DOI:10.1478/C1A0601007

*Atti dell'Accademia Peloritana dei Pericolanti*
*Classe di Scienze Fisiche, Matematiche e Naturali*
*Vol. LXXXIV, C1A0601007 (2006)*
*Adunanza del 28 novembre 2005*

# CONDITIONS ($S_q$) AND ($G_q$) ON GRADED RINGS

MONICA LA BARBIERA
(Nota presentata dal Socio Ordinario Gaetana Restuccia)

ABSTRACT. Let $R$ be a commutative noetherian graded ring. We study Serre's condition and Ischebeck-Auslander's condition in the graduate case. Let $q$ be an integer, we characterize the $^*q$-torsion freeness of graded modules on a $^*G_q$-ring.

## 1. Introduction

Serre's condition ($S_q$) and Ischebeck-Auslander's condition ($G_q$) have been studied for a commutative noetherian ring $R$ ([8]). These conditions have an important role for studying properties of $R$, such as the regularity and the normality ([7]). The Ischebeck-Auslander's condition is less known, it is introduced by Ischebeck in 1967 and it is very interesting to study the $q$-torsion of finitely generated $R$-modules ([5]).
A noetherian ring that satisfies the Ischebeck-Auslander's condition is said $G_q$-ring. In [4] the author observes that the $G_2$-rings are a larger class than the class of the integrally closed rings. In [6] it is proved that if a finitely generated $R$-module $M$ satisfies Samuel's condition then $M$ is $q$-torsion free if $R$ is a $G_q$-ring.
The aim of this paper is to investigate these properties in the graduate case, that is introduced in [3].
Let $R$ be a graded ring and $I$ be an arbitrary ideal of $R$. $I^*$ is the graded ideal generated by all homogeneous elements of $I$ and $I^*$ is the largest graded ideal contained in $I$. In particular, for each prime ideal $\wp \subset R$, the graded ideal $\wp^*$ is a prime ideal too. In [2] there are theorems that link theoretic properties of $\wp$ and $\wp^*$ by localizations of finitely generated graded $R$-modules, that is $dimM_{\wp} = dimM_{\wp^*} + 1$ and $depthM_{\wp} = depthM_{\wp^*} + 1$ for all not graded prime ideal $\wp$ of $R$.

In the section 1 of this paper we consider a graded ring $R$ and we introduce the conditions ($^*S_q$) and ($^*G_q$) generalizing the classic conditions for graded prime ideals $\wp^*$ and we study connections with the classic conditions ($S_q$) and ($G_q$).
Moreover, the Ischebeck-Auslander's condition ($^*G_q$) is linked to the $^*q$-torsion freeness of graded $R$-modules. More precisely, a finitely generated graded $R$-module $M$ is $^*q$-torsion free if each homogeneous $R$-sequence of length $q$ is a homogeneous $M$-sequence. In the second part of this work we give a characterization of the $^*q$-torsion freeness of graded modules on $^*G_q$-rings. This characterization is studied in [8] for not graded rings. We find some properties related to the $^*q$-torsion freeness for graded modules on $^*G_q$-rings.

We like to thank Professor Gaetana Restuccia for useful discussions about the results of this work.

## 2. Conditions ($^*S_q$) and ($^*G_q$)

Let $R$ be a commutative noetherian graded ring. In [2] there are some definitions related to the graded ideals of $R$ and some properties on the dimension and the depth of some localizations of finitely generated graded $R$-modules.

**Definition 2.1.** *Let $I \subset R$ be an ideal (not necessarily graded). $I^*$ is the graded ideal of $R$ generated by all the homogeneous elements of I.*

The ideal $I^*$ is the largest graded ideal contained in $I$.

**Remark 2.1.** *Let $R$ be a commutative noetherian graded ring and $I \subset R$ be an ideal. If I is homogeneous, then $I = I^*$.*

**Remark 2.2.** *Let $R$ be a commutative noetherian graded ring and $\wp \subset R$ be a prime ideal, then $\wp^*$ is a prime ideal too (*[2]*, 1.5.6).*

**Example 2.1.** *Let $I = (X^2 - 1, XY - 1) \subset R = K[X, Y]$. The homogeneous elements of I of degree i are $f \in I_i = I \cap R_i$. We have:*

$$I_0 = I \cap R_0 = (0)$$

$$I_1 = I \cap R_1 = (0)$$

$$I_2 = I \cap R_2 = \langle X^2 - XY \rangle,$$

*in fact $f = X^2 - 1 - (XY - 1) = X^2 - XY \in I_2$. In general, for $i > 2$*

$$I_i = I \cap R_i = \langle g(X^2 - XY) \rangle,$$

*with $\deg(g) = i - 2$. It follows that $I^* = (X^2 - XY) \subset I$.*

**Proposition 2.1.** *Let $R$ be a commutative noetherian graded domain and $I = (f) \subset R$ be a principal ideal. $I^* = (0) \Longleftrightarrow f$ is not a homogeneous element.*

**Proof:** $\Rightarrow$) If $I^* = (0)$, then there aren't homogeneous elements in $I$, by definition of $I^*$. It follows that the generator $f \in I$ is not a homogeneous element.
$\Leftarrow$) Let $I = (f)$, with $f$ not homogeneous element. We prove that in $I$ there aren't homogeneous elements. Each element $g \in I$ is written as $g = af$, $a \in R$. We prove that: if $f$ is not homogeneous then $g = af$ is not a homogeneous element of $R$, $\forall a \in R$. We can suppose $f = f_i + f_{i+1}$, $i \geq 1$. We have two cases.
I) $a \in R$ is a homogeneous element: $a = a_j \in R_j$ with $a \neq 0$ and $f = f_i + f_{i+1}$ with $f_i, f_{i+1} \neq 0$, $f_i \in R_i$, $f_{i+1} \in R_{i+1}$. We have $af = a_j f_i + a_j f_{i+1}$, where $a_j f_i \in R_{j+i}$, $a_j f_{i+1} \in R_{j+i+1}$ and $a_j f_i \neq 0$, $a_j f_{i+1} \neq 0$. It follows that $g = af \in I$ is not homogeneous.
II) $a \in R$ is not homogeneous: $a = a_{i_1} + \cdots + a_{i_n}$, $0 \leq i_1 < \ldots < i_n$, where $a_{i_j} \in R_{i_j}$ with $a_{i_1}, \cdots, a_{i_n} \neq 0$ and $f = f_i + f_{i+1}$ with $f_i, f_{i+1} \neq 0$, $f_i \in R_i$, $f_{i+1} \in R_{i+1}$. We have: $af = a_{i_1} f_i + \cdots + a_{i_n} f_i + a_{i_1} f_{i+1} + \cdots + a_{i_n} f_{i+1}$, where $a_{i_1} f_i \in R_{i_1+i}$, $a_{i_n} f_{i+1} \in R_{i_n+i+1}$, $a_{i_1} f_i \neq 0$ and $a_{i_n} f_{i+1} \neq 0$ because they are the only elements in

their degree. It follows that $g = af \in I$ is not homogeneous $\forall a$ not homogeneous.
$I^* = (0)$.

The previous proposition is not true if the ring $R$ is not a domain as the following example shows.

**Example 2.2.** *Let $R = \mathbb{Z}_6[X]$ be the graded ring with standard graduation and consider $I = (X - 2)$ the ideal of $R$ generated by a not homogeneous element. In $I$ there are homogeneous elements, hence $I^* \neq (0)$. In fact, let $a = 3X \in R_1$ be a homogeneous element of $R$, we have $af = 3X^2 \in I$ that is a homogeneous element of degree two. We have $I^* = (3X^2)$.*

**Remark 2.3.** *Let $R$ be graded ring that is not a domain and $\wp = (f)$ a prime ideal of $R$ generated by a not homogeneous element. Since $\wp^*$ is a prime ideal ([2], 1.5.6), it follows that $\wp^* \neq (0)$ because $(0)$ is not prime if $R$ is not a domain. Hence in the Proposition **2.1** the condition $\Leftarrow)$ is not true.*

Now we investigate some conditions on commutative noetherian rings in the graduate case. In [8] the Serre's condition $(S_q)$ is studied.

**Definition 2.2.** *Let $R$ be a commutative noetherian ring, q be an integer. $R$ satisfies Serre's condition $(S_q)$ if for all prime ideal $\wp$ of $R$*

$$\mathrm{depth} R_\wp \geq \min\{\mathrm{q}, \dim R_\wp\}.$$

In connection to the classical condition $(S_q)$, we are able to give the following properties for prime ideals of graded rings.

**Definition 2.3.** *Let $R$ be a commutative noetherian graded ring, q be an integer. $R$ satisfies Serre's condition $(^*S_q)$ if for all prime ideal $\wp$ of $R$*

$$\mathrm{depth} R_{\wp^*} \geq \min\{\mathrm{q}, \dim R_{\wp^*}\}.$$

**Definition 2.4.** *Let $R$ be a commutative noetherian graded ring, q be an integer. $R$ satisfies Serre's condition $(\widetilde{S_q})$ if for all not graded prime ideal $\wp$ of $R$*

$$\mathrm{depth} R_\wp \geq \min\{\mathrm{q}, \dim R_\wp\}.$$

In [3](Cor.1.3) it is proved that a graded ring $R$ satisfies $(S_q)$ if and only if for all homogeneous prime ideal $\wp$ of $R$ $\mathrm{depth} R_\wp \geq \min\{\mathrm{q}, \dim R_\wp\}$. Now we reformulate the same result in terms of the condition $(^*S_q)$ and also the proof of this result uses the condition $(^*S_q)$.

**Theorem 2.1.** *Let $R$ be a commutative noetherian graded ring, q be an integer.*
*$R$ satisfies the condition $(S_q) \Leftrightarrow R$ satisfies the condition $(^*S_q)$.*

**Proof:** $\Leftarrow)$ Let $\wp$ be a not graded prime ideal, we have the relations ([2], 1.5.8 and 1.5.9):

$$\dim R_{\wp^*} = \dim R_\wp - 1$$
$$\mathrm{depth} R_{\wp^*} = \mathrm{depth} R_\wp - 1,$$

then: $\mathrm{depth} R_{\wp} = \mathrm{depth} R_{\wp^*} + 1 \geq \min\{\mathrm{q}, \dim R_{\wp^*}\} + 1 \geq \min\{\mathrm{q}, \dim R_{\wp^*} + 1\} = \min\{\mathrm{q}, \dim R_{\wp}\}$, that is the condition $(S_q)$.
$\Rightarrow$) It is trivial.

**Remark 2.4.** *For a graded ring $R$ we have:*
*$(S_q) \Rightarrow (\widetilde{S_q})$, since in the Definition* ***2.2*** *$\wp$ is an arbitrary prime ideal.*
*In general, $(\widetilde{S_q})$ does not imply $(S_q)$. It follows by previous definitions.*

Now we study the connection between the conditions $({}^*S_q)$ and $(\widetilde{S_q})$.

**Remark 2.5.** *Let $R$ be a commutative noetherian graded ring, $q$ be an integer.*
*1) $R$ satisfies the condition $({}^*S_q) \Longrightarrow R$ satisfies the condition $(\widetilde{S_{q+1}})$.*
*2) $R$ satisfies the condition $(\widetilde{S_{q+1}}) \Longrightarrow \forall \wp$ not graded prime ideal of $R$ $\mathrm{depth} R_{\wp^*} \geq \min\{\mathrm{q}, \dim R_{\wp^*}\}$.*

**Proof:** 1) If $\wp$ is not a graded prime ideal, we have the relations ([2]):

$$\dim R_{\wp^*} = \dim R_{\wp} - 1$$

$$\mathrm{depth} R_{\wp^*} = \mathrm{depth} R_{\wp} - 1.$$

Because $R$ satisfies the condition $({}^*S_q)$ one has

$$\mathrm{depth} R_{\wp^*} \geq \min\{\mathrm{q}, \dim R_{\wp^*}\}.$$

Using the previous relations we have $\mathrm{depth} R_{\wp} \geq \min\{\mathrm{q}, \dim R_{\wp} - 1\} + 1$, it follows $\mathrm{depth} R_{\wp} \geq \min\{\mathrm{q}+1, \dim R_{\wp}\}$, for all not graded prime ideal $\wp \in \mathrm{Spec}(R)$,
that is $R$ satisfies the condition $(\widetilde{S_{q+1}})$.
These relations are not valid also for graded ideals, so $R$ satisfies only the condition $(\widetilde{S_{q+1}})$ and not $(S_{q+1})$.
2) We suppose that $R$ satisfies the condition $(\widetilde{S_{q+1}})$, that is
$\mathrm{depth} R_{\wp} \geq \min\{\mathrm{q}+1, \dim R_{\wp}\}$, for all not graded prime ideal $\wp \in \mathrm{Spec}(R)$. We have the relations:

$$\dim R_{\wp} = \dim R_{\wp^*} + 1$$

$$\mathrm{depth} R_{\wp} = \mathrm{depth} R_{\wp^*} + 1,$$

it follows $\mathrm{depth} R_{\wp^*} \geq \min\{\mathrm{q}+1, \dim R_{\wp^*}+1\} - 1$, hence: $\mathrm{depth} R_{\wp^*} \geq \min\{\mathrm{q}, \dim R_{\wp^*}\}$ for all not graded prime ideal $\wp \in \mathrm{Spec}(R)$.
This condition is not $({}^*S_q)$ because the previous relations are not valid for graded prime ideals.

In [8] it is studied the condition $(G_q)$ of Ischebeck-Auslander.

**Definition 2.5.** *Let $R$ be a commutative noetherian ring and $q > 0$ be an integer. $R$ satisfies the condition $(G_q)$ of Ischebeck-Auslander (that is $R$ is a $G_q$-ring) if:*
*1) $R$ satisfies the condition $(S_q)$;*
*2) For all prime ideal $\wp$ of $R$ such that $\dim R_{\wp} < q$, $R_{\wp}$ is a Gorenstein ring.*

For a graded ring we give the following definitions.

**Definition 2.6.** *Let $R$ be a commutative noetherian graded ring and $q > 0$ be an integer. $R$ satisfies the condition ($^*G_q$) of Ischebeck-Auslander (that is $R$ is a $^*G_q$-ring) if:*
*1) $R$ satisfies the condition ($^*S_q$);*
*2) For all prime ideal $\wp$ of $R$ such that $\dim R_{\wp^*} < q$, $R_{\wp^*}$ is a Gorenstein ring.*

**Definition 2.7.** *Let $R$ be a commutative noetherian graded ring and $q > 0$ be an integer. $R$ satisfies the condition ($\widetilde{G_q}$) of Ischebeck-Auslander (that is $R$ is a $\widetilde{G_q}$-ring) if:*
*1) $R$ satisfies the condition ($\widetilde{S_q}$);*
*2) For all not graded prime ideal $\wp$ of $R$ such that $\dim R_\wp < q$, $R_\wp$ is a Gorenstein ring.*

In [3](Cor.1.3) it is proved that a graded ring $R$ satisfies $(G_q)$ if and only if for all homogeneous prime ideal $\wp$ of $R$ such that $\dim R_\wp < q$, $R_\wp$ is a Gorenstein ring. Now we reformulate the same result in terms of the condition ($^*G_q$).

**Theorem 2.2.** *Let $R$ be a commutative noetherian graded ring, $q > 0$ be an integer. $R$ satisfies the condition ($G_q$) $\Leftrightarrow$ $R$ satisfies the condition ($^*G_q$).*

**Proof:** $\Leftarrow$) By Theorem **2.1** $(^*S_q) \Rightarrow (S_q)$.
For all graded prime ideals $R$ satisfies the condition $(G_q)$. Let $\wp$ a not graded prime ideal such that $\dim R_\wp < q$. Then $\dim R_{\wp^*} = \dim R_\wp - 1 < q$, hence by hypothesis $R_{\wp^*}$ is a Gorenstein ring. By [2](§3.6) $R_{\wp^*}$ is a Gorenstein ring implies that $R_\wp$ is a Gorenstein ring.
$\Rightarrow$) It is trivial.

**Remark 2.6.** *For a graded ring $R$ we have:*
*($G_q$) $\Rightarrow$ ($\widetilde{G_q}$), since in the condition $(G_q)$ $\wp$ is an arbitrary prime ideal.*
*($\widetilde{G_q}$) does not imply ($G_q$). It follows by previous definitions.*

**Remark 2.7.** *Let $R$ be a commutative noetherian graded ring and $q > 0$ be an integer.*
*1) $R$ is a $^*G_q$-ring $\Rightarrow$ $R$ is a $\widetilde{G_{q+1}}$-ring.*
*2) If $R$ is a $\widetilde{G_{q+1}}$-ring then:*
*i) $\forall$ $\wp$ not graded prime ideal of $R$ $\mathrm{depth} R_{\wp^*} \geq \min\{\mathrm{q}, \dim R_{\wp^*}\}$;*
*ii) For all not graded prime ideal $\wp$ of $R$ such that $\dim R_{\wp^*} < q$, $R_{\wp^*}$ is a Gorenstein ring.*

**Proof:** 1) We suppose that $R$ is a $^*G_q$-ring. By Remark **2.5**, if $R$ satisfies the condition $(^*S_q)$ then $R$ satisfies the condition $(\widetilde{S_{q+1}})$. As $R$ is a $^*G_q$-ring we have: for all prime ideal $\wp$ of $R$ such that $\dim R_{\wp^*} < q$, $R_{\wp^*}$ is a Gorenstein ring. If $\wp$ is not graded, we have $\dim R_\wp = \dim R_{\wp^*} + 1 < q + 1$, and $R_{\wp^*}$ Gorenstein $\Rightarrow$ $R_\wp$ Gorenstein ([2], §3.6). Then the previous condition become: for all not graded prime ideal $\wp$ of $R$ such that $\dim R_\wp < q + 1$, $R_\wp$ is a Gorenstein ring. Hence $R$ is a $\widetilde{G_{q+1}}$-ring.
2) We suppose that $R$ is a $\widetilde{G_{q+1}}$-ring.
By Remark **2.5**(2), if $R$ satisfies the condition $(\widetilde{S_{q+1}})$, then $\forall \wp$ not graded prime ideal of $R$ $\mathrm{depth} R_{\wp^*} \geq \min\{\mathrm{q}, \dim R_{\wp^*}\}$, that is $i$).
If $\wp$ is not graded, then: $\dim R_{\wp^*} = \dim R_\wp - 1$, and $R_\wp$ Gorenstein $\Rightarrow$ $R_{\wp^*}$ Gorenstein ([2], §3.6).
By hypothesis, for all not graded prime ideal $\wp$ of $R$ such that $\dim R_\wp < q + 1$, $R_\wp$ is a Gorenstein ring, so we have: for all not graded prime ideal $\wp$ of $R$ such that $\dim R_{\wp^*} =$

$\dim R_{\wp} - 1 < q$, $R_{\wp^*}$ is a Gorenstein ring, that is $ii)$.
These conditions do not mean that $R$ is a ${}^*G_q$-ring because the relations used are not valid for graded prime ideals.

**Example 2.3.** *Let $R = K[X_1, \ldots, X_n]$ be the polynomial ring over a field $K$. $R$ is a graded ring with standard graduation.*
*$R$ is a ${}^*G_q$-ring for all $q > 0$ because it is a regular ring.*

**Definition 2.8.** *Let $R$ be a commutative noetherian graded ring, $q$ be an integer. A graded finitely generated $R$-module $E$ satisfies Serre's condition (${}^*S_q$) if for all prime ideal $\wp$ of $R$*

$$\mathrm{depth} E_{\wp^*} \geq \min\{\mathrm{q}, \dim E_{\wp^*}\}.$$

**Definition 2.9.** *Let $R$ be a commutative noetherian graded ring and $q > 0$ be an integer. A graded finitely generated $R$-module $E$ satisfies the condition (${}^*G_q$) of Ischebeck-Auslander (that is $E$ is a ${}^*G_q$-module) if:*
*1) $E$ satisfies the condition (${}^*S_q$);*
*2) For all prime ideal $\wp \in Supp(E)$ such that $\dim E_{\wp^*} < q$, $E_{\wp^*}$ is a Gorenstein $R_{\wp^*}$-module.*

**Theorem 2.3.** *Let $R$ be a commutative noetherian graded ring, $q$ be an integer.*
$1)$ *$E$ satisfies the condition (${}^*S_q$) $\Leftrightarrow$ $E$ satisfies the condition ($S_q$), $q \geq 0$.*
$2)$ *$E$ satisfies the condition (${}^*G_q$) $\Leftrightarrow$ $E$ satisfies the condition ($G_q$), $q > 0$.*

**Proof:** (See Theorems **2.1** and **2.2**).

**Proposition 2.2.** *Let $R$ be a commutative noetherian graded ring, $q > 0$ be an integer and $E$ be a finitely generated $R$-module. The following conditions are equivalent:*
*1) $E$ satisfies the condition (${}^*G_q$);*
*2) For all prime ideal $\wp$ of $R$ such that $\mathrm{depth} E_{\wp^*} < q$, $E_{\wp^*}$ is a graded $R_{\wp^*}$-module of Gorenstein.*

**Proof:** $1) \Rightarrow 2)$ By hypothesis $E$ satisfies (${}^*G_q$). We suppose there exists $\wp$ of $R$ such that $\mathrm{depth} E_{\wp^*} < q$ and $E_{\wp^*}$ is not a Gorenstein $R_{\wp^*}$-module, then by hypothesis (see Definition **2.9**) $\dim E_{\wp^*} \geq q$. Moreover $E$ satisfies the condition (${}^*S_q$), then $\mathrm{depth} E_{\wp^*} \geq \min\{\mathrm{q}, \dim E_{\wp^*}\}$. But $\dim E_{\wp^*} \geq q$, so we have:
$\mathrm{depth} E_{\wp^*} \geq \min\{\mathrm{q}, \dim E_{\wp^*}\} = q \Rightarrow \mathrm{depth} E_{\wp^*} \geq q$.
Hence we obtain a contradiction. It follows that $E_{\wp^*}$ is a graded Gorenstein $R_{\wp^*}$-module.
$2) \Rightarrow 1)$ It is sufficient to prove that $E$ satisfies the condition (${}^*S_q$). We suppose that $E$ doesn't satisfy (${}^*S_q$). If there exists $\wp$ of $R$ such that $\mathrm{depth} E_{\wp^*} < \min\{\mathrm{q}, \dim E_{\wp^*}\}$, then $E_{\wp^*} \neq 0$. Hence we have two case:
I) If $\dim E_{\wp^*} \leq q$, $\mathrm{depth} E_{\wp^*} < \dim E_{\wp^*} \leq q$. Hence $E_{\wp^*}$ is not Gorenstein and that is a contradiction for $2)$.
II) If $\dim E_{\wp^*} > q$, $\mathrm{depth} E_{\wp^*} < q$ and $E_{\wp^*}$ is not Gorenstein and that is a contradiction too. It follows that $E$ satisfies the condition (${}^*S_q$).

## 3. ${}^*G_q$-rings and ${}^*q$-torsion

Let $R$ be a commutative noetherian ring, the Samuel's condition ($a_q$) is studied in [6], [8]. We give the following definitions for graded rings.

**Definition 3.1.** *Let $R$ be a commutative noetherian graded ring, $q > 0$ be an integer and $E$ be a finitely generated graded $R$-module. $E$ satisfies Samuel's condition $(^*a_q)$ (or $E$ is an $a_q$-module) if each homogeneous $R$-sequence of length less or equal than q made of not invertible elements is a homogeneous $E$-sequence.*

**Definition 3.2.** *Let $R$ be a commutative noetherian graded ring and $q > 0$ be an integer. Let $E$ be a finitely generated graded $R$-module. $E$ is a $q$-th module of syzygies if there exists an exact sequence:*

$$0 \rightarrow E \rightarrow P_1 \rightarrow \cdots \rightarrow P_q,$$

*where each $P_i$ is a graded projective $R$-module.*

**Definition 3.3.** *Let $R$ be a commutative noetherian graded ring, $E$ be a finitely generated graded $R$-module and $q > 0$ be an integer. $E$ is $^*q$-torsion free if each homogeneous $R$-sequence of length $q$ is a homogeneous $E$-sequence.*

**Remark 3.1.** *If $E$ is $q$-torsion free then $E$ is $^*q$-torsion free.*

**Theorem 3.1.** *Let $R$ be a $^*G_q$-ring, $E$ be a graded finitely generated $R$-module and $q > 0$ be an integer. The following conditions are equivalent:*

1) *$E$ satisfies Samuel's condition $(^*a_q)$;*
2) *$E$ is a $q$-th module of syzygies;*
3) *$E$ is $^*q$-torsion free.*

**Proof:** $1) \Rightarrow 2)$ Each $^*a_q$-module on a $^*G_q$-ring is a $q$-th module of syzygies ([5], 4.6).
$2) \Rightarrow 3)$ $R$ is a $^*G_q$-ring if and only if each $(q+1)$-th module of syzygies of $E$ $(Syz_{q+1}(E))$ is $^*(q+1)$-torsion free ([8], 4.3). If $Syz_{q+1}(E)$ is $^*(q+1)$-torsion free then $Syz_j(E)$ is $^*j$-torsion free for all $j = 1, \ldots, q+1$ ([8], 4.2). It follows that $Syz_q(E)$ is $^*q$-torsion free. By hypothesis $E$ is a $q$-th module of syzygies, hence $E$ is $^*q$-torsion free.
For a generic graded ring $R$ we prove that $3) \Rightarrow 2) \Rightarrow 1)$.
$3) \Rightarrow 2)$ The two conditions are equivalent for graded modules of finite projective dimension ([1], 4.25).
$2) \Rightarrow 1)$ Each $q$-th graded module of syzygies is an $^*a_q$-module ([5], 4.4).

**Corollary 3.1.** *Let $R$ be a $^*G_q$-ring, $E$ be a graded finitely generated $R$-module.*
*$E$ is $^*q$-torsion free $\Longrightarrow \mathrm{depth} E_\wp \geq \min\{\mathrm{q}+1, \mathrm{depth} R_\wp\}$, $\forall \wp \in Spec(R)$ not graded.*

**Proof:** If $E$ is $^*q$-torsion free then $E$ is $^*a_q$-module by Theorem **3.1**. This implies that for all not graded prime ideal $\wp \in Spec(R)$ $\mathrm{depth} E_{\wp^*} \geq \min\{\mathrm{q}, \mathrm{depth} R_{\wp^*}\}$ ([5], 4.2). We have the relations ([2], 1.5.9)

$$\mathrm{depth} R_{\wp^*} = \mathrm{depth} R_\wp - 1$$

$$\mathrm{depth} E_{\wp^*} = \mathrm{depth} E_\wp - 1.$$

So we can write: $\mathrm{depth} E_\wp \geq \min\{\mathrm{q}, \mathrm{depth} R_\wp - 1\} + 1$.
Hence for all not graded prime ideal $\wp \in Spec(R)$ we have:

$$\mathrm{depth} E_\wp \geq \min\{\mathrm{q}+1, \mathrm{depth} R_\wp\}.$$

**Remark 3.2.** *If $E$ is $(q+1)$-torsion free, then $E$ is $q$-torsion free (*[8]*, 4.11). Hence by definition $E$ is $^*q$-torsion free.*

**Proposition 3.1.** *Let $R$ be a commutative noetherian graded ring, $E$ be a graded finitely generated $R$-module and $q > 0$ be an integer. If $E$ is $^*q$-torsion free, then $E$ satisfies the following conditions:*
$1)$ *For all prime ideal $\wp$ of $R$ such that $\mathrm{depth} R_{\wp^*} < q$, $E_{\wp^*}$ is $^*q$-torsion free as $R_{\wp^*}$-module;*
$2)$ *For all prime ideal $\wp$ of $R$ such that $\mathrm{depth} R_{\wp^*} \geq q$, $\mathrm{depth} E_{\wp^*} \geq q$.*

**Proof:** We suppose that $E$ is $^*q$-torsion free. If $\wp \in Spec(R)$, $E_{\wp^*}$ is a graded $R_{\wp^*}$-module and $E_{\wp^*}$ is $^*q$-torsion free because $R_{\wp^*}$ is flat. It follows that the condition $1)$ is verified.
Moreover $E_{\wp^*}$ satisfies the condition $(^*a_q)$ ([5], 4.4), then each $R_{\wp^*}$-sequence of length $q$ is an $E_{\wp^*}$-sequence. As a consequence:

$$\mathrm{depth} R_{\wp^*} \geq q \Rightarrow \mathrm{depth} E_{\wp^*} \geq q,$$

that is the condition $2)$.

**Proposition 3.2.** *Let $R$ be a $^*G_q$-ring, $E$ be a graded finitely generated $R$-module and $q > 0$ be an integer. The following conditions are equivalent:*
$1)$ *$E$ is $^*q$-torsion free;*
$2)$ *There exists an exact sequence:*

$$0 \rightarrow E \rightarrow F_1 \rightarrow \cdots \rightarrow F_q,$$

*where each $F_i$ is a free graded finitely generated $R$-module;*
$3)$ *$E$ is $^*(q-1)$-torsion free and for all $\wp \in Spec(R)$ such that $\mathrm{ht} \wp^* \geq q$, we have $\mathrm{depth} E_{\wp^*} \geq q$.*

**Proof:** $1) \Leftrightarrow 2)$ It follows by Theorem **3.1**.
$1) \Rightarrow 3)$ If $E$ is $^*q$-torsion free then $E$ is $^*(q-1)$-torsion free ([8], 4.2). Let $\wp \in Spec(R)$ such that $\mathrm{ht} \wp^* \geq q$. We have $\mathrm{depth} R_{\wp^*} \geq q$ because $R$ satisfies $(^*S_q)$ condition. By Proposition **3.1** we have:

$$\mathrm{depth} R_{\wp^*} \geq q \Rightarrow \mathrm{depth} E_{\wp^*} \geq q.$$

$3) \Rightarrow 1)$ $E$ satisfies the condition $(^*a_{q-1})$ because $E$ is $^*(q-1)$-torsion free (by Theorem **3.1**). Then for all $\wp \in Spec(R)$

$$\mathrm{depth} E_{\wp^*} \geq \min\{\mathrm{q}-1, \mathrm{depth} R_{\wp^*}\}, \quad ([5], 4.2).$$

We prove that $E$ satisfies the condition $(^*a_q)$, that is

$$\mathrm{depth} E_{\wp^*} \geq \min\{\mathrm{q}, \mathrm{depth} R_{\wp^*}\}.$$

If $\mathrm{depth} R_{\wp^*} < q$, then $\mathrm{depth} E_{\wp^*} \geq \min\{\mathrm{q}-1, \mathrm{depth} R_{\wp^*}\} = \mathrm{depth} R_{\wp^*} \Rightarrow \mathrm{depth} E_{\wp^*} \geq \mathrm{depth} R_{\wp^*}$. It follows that $\mathrm{depth} E_{\wp^*} \geq \min\{\mathrm{q}, \mathrm{depth} R_{\wp^*}\}$.
If $\mathrm{depth} R_{\wp^*} \geq q$, then $\mathrm{ht} R_{\wp^*} \geq \mathrm{depth} R_{\wp^*} \geq q$ and by hypothesis it follows that $\mathrm{depth} E_{\wp^*} \geq q = \min\{\mathrm{q}, \mathrm{depth} R_{\wp^*}\}$. Hence $E$ satisfies the condition $(^*a_q)$ and, as $R$ is a $^*G_q$-ring, $E$ is $^*q$-torsion free (by Theorem **3.1**).

## References

[1] M. Auslander and M. Bridger, *Stable module theory* (Soc. 94, Providence, R.I., 1969)
[2] W. Bruns and J. Herzog, *Cohen-Macaulay Rings* (Cambridge studies in advanced mathematics, 1998)
[3] M.P. Cavaliere and G. Niesi, "On Serre's conditions in the form ring of an ideal", *J. Math. Kyoto Univ.* **21**, 537 (1981).
[4] R. Fossum, "Duality over Gorenstein rings", *Math.Scand.* **26**, 165 (1970)
[5] F.Ischebeck, "Eine Dualität zwischen den Funktoren Ext und Tor", *J. Algebra* **11**, 510 (1969)
[6] M. Malliavin "Condition $a_q$ de Samuel et $q-$torsion", *Bull.Soc.Math. France* **96**, 193 (1968)
[7] H.Matsumura, *Commutative Algebra* (Benjamin-Cummings, Reading, MA , 1980)
[8] M. Paugam, "Condition $G_q$ de Ischebeck-Auslander et condition $S_q$ de Serre", *Thèse de Doctorat d'état Université de Monpellier* (1973)

Monica La Barbiera
Università degli Studi di Messina
Dipartimento di Matematica
Salita Sperone, 31
98166 Messina, Italy
**E-mail**: monicalb@dipmat.unime.it

Presented: November 28, 2005
Published on line on November 9, 2006