u \mid A^{-1}w]A^{-1}w,$$

applying A,

$$Au = \int_{\mathbb{R}^m} [u \mid A^{-1}w] A A^{-1} w = \int_{\mathbb{R}^m} [u \mid A^{-1}w] w,$$

so, Au belongs to ${}^{\mathcal{S}}\mathrm{span}(w)$ and

$$[Au \mid w] = [u \mid A^{-1}w]. \blacksquare$$

Theorem. *Let V and W be two observables of a system with $\mathcal{S}$ -eigenbases v and w respectively. Let S be an operator-valued propagator of the system. Then, for every triple of times t_0, t_1, t and, for every state u, we have*

$$[S(t_0, t)u \mid w] = \int_{\mathbb{R}^n} [S(t_0, t_1)u \mid v]\, [S(t_1, t)v \mid w]\,.$$

Proof. Applying the preceding theorem and the change of basis theorem

$$\begin{aligned} [S(t_0, t)u \mid w] &= [S(t_0, t_1)S(t_1, t)u \mid w] = \\ &= [S(t_0, t_1)u \mid S(t, t_1)w] = \\ &= \int_{\mathbb{R}^n} [S(t_0, t_1)u \mid v]\, [v \mid S(t, t_1)w] = \\ &= \int_{\mathbb{R}^n} [S(t_0, t_1)u \mid v]\, [S(t_1, t)v \mid w]\,. \blacksquare \end{aligned}$$

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David Carfì
Faculty of Economics, University of Bergamo
via Dei Caniana 2, Bergamo
Faculty of Economics, University of Messina
via dei Verdi 75, Messina
E-mail: davidcarfi@eniware.it

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