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ELASTIC WAVELETS AND THEIR APPLICATION TO PROBLEMS OF SOLITARY WAVE PROPAGATION

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ABSTRACT. The paper can be referred to that direction in the wavelet theory, which was called by Kaiser "the physical wavelets". He developed the analysis of first two kinds of physical wavelets - electromagnetic (optic) and acoustic wavelets. Newland developed the technique of application of harmonic wavelets especially for studying the harmonic vibrations. Recently Cattani and Rushchitsky proposed the 4th kind of physical wavelets - elastic wavelets. This proposal was based on three main elements: 1. Kaiser's idea of constructing the physical wavelets on the base of specially chosen (admissible) solutions of wave equations. 2. Developed by one of authors theory of solitary waves (with profiles in the form of Chebyshev-Hermite functions) propagated in elastic dispersive media. 3. The theory and practice of using the wavelet "Mexican Hat" system, the mother and father wavelets (and their Fourier transforms) of which are analytically represented as the Chebyshev-Hermite functions of different indexes. An application of elastic wavelets to studying the evolution of solitary waves of different shape during their propagation through composite materials is shown on many examples.

1. Introduction

One of interesting directions in the wavelet theory is associated with "physical wavelet". The last term was introduced by G. Kaiser [1, 2]. At present, four types of wavelets can be referred to physical wavelets. The first two kinds of physical wavelets, electromagnetic (optic) and acoustic wavelets were considered by Kaiser [3, 4]. Other physical wavelets, harmonic wavelets, were applied by Newland to the analysis of harmonic vibrations [5]. More recently the elastic wavelets [6–10] were proposed by Cattani and Rushchitsky as a new (wavelet based) tool for the analysis of wave propagation in elastic materials.

The main feature of this wavelets is that they are solution of some (physical) differential equations thus having a direct physical meaning.

In particular, *the electromagnetic (optic) wavelets* [3, 4] are so-called because they satisfy both the linear wave equations of optics and the main axioms of wavelet theory [5, 7, 11–17]. In particular, they are smooth, symmetric, they form a frame and have a compact support (or at least the finite weight).

The acoustic wavelets [3, 4] fulfill the linear wave equations in acoustics. They describe the so-called chirp-signals (practically used in many acoustic antennas systems) which is very similar to a wavelet. But for representing the chirp-signal by wavelets some additional conditions are needed.

The harmonic wavelets were proposed by Newland [5] (see also [18–20]) for studying the vibration processes. They can be obtained by some generalization of Shannon wavelets [13, 20–22]

$$\psi_{T,k}(x) = \left(\sin \frac{\pi}{T}(2x-k)\right) \Big/ \left(\frac{\pi}{T}(2x-k)\right) \tag{1}$$

which are real-valued wavelets, by transition from the real domain to the complex domain. The harmonic wavelet, wavelet family and family of scaling functions are [5, 18–20]

$$\psi(x) = \left(e^{i4\pi x} - e^{i2\pi x}\right) \big/ (i2\pi x),$$

$$\psi_{j,k}(x) = \psi(2^j x - k) = \left(e^{i4\pi(2^j x-k)} - e^{i2\pi(2^j x-k)}\right) \Big/ \left(i2\pi(2^j x-k)\right), \tag{2}$$

$$\varphi_{j,k}(x) = \varphi(2^j x - k) = \left(e^{i2\pi(2^j x-k)} - 1\right) \Big/ \left(i2\pi(2^j x-k)\right). \tag{3}$$

The elastic wavelets, proposed by Cattani and Rushchitsky in [6] (see also [7–10]) are based on three elements:

1. Kaiser's idea of constructing the physical wavelets as solutions of wave equations
2. The theory of solitary waves (with profiles in the form of Chebyshev-Hermite functions) propagation in elastic dispersive media [23–25].
3. The link between the wavelet "Mexican hat" (MH) system and the Chebyshev-Hermite functions [1, 26–30] of different indexes.

In our case, the elastic wavelet is a solution of the wave equation for linear elastic dispersive medium. The two-phase elastic mixture [9, 10, 23–25, 31–37] is used as the basic medium and the wave propagation equations in composites are considered. The solution can be represented through the Chebyshev-Hermite functions which are linked with MH-wavelets. Thus MH-wavelets are the first natural candidate as elastic wavelets.

Chebyshev-Hermite functions and their link with MH wavelets are given in sections 2 and 3. Section 4 deals with dispersive media and dispersive waves. Solitary elastic waves are described in sect. 5, and their representation in terms of MH wavelets is discussed in section 6 on some examples.

2. Chebyshev-Hermite functions

The functions named after P.L. Chebyshev and C. Hermite are the functions of [1, 26–30]

$$T_n(x) = e^{-z/2} H_n^*(x), \tag{4}$$

where $H_n^*(z)$ are the Chebyshev-Hermite polynomials

$$H_n^*(x) = 2^{n/2} H_n\left(\sqrt{2}x\right); \quad H_n(x) = (-1)^n e^{x^2/2} \frac{d^n}{dx^n}\left(e^{-x^2/2}\right) \tag{5}$$

These polynomials are determined by the generating function

$$e^{tx-\frac{t^2}{2}} = \sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!}.$$

The general form of polynomials is given by the formula

$$H_n(x) = x^n - \frac{1}{2}\frac{n(n-1)}{1}x^{n-2}+$$

$$+\frac{1}{2^2}\frac{n(n-1)(n-2)(n-3)}{1\cdot 2}x^{n-4} + ... = \sum_{j=0}^{j\le\frac{n}{2}} \frac{(-1)^j(2j)!}{2^j j!}\binom{n}{2j} x^{(n-2j)}$$

Recurrent relationships have the form [26, 30]

$$H_{n+1}(x) = 2x\,H_n(x) - 2n\,H_{n-1}(x)\ ; \qquad \frac{dH_n(x)}{dx} = 2n\,H_{n-1}(x)\ .$$

Let us write also some useful facts from the theory

$$\begin{array}{l} H_{2n}(-x) = H_{2n}(x)\ ; \quad H_{2n+1}(-x) = -H_{2n+1}(x); \\ H_0^*(z) = 1\ ; \quad H_1^*(z) = z\ ; \quad H_2^*(z) = z^2 - 1\ ; \quad H_3^*(z) = z^3 - 3z\ ; \\ H_4^*(z) = z^4 - 6z^2 + 3\ ; \quad H_5^*(z) = z^5 - 10z^3 + 15z. \end{array} \tag{6}$$

For real values of an argument the functions are real, and they are orthogonal with the weight 1 on the real axis $(-\infty, \infty)$

$$\int_{-\infty}^{\infty} \psi_m(x)\psi_n(x)dx = \left\{ \begin{array}{ll} 0 & m \neq n\ ; \\ 1 & m = n\ . \end{array} \right.$$

These functions are solutions of the Weber's differential equation [1, 26–30]

$$T_n^{''}(x) + (1 + 2n - x^2)T_n(x) = 0, \tag{7}$$

and polynomials $H_n^*(z)$ are solutions of the Chebyshev-Hermite differential equation

$$w^{''} - 2zw^{'} + 2nw = 0. \tag{8}$$

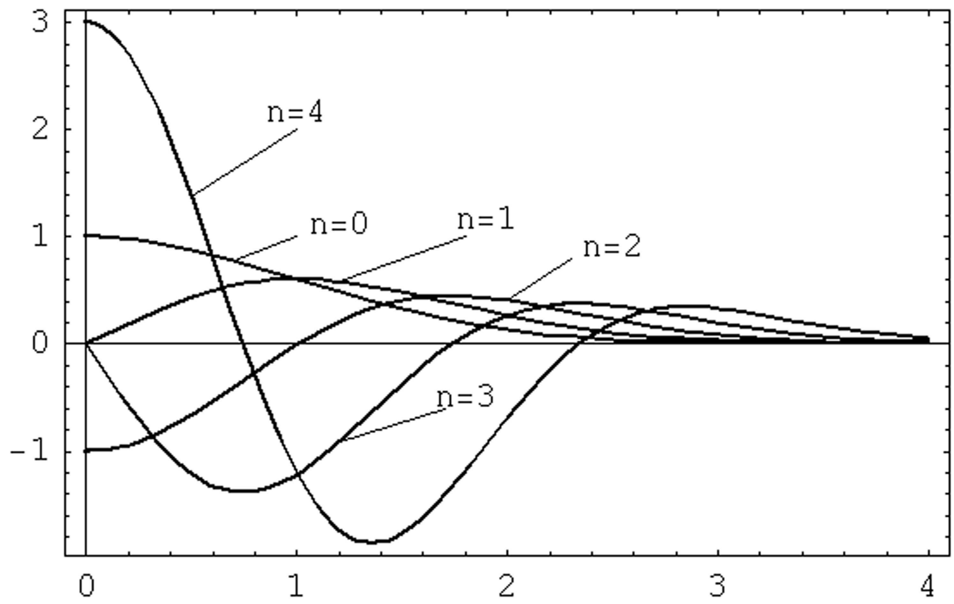

Fig.1 Chebyshev-Hermite functions

It is clear from Fig. 1 that functions $T_n\,(x)$ are defined over the infinite interval, but with a finite mass.

3. Mexican Hat (MH) wavelets and their link with Chebyshev-Hermite functions

The MH-wavelet (mother wavelet) and Fourier transformed MH-wavelet can be written as follows

$$\psi(x) = -\frac{2}{\sqrt[4]{\pi}\sqrt{3\sigma}}\left(x^2/\sigma^2 - 1\right)\, e^{-\frac{x^2}{2\sigma^2}}, \tag{9}$$

$$\hat{\psi}(\omega) = \frac{2\sqrt{2\sigma^5}\,\sqrt[4]{\pi}}{\sqrt{3}}\omega^2\, e^{-\frac{\sigma^2\omega^2}{2}}. \tag{10}$$

The scaling function (father wavelet) cannot be represented analytically, whereas its Fourier transform has the exact analytical representation (see Fig. 2, with $\sigma = 1$)

$$\hat{\varphi}(\omega) = \frac{2\sqrt[4]{\pi}\sqrt{\sigma}}{\sqrt{3}}\sqrt{\omega^2\sigma^2 + 1}\, e^{-\frac{\sigma^2\omega^2}{2}}. \tag{11}$$

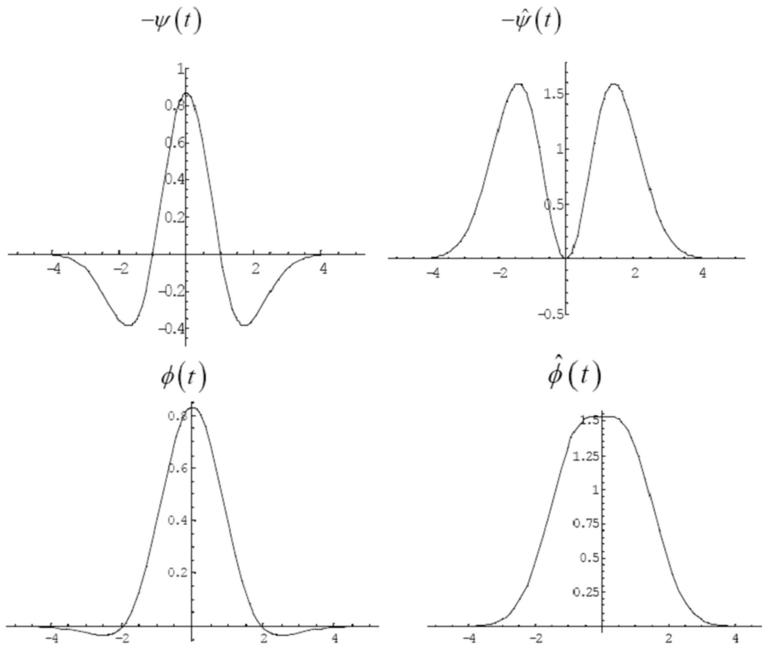

Fig.2 Mexican Hat: wavelet and scaling functions

Formulas (9),(10) can be rewritten through Chebyshev-Hermite functions

$$\psi(x) = \frac{1}{2\sqrt[4]{\pi}\sqrt{3\sigma}}\left(T_2\left(x/\sigma\right) - 2\,T_0\left(x/\sigma\right)\right), \tag{12}$$

$$\hat{\psi}(\omega) = \frac{\sqrt{\sigma^9}\sqrt[4]{\pi}}{\sqrt{6}}\left[T_2\left(\omega/\sigma\right) + 2\,T_0\left(\omega/\sigma\right)\right]; \tag{13}$$

being

$$T_0\left(x\right) = e^{-x^2/2}, \quad T_2\left(x\right) = e^{-x^2/2}\left(-2 + 4x^2\right).$$

Therefore the MH wavelets, on the LHS of Eqs. (12)-(13), can be expressed as linear combination of Chebyshev-Hermite functions on the RHS.

It is known (see e.g. [5, 7, 11–17]) that wavelets depend on two parameters: the translation and scale factor. Therefore, an efficient application of the MH wavelet family $\psi_{j,k}(x) = \psi\left(a^j x - u_o k\right)$ depends on the choice of the translation step $u_o$and the scale base a. The most convenient choice of u_o, a is when the system $\{\psi_{j,k}(x)\}_{j,k\in Z}$ forms a tight frame [7, 17]

$$\sum_{j,k\in Z} \left|\langle f, \psi_{j,k}\rangle\right|^2 = A\,\left\|f\right\|^2, \tag{14}$$

where $f(x)$ is a given function and is the frame bound. The quantity C_ψ is conventionally treated as the admissibility condition

$$C_\psi = \int_0^\infty \left[\left(\hat{\psi}(\omega)\right)^2 \Big/ \omega \right] d\omega < \infty.$$

In the following, we will take as wavelet $\psi(x) = \left(2/\sqrt{3}\right) \pi^{-1/4} \left(1 - x^2\right) e^{-x^2/2}$ so that $a = 2$ with the unit $u_o = 1.0$ or half $u_o = 0.5$ step of translation. There follows $A \approx 6.819$ or $A \approx 3.409$, $C_\psi = (4/3)\sqrt{\pi}$.

With this choice, an approximate (by a finite number of scales from j_o to j_{oo}) representation of a function-signal with the MH wavelets $\psi_{j,k}(x) = 2^{-j/2}\psi\left(2^{-j}x - 0.5k\right)$ is as follows

$$F(x) \approx (1/A) \sum_{j=j_o}^{j=j_{oo}} \sum_{k \in Z} d_{j,k}\psi_{j,k}(x), \quad d_{j,k} = \int_{-\infty}^{\infty} F(x)\psi_{j,k}(x)dx. \tag{15}$$

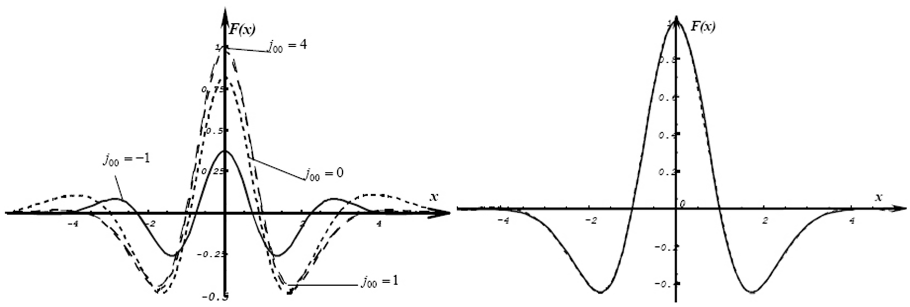

Fig.3 Wavelet representation

Let us take a function $F(x)$ symmetric, with a sufficiently fast decays to zero, localized in the origin, that is the effective support or bottom length (the interval in which the weight function is concentrated) is the interval $(-5, 5)$. The MH wavelet reconstruction of this function (Fig. 3), by using (15) can be done by a limited number of terms in the series expansion. In the example of Fig. 3, it was found to be sufficient to work with the scale levels from $j_o = -4$ to $j_{oo} = 4$ in order to get a good reconstruction of the function (with $j \in \{-4; -3; -2; -1; 0; 1; 2; 3; 4\}$and $k \in \{-20; -19; ...; 19; 20\}$). On Fig. 3 (left) the different approximations corresponding to the transition from the coarse scale $j_{oo} = -1$ to the sufficiently fine scale $j_{oo} = -4$ are shown. The approximations are given by

$$F_{-1} = \frac{1}{A} \sum_{j=-4}^{-1} \sum_{k=-20}^{20} d_{jk} 2^{-\frac{j}{2}} \psi\left(2^{-j}x - 0.5k\right),$$

$$F_0 = \frac{1}{A} \sum_{j=-4}^{0} \sum_{k=-20}^{20} d_{jk} 2^{-\frac{j}{2}} \psi\left(2^{-j}x - 0.5k\right),$$

$$F_1 = \frac{1}{A} \sum_{j=-4}^{1} \sum_{k=-20}^{20} d_{jk} 2^{-\frac{j}{2}} \psi \left(2^{-j} x - 0.5k\right),$$

$$F_4 = \frac{1}{A} \sum_{j=-4}^{4} \sum_{k=-20}^{20} d_{jk} 2^{-\frac{j}{2}} \psi \left(2^{-j} x - 0.5k\right).$$

Figure 3 (right) shows that the difference between the exact plot of the function $F(x)$ (solid line) and its approximation at the scale $(j_{oo} = -4)$ (dotted line) doesn't exceed 1%. Besides that, usually not the all set of wavelet coefficients $d_{j,k}$ are considered. In fact, according to the standard method of estimating by threshold coefficients, the small coefficients not exceeding certain given value ε, can be neglected. The method is based on the fact that wavelet coefficients for the sufficiently smooth function are very small for fine scales (the absolute value of wavelet coefficients $d_{j,k}$ depends on the local regularity of the function in the neighborhood of the point $2^j k$).

4. Dispersive media (composite materials) and dispersive waves

The area of application of MH wavelets is the propagation of elastic waves in composite materials. Elastic waves are thought as waves in elastic media. In our case these media are the models of composite materials, and they are not necessarily linear and one-phase [7, 23, 31–33, 37] the waves we will consider below are no-classical waves. Let us shortly remind the main characteristic of elastic waves.

The classical harmonic elastic waves (periodic, nondispersive) are described by the classical wave equation and they have (in the case of plane front) the analytical representation

$$u(x,t) = Ae^{ik(x - v_{ph} t)}, \quad (v_{ph} = \omega / k). \tag{16}$$

The last formula includes besides the phase velocity, the constant amplitude, as well as some additional parameters characterizing the periodicity, the wave number k or the frequency ω. These waves propagate with constant phase velocity and doesn't undergo during propagation any change in its sinusoidal profile. This type is the simplest one.

A second type of waves includes the classical elastic waves with arbitrary profile (D'Alembert waves). They have (in the case of propagation in positive direction) the analytical representation

$$u(x,t) = u_o (x - v_{ph} t). \tag{17}$$

D'Alembert waves propagate with a constant velocity and doesn't change its, initially given, profile. But arbitrariness of the profile stands out against this wave and holds up it on the special place among waves in materials.

A third kind of waves are the periodic harmonic dispersive elastic waves, i.e. waves with the phase velocity depending on the frequency (due to the dispersivity of the medium). Dispersive waves are defined by the nonlinear functional dependence of frequency on the wave number

$$\omega = W(k) \quad (W'(k) \neq 0). \tag{18}$$

This wave has as many modes as the roots of (18), or better $v_{ph}\, k - W(k) = 0$. The modes are meant as independently existing waves with distinct phase velocities. Each mode is characterized by its own dispersion curve, i.e. the graph of the dependence of the wave number (or the phase velocity) on frequency.

Owing to (18) there follows that particles in different points of the wave profile move with different phase velocities and the profile is deformed at some given time. Such understanding of the dispersion needs some further comments. In fact, the linear harmonic wave with a given frequency doesn't deform at all. The question is about the profile of the wave formed as the superposition of a few linear harmonic waves with closed frequencies. Just such a profile will be changed owing to dispersivity of the resulting wave:

$$u(x,t) = A\,e^{ik[x - v_{ph}(k)t]}\left(v_{ph}(k) = \frac{\omega}{k} = \frac{W(k)}{k}\right). \tag{19}$$

Some classical procedure are known for the analytical representation of dispersivity. Let us explain it on a few examples. Let us first consider the linear Klein-Gordon equation [7,23,31–33]

$$\rho\,\varphi''_{,tt} - \alpha^2\varphi''_{,xx} \pm \beta^2\varphi = 0, \tag{20}$$

which has the solution in the form of harmonic dispersive waves $\varphi(x,t) = A\,e^{i(kx-\omega t)}$. The dispersion law is written as $\omega = \pm\sqrt{\alpha^2k^2 \pm \beta^2}$, but sometimes it is convenient to write it as the dependence of a phase velocity on a wave number

$$v_{ph}\,(k) = \pm\sqrt{\alpha^2k^2 \pm \beta^2}\Big/k. \tag{21}$$

Let us take the second example from the structural theory of composite materials, i.e. the linear theory of two-phase mixture of materials. For this purpose, choose the plane wave in a linear elastic mixture as the most characteristic. The basic system of equations for plane waves has the form of three uncoupled systems, each of two coupled equations [7,23,31–33]

$$\rho_{\alpha\alpha}u^{(\alpha)}_{k,tt} - a^{(k)}_{\alpha}u^{(\alpha)}_{k,xx} - a^{(k)}_{3}u^{(\delta)}_{k,xx} - \beta\left(u^{(\alpha)}_{k} - u^{(\delta)}_{k}\right) = 0 \tag{22}$$

$$\left(\,a^{(1)}_{m} = \lambda_m + 2\mu_m\ ,\ a^{(2)}_{m} = a^{(3)}_{m} = \mu_m\,\right).$$

This system has a solution in the form of harmonic dispersive waves

$$u^{(\alpha)}_{m}(x_1,t) = A^{(\alpha)}_{om}e^{-i\left(k^{(m)}_{\alpha}x - \omega\ t\right)} + l(k^{(m)}_{\delta})A^{(\delta)}_{om}e^{-i\left(k^{(m)}_{\delta}x - \omega\ t\right)}. \tag{23}$$

The necessity for dispersive nonlinear dependence of wave numbers on frequency is (see e.g. [7])

$$M^{(m)}_1k^4 - 2M^{(m)}_2k^2\omega^2 + M^{(m)}_3\omega^4 = 0;$$

with

$$\begin{cases} M^{(m)}_1 = a^{(m)}_1a^{(m)}_2 - \left(a^{(m)}_3\right)^2 \\ 2M^{(m)}_2 = a^{(m)}_1\rho_{11} + a^{(m)}_2\rho_{22} - (a^{(m)}_1 + a^{(m)}_2 + 2a^{(m)}_3)\left(\dfrac{\beta}{\omega^2} + \rho_{12}\right) \\ M^{(m)}_3 = \rho_{11}\rho_{22} - (\rho_{11} + \rho_{22})\left(\dfrac{\beta}{\omega^2} + \rho_{12}\right) \end{cases}$$

The solutions are
(24)

$$k_{1,2}^2 = \frac{\omega^2}{M_1^{(m)}} \left\{ M_2^{(m)} \pm \sqrt{\left(M_2^{(m)}\right)^2 - M_1^{(m)} M_3^{(m)}} \right\} = \omega^2 \left\{ s_1 + \frac{s_2}{\omega^2} \pm \sqrt{s_3 + \frac{s_4}{\omega^2} + \frac{s_5}{\omega^4}} \right\}$$

and the coefficients of the matrix of amplitude distribution are determined by the simple algebraic formula

$$l(k_\alpha^{(m)}) = \left[-\frac{a_\alpha^{(m)} \left(k_\alpha^{(m)}\right)^2 + \beta - \rho_{\alpha\alpha}\omega^2}{a_3^{(m)} \left(k_\alpha^{(m)}\right)^2 - \beta} \right]^{(-1)^\alpha} .$$

5. Solitary elastic waves in dispersive (composite) materials

Let us consider now two new kind of waves: the solitary (nonperiodic) elastic waves with a given profile and with the phase velocity depending on the phase, and the simple elastic waves, with the phase velocity depending on the amplitude.

The last waves are a classical ones. They were described by Riemann and were intensively studied in the last two centuries [7, 23, 31–33]. The first waves were discovered and studied just recently. The solitary elastic waves arise in the microstructural theory of composite materials of the second order, the linear theory of elastic mixtures. Such medium of propagation of waves is dispersive one, and this propagation is regulated by the basic equations (22).

Let us explain the notion of solitary elastic waves on a Cauchy problem for (22). We will assume the solution be given at the initial time in the form of a solitary profile, that is a function mostly defined over a finite interval (function with a finite support) or almost located over the finite area (function with a finite weight). If this solution will hold its initial form at the next times (during some time interval) and the solution will depend on phase coordinate $z = x - v_{ph}t$, then we can say that the given profile is propagating as a solitary wave [24, 25].

From hyperbolicity of equations (22) follows that the initial finite pulse will propagate in the form of a wave, but the dispersivity will be weak for the most values of frequencies excepting some small area around the second mode cut frequency. This means that the pulse will be distorted slowly. The solitary elastic wave can exist, in a finite time-interval, only up to the limit when the interaction in the mixture is weak.

Let us consider the possibility of representing the solution of system (22) in the form of Chebyshev-Hermite functions: $u_1^{(\alpha)}(x,t) = A^{(\alpha)}\psi_0(z) \equiv A^{(\alpha)} e^{-z^2/2}$ This solution can be considered as the approximate one if the phase velocity is assumed as

(25)
$$v^{(\alpha)}(z) = \left(M_1 - (-1)^\alpha \sqrt{M_1^2 - M_2} \right)^{1/2},$$

$$M_1 = \frac{\lambda_1 + 2\mu_1}{\rho_{11}} + \frac{\lambda_2 + 2\mu_2}{\rho_{22}} - \frac{\rho_{11} + \rho_{22}}{\rho_{11}\rho_{22}} \frac{\beta}{1 + 2n - z^2},$$

$$M_2 = \frac{\lambda_1 + 2\mu_1}{\rho_{11}} \frac{\lambda_2 + 2\mu_2}{\rho_{22}} - \frac{(\lambda_3 + 2\mu_3)^2}{\rho_{11}\rho_{22}} +$$
$$+ \frac{(\lambda_1 + 2\mu_1) + (\lambda_2 + 2\mu_2) + 2(\lambda_3 + 2\mu_3)}{\rho_{11}\rho_{22}} \frac{\beta}{1 + 2n - z^2}.$$

So, the formula (25) expresses some new and inexpedient fact that the phase velocity of propagating through elastic composite material solitary elastic wave depends on the phase.

Now the solution of system (22) can be written in the form of two solitary waves with profiles described by Chebyshev-Hermite functions of arbitrary indices [24, 25]

$$u_1^{(\alpha)}(x,t) = A^{(\alpha)}\psi_n(z^{(\alpha)}) + p(z^{(\delta)})A^{(\delta)}\psi_n(z^{(\delta)}), \tag{26}$$

$$p(z^{(\alpha)}) = \left(\frac{a_3 + \beta\big/[1 + 2n - z^{(\delta)^2}]}{a_\alpha + v(z^{(\alpha)})\rho_{\alpha\alpha} - \beta\big/[1 + 2n - z^{(\alpha)^2}]} \right)^{(-1)^\alpha}.$$

The phase in (26) depends on an index, because only one (from the two distinct phase velocities $v_{ph}^{(\alpha)}$ in (24)) has to be used for writing the phases.

The representation $u_1^{(\alpha)}(x,t) = A^{(\alpha)}\psi_0(z) \equiv A^{(\alpha)}e^{-z^2/2}$ has been recently generalized and the initial profile is assumed in the form

$$u_1^{(\alpha)}(x,t) = A^{(\alpha)}\psi_0(z) \equiv A^{(\alpha)}e^{-z^2/2},$$

where the initially known function $F(x)$ can be taken in the form of the MH-function or some experimentally observed initial profiles of shock waves in materials [6–10].

So, we linked the MH wavelet with Chebyshev - Hermite functions, further we have shown that these functions can be chosen as the wave solution for some dispersive media. In our next steps we will concentrate on the MH wavelets. But we did notconsidered yet the question: will the translated and scaled Chebyshev-Hermite functions still be solutions of the basic mixture wave equations? The positive answer about a translation is obviously.

When we introduce the scaling factor k into the Chebyshev-Hermite function argument, then the basic formula (25) for phase velocities is still valid, but some of its components will be different

$$2M_1 = \frac{a_1}{\rho_{11}} + \frac{a_2}{\rho_{22}} - \frac{\rho_{11} + \rho_{22}}{\rho_{11}\rho_{22}} K(z), \tag{27}$$

$$M_2 = \frac{a_1}{\rho_{11}} \frac{a_2}{\rho_{22}} - \frac{a_3^2}{\rho_{11}\rho_{22}} - \frac{a_1 + a_2 + 2a_3}{\rho_{11}\rho_{22}} K(z), \quad K(z) = \frac{\beta k^2}{(1 + 2n) - z^2/k^2}.$$

The initial pulse in the composite material can't be chosen arbitrarily, if we try to take into account the microstructure (that is, we go out from the so-called long waves). Then the factor k can be treated as the length of the initial pulse bottom. The experience in studying the harmonic waves shows that the wave length must exceed the characteristic size of the microstructure (CSM) on one order (practically 7-10 times). This ensures the possibility of the continuum approach to the harmonic waves study. Otherwise, the microstructure can't be taken into account in the averaged form, and the two-phase material should be considered as the piecewise continuum.

Obviously, also in the case of solitary waves the exceeding the CSM by the bottom on one order should be considered as the threshold one, getting across which is interdicted. This threshold value gives, for computer modeling of the wave profile evolution, the smallest value for the bottom length and the possibility for the most displaying of evolution. The most value of the bottom length means as such one when the evolution is yet anything remarkable.

6. MH wavelets for the analysis of solitary waves in composite materials

A solitary wave might be considered as a signal with a finite bottom, therefore when this function is analytically expressed as an infinite series, we can approximate the profile with a finite sum. This can be easily achieved by a threshold for the wavelet coefficients.

The main idea, in the application of elastic wavelets to the evolution of solitary waves, consists not only in the wavelet representation of the wave, but also in the assumption that during the propagation (in the weakly dispersive medium) the initial profile will slowly change. Therefore, the wavelet coefficients are assumed to be constant and the distortion of the initial profile is described owing to the nonlinear change of the phase velocities.

The developed technique of application of elastic wavelets to the analysis of solitary waves in composite materials is very effective and permits to apply wavelets to waves with arbitrary initial profiles as well as to describe many wave effects arising in the process of solitary wave propagation.

Let us consider, for example, the case of an initial pulse in the form shown on Fig. 3 (right) and introduce the normalizing coefficient l responsible for the bottom length of the initial profile, so that the pulse practically vanishes when $\frac{|x|}{l} > 1$. The wavelet representation of the initial profile is assumed in the form

$$u\left(\frac{x}{l},0\right) = F\left(\frac{x}{l}\right) = \frac{1}{A}\sum_{j=-4}^{4}\sum_{k=-20}^{20} d_{jk}2^{-\frac{j}{2}}\psi\left(2^{-j}\frac{x}{l} - 0.5k\right), \tag{28}$$

$$d_{jk} = \frac{1}{l}\int_{-\infty}^{\infty} F\left(\frac{x}{l}\right)2^{-\frac{j}{2}}\psi\left(2^{-j}\frac{x}{l} - 0.5k\right)dx, \quad \psi(x) = \frac{2}{\sqrt{3}}\pi^{-\frac{1}{4}}\left(1 - x^2\right)e^{-\frac{x^2}{2}},$$

and the solution is

$$u(x,t) = (1/A)\sum_{j=-4}^{4}\sum_{k=-20}^{20}\left\{B_\alpha f_{(\alpha,j)} + p\left(z^{(\delta,j)}\big/l\right)B_\delta f_{(\delta,j)}\right\}, \tag{29}$$

$$f_{(\alpha,j)} = d_{jk}2^{-\frac{j}{2}}\psi\left[2^{-j}\left(z^{(\delta,j)}\big/l\right) - 0.5k\right], \quad z^{(\alpha,j)} = x - v_{ph}^{(\alpha,j)}t.$$

In the solution (29) the different phase velocities $v_{ph}^{(\alpha,j)}$ and different coefficients $p\left(z^{(\delta,j)}/l\right)$ corresponds to different scales

$$v^{(\alpha)}(z) = \sqrt{M_1 - (-1)^\alpha\sqrt{(M_1)^2 - M_2}}; \tag{30}$$

$$2M_1 = \frac{a_1}{\rho_{11}} + \frac{a_2}{\rho_{22}} - \frac{\rho_{11} + \rho_{22}}{\rho_{11}\rho_{22}} \cdot Z^{(l,j,k)};$$

$$M_2 = \frac{a_1}{\rho_{11}} \frac{a_2}{\rho_{22}} - \frac{a_3^2}{\rho_{11}\rho_{22}} - \frac{a_1 + a_2 + 2a_3}{\rho_{11}\rho_{22}} \cdot Z^{(l,j,k)};$$

(31) $$p\left(z^{(\alpha,j)} \Big/ l\right)^{(-1)^{\alpha}} = \frac{-a_\alpha + v^2\left(z^{(\alpha,j)}\right)\rho_{\alpha\alpha} - Z^{(l,j,k)}}{a_3 - Z^{(l,j,k)}};$$

$$Z^{(l,j,k)} = \frac{2^{2j} \cdot \beta \cdot l^2}{\left(2^{-j}\frac{z}{l} - k\right)^2 - \frac{2}{\left(2^{-j}\frac{z}{l} - k\right)^2 - 1} - 5}.$$

Thus, each term of the sum (29) characterized by the indices j and k , will have a corresponding phase velocity. This situation is similar to the signal decomposition by harmonic components, where the dispersion equation defines the phase velocity of each separate component. The velocities (30) have some common singularities in the inflexion points of wave profiles, going to infinity at the points $z_{j,k}^{\inf} = 2^j\left(\pm\sqrt{3 \pm \sqrt{6}} + k\right)$.

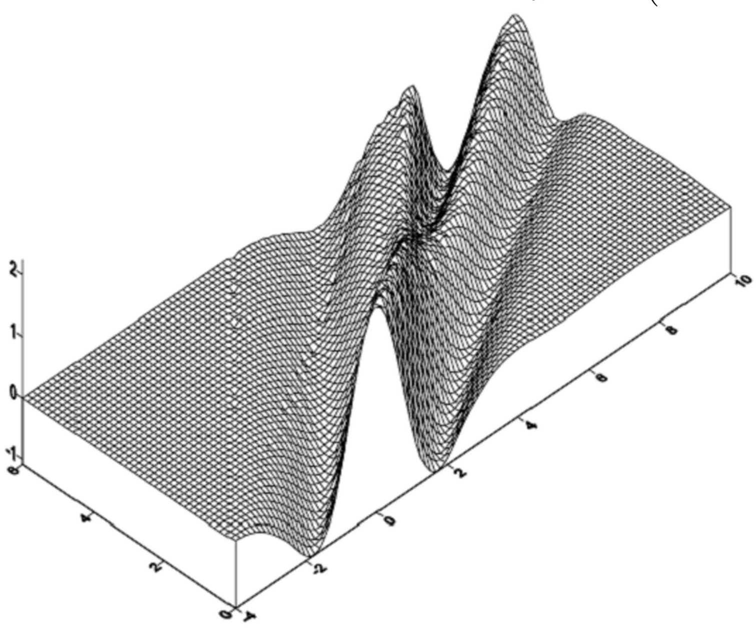

Fig.4 Break-up of the initial profile into two different waves

On Fig. 4 it is shown the break-up of the initial profile into two different waves with different profiles and different phase velocities. In the plot, the abscissa axis corresponds to distance x in cm, the ordinate axis corresponds to the time of propagation in μs, the applicate axis corresponds to the wave amplitude in 100 μm. The bottom length exceeds the characteristic length of structure ten times more.

In the next example, the initial profile is similar to the data list (Fig. 5) detected in some classical experiments on propagation of impacts in materials [2,38,39]. Experiments describe the impact on a long rod in different conditions: symmetrical impact of two rods, the impact using the Hopkinson bar and the impact obtained by using a falling hammer.

An excitation in a two-phase medium of the initial displacement impulse means that two identical displacement pulses are excited simultaneously in two phases of the medium. Usually each of initial waves is breaking up into two similar waves. As the initial pulse we take a profile similar to the one shown in Fig. 6 and approximated by the MH wavelets.

The initial impulse is shown on Fig. 6 by the solid line. The bottom lengths are varying from 6.48 cm to 6.48 mm. The dashed line corresponds to the approximate representation $f_\varepsilon(x)$.

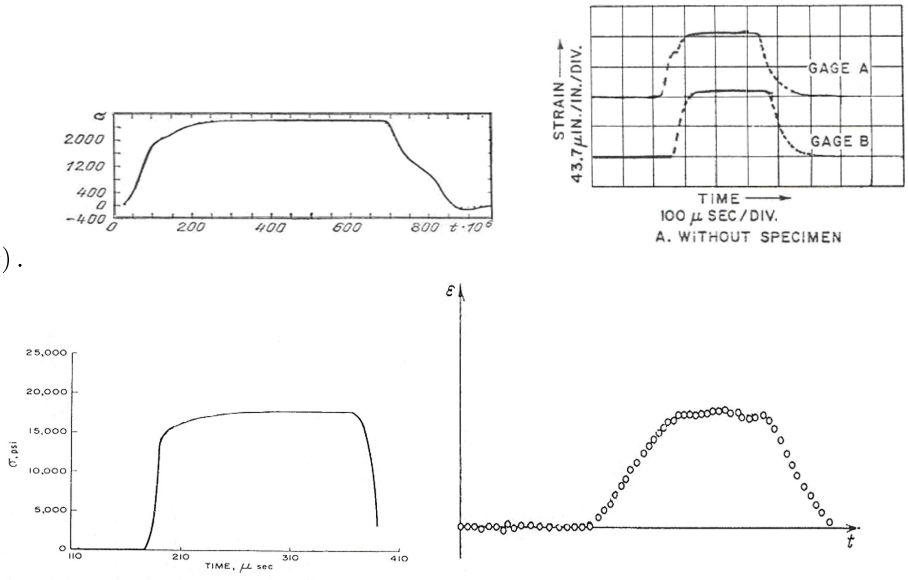

Fig.5 Experimental data on impacts in materials

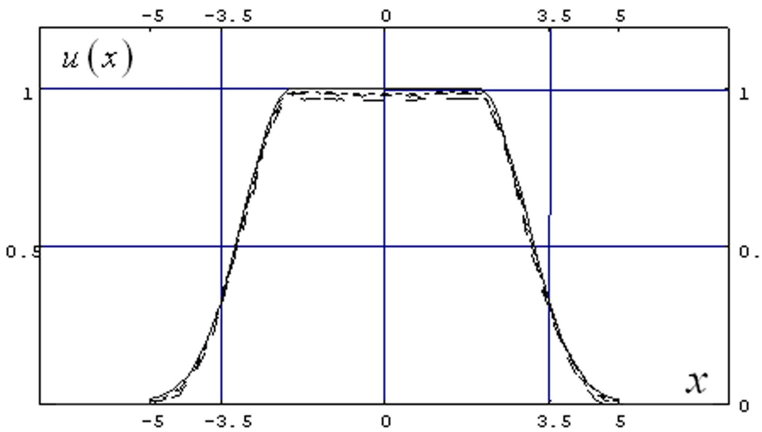

Fig.6 Initial pulse and its wavelet approximation

The wavelet approximation is obtained by neglecting those wavelet coefficients which absolute values are less than 10^{-4} and it is as follows

$$f_\varepsilon(x) \approx \frac{1}{A} \sum_{j=-2}^{9} \sum_{k=-14}^{14} d_{j,k} \psi_{j,k}(x) . \tag{32}$$

If the sum in (32) is carried out by the coefficients $|d_{j,k}| > \varepsilon$ and the number of neglected coefficients is n_0, then

$$\|f(x) - f_\varepsilon(x)\| < \varepsilon \, n_0^{1/2}. \tag{33}$$

In our case the total number of coefficients is 348 and the neglected coefficients 202. Then the estimate (33) is

$$\|f(x) - f_\varepsilon(x)\| < 1.42 \cdot 10^{-3}.$$

On Fig. 6, two more coarse approximations of $f(x)$ are shown (by dotted lines), corresponding to sums over j from - 2 to 8 and from - 2 to 7.

The propagating pulse, through the composite material, is represented as

$$u^{(\alpha)}(x,t) = \frac{1}{A} \sum_{j=-2}^{9} \sum_{k=-14}^{14} \left(B^{(\alpha)} \tilde{f}_{(\alpha,l)} + p\left(z^{(\delta,j)} \big/ l \right) B^{(\delta)} \tilde{f}_{(\delta,l)} \right), \tag{34}$$

$$\tilde{f}_{(\alpha,l)} = d_{j,k} 2^{j/2} \psi \left(2^j \left(z^{(\alpha,j)} \big/ l \right) - k \right), \quad z^{(\alpha)} = x - v_{ph}^{(\alpha)} t. \tag{35}$$

The plots of the evolution of the first phase of composite material (matrix) are shown in Fig. 7. Three cases of exceeding the wave bottom lengths of the characteristic length of microstructure are considered (100, 50, and 10 times more). Here the profiles are built for five successive moments (in μs): $t_{11} = 25$, $t_{12} = 51$,$t_{13} = 76$,$t_{14} = 100$,$t_{15} = 120$, $t_{21} = 12.5$, $t_{22} = 25$, $t_{23} = 37.5$, $t_{24} = 50$, $t_{25} = 62.5$, $t_{31} = 2.5$, $t_{32} = 5.1$, $t_{33} = 7.6$, $t_{34} = 10$, $t_{35} = 12$.

In the case of long bottoms the wave dispersion is displayed not right away similar to the case of long harmonic waves. The second mode is cut, and the first mode propagates without essential distortion (see the first picture of Fig. 7). For short bottoms the influence of the microstructure is easily displayed: the initial profile is broken up on two modes, propagating with different phase velocities (see the third picture of Fig.7).

In the process of evolution of solitary waves there are three characteristic stages: On the first stage, the profile moves for some time without essential distortions; at the second stage, there is a splitting of the signal into two ones propagating with differing velocities. The profile is distorted on the rise-up portion, after that the separation of the second impulse is started. The final stage sees the separate motion of both impulses with different velocities and amplitudes.

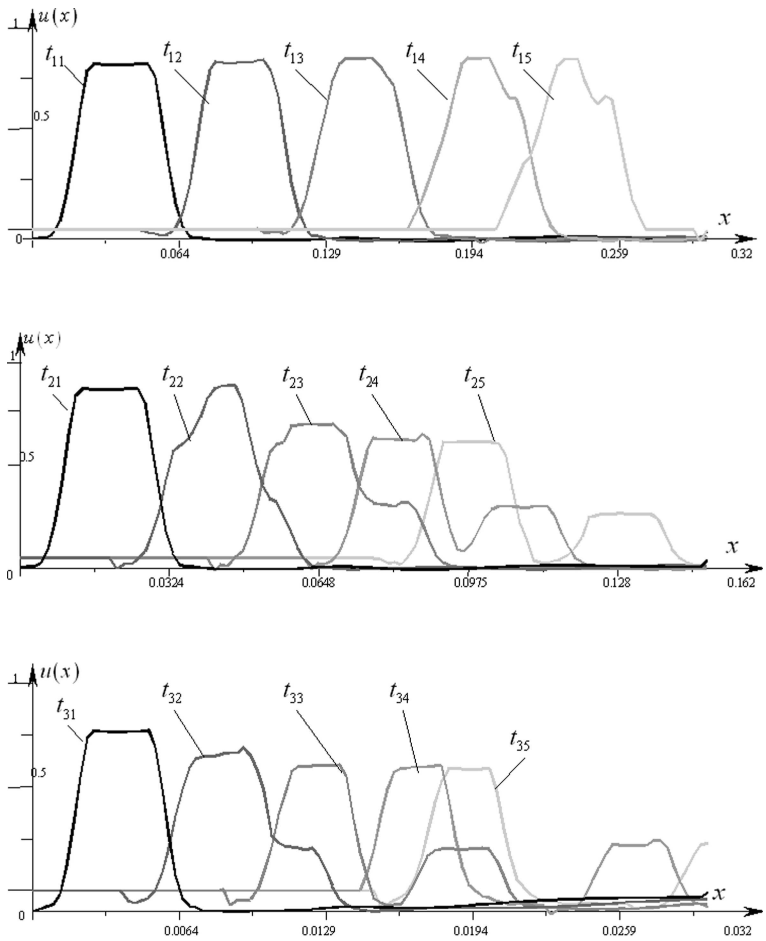

Fig.7 Evolution of the pulse

All three stages are present for arbitrary choice of the bottom, but they have different durability. For the case of long bottom the first stage is displayed during motion on the distance of three bottoms. For middle bottoms and for the same distances, the second stage is already characteristic. For short bottoms these two stages are displayed on the distance in two bottoms.

Conclusions

We considered in this paper the elastic wavelets for studying the evolution of solitary profiles. These wavelets are based on the "Mexican hat" wavelet system, which can be represented using the Chebyshev-Hermite functions of different indexes. The essential peculiarity of the elastic wavelets is that they are solutions of the corresponding wave equations for elastic medium, thus showing that some wavelets are solutions of physical-mechanical problems.

The application of elastic wavelets to studying the evolution of solitary waves of different shape during their propagation through composite materials shows on many examples the high capability to describe the solitary elastic wave evolution.

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