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MULTITIME OPTIMAL CONTROL WITH SECOND-ORDER PDES CONSTRAINTS

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ABSTRACT. In this paper we study a simplified version of multitime optimal control problem for linear second-order partial differential equations (PDE). The multitime multiple integral functional is of Lagrange type. Necessary optimality conditions of multitime maximum principle type are derived. The multitime optimal control with second-order PDE constraints can be analyzed in three ways: as problems governed by (i) explicit m-flows, (ii) implicit m-flows, and (iii) second-order PDEs. All directions lead to variants of multitime maximum principle. The theoretical results are confirmed by solving two significant problems.

1. Introduction

Partial differential equations (PDE) constrained optimization is a very active area, as indicated by the large number of talks/symposia and papers. In this paper, our aim is to prove a simplified multitime maximum principle for optimal control problems governed by linear second-order PDEs. The control is distributed and takes values in an interval. Although the problem is known (Bardi and Capuzzo-Dolcetta 1997; Becker, Kapp, and Rannacher 2000; Hinze *et al.* 2009; Lions 1971; Tröltzsch 2010; Yong 1992, 1993), our technique is a simpler solution than that of published works (see also Udrişte 2008; Udrişte 2009[a],[b], 2010, 2011; Udrişte and Bejenaru 2011; Udrişte and Ţevy 2009, 2010).

In a PDE constrained optimization problem there are four basic elements: (i) A control u that we can handle according to our interests, which can be chosen among a family of feasible controls $\mathscr{U}$. (ii) The state of the system x to be controlled, which depends on the control. Some limitations can be imposed on the state, in mathematical terms $x \in C$, which means that not every possible state of the system is satisfactory. (iii) A state PDE that establishes the dependence between the control and the state. (iv) A functional $J(u(\cdot))$ to be extremized, called the objective functional or the cost functional, depending on the control and the state.

In the next sections the state equation will be a linear second-order PDE (partial differential equation), $x(t)$ being the solution of the equation and $u(t)$ a control arising in the equation so that any change in the control $u(t)$ produces a change in the solution $x(t)$. The objective is to determine an admissible control, called optimal control, that provides a satisfactory state for us and that extremizes the value of functional $J(u(\cdot))$. The basic questions to study are the necessary conditions, the existence of solution and its computation.

2. Setting of optimal control problem

Let $\Omega_{t_0t_1} \in R^m_+$ be a hyper-parallelepiped determined by the diagonal opposite points $t_0, t_1 \in R^m_+$. In $\Omega_{t_0t_1}$ we will consider the controlled linear second-order PDE

$$\frac{1}{2}h^{\alpha\beta}(t)\frac{\partial^2 x}{\partial t^\alpha \partial t^\beta}(t) + h^\alpha(t)\frac{\partial x}{\partial t^\alpha}(t) + x(t) = u(t). \tag{1}$$

If $(h^{\alpha\beta}) \in C^2(\Omega_{t_0t_1}),(h^\alpha) \in L^2(\Omega_{t_0t_1})$ and $u \in L^2(\Omega_{t_0t_1})$, then the Dirichlet problem fixed by $x = 0$ on $\partial\Omega_{t_0t_1}$, and some other additional conditions depending on the type of equation (elliptic, parabolic, hyperbolic), has a unique solution $x \in H^1_0(\Omega_{t_0t_1}) \cap L^\infty(\Omega_{t_0t_1})$.

It is supposed that the Lagrangian $L : \Omega_{t_0t_1} \times A \times U \to R$ is a C^2 function, $A \subset R^n$ is a bounded and closed subset, which contains the m-sheet $x(t)$, $t \in \Omega_{t_0t_1}$ of controlled PDE, the function $u : \Omega_{t_0t_1} \to [a,b]$ is the control and $dt^1 \cdots dt^m$ is the volume element.

Problem: *find*

$$\max_{u(\cdot)} \int_{\Omega_{t_0t_1}} L(t,x(t),u(t))\, dt^1...dt^m \tag{2}$$

constrained by the PDE (1).

3. Multitime maximum principle via first order constraints

For solving the foregoing problem, using simplifying reasonings, we prefer to transform the second-order PDE into a first order PDE system as in our papers (Udrişte 2008; Udrişte 2009[a],[b], 2010, 2011; Udrişte and Bejenaru 2011; Udrişte and Ţevy 2009, 2010):

$$\frac{\partial x}{\partial t^\alpha}(t) = v_\alpha(t),\ \frac{\partial v_\alpha}{\partial t^\beta}(t) = \frac{\partial v_\beta}{\partial t^\alpha}(t)$$

$$\frac{1}{2}h^{\alpha\beta}(t)\frac{\partial v_\alpha}{\partial t^\beta}(t) + h^\alpha(t)v_\alpha(t) + x(t) = u(t). \tag{3}$$

We use the generalized Lagrangian

$$\mathscr{L} = L + p^\alpha(t)\left(v_\alpha(t) - \frac{\partial x}{\partial t^\alpha}(t)\right) + p^{\alpha\beta}\left(\frac{\partial v_\alpha}{\partial t^\beta}(t) - \frac{\partial v_\beta}{\partial t^\alpha}(t)\right)$$

$$+q(t)\left(\frac{1}{2}h^{\alpha\beta}(t)\frac{\partial v_\alpha}{\partial t^\beta}(t) + h^\alpha(t)v_\alpha(t) + x(t) - u(t)\right)$$

or simplified

$$\mathscr{L} = L + (p^\alpha + qh^\alpha)v_\alpha - p^\alpha\frac{\partial x}{\partial t^\alpha} + (p^{\alpha\beta} - p^{\beta\alpha} + \frac{1}{2}qh^{\alpha\beta})\frac{\partial v_\alpha}{\partial t^\beta} + q(x-u).$$

The attached Hamiltonian is

$$H = L + (p^\alpha + qh^\alpha)v_\alpha + q(x-u).$$

The Lagrange multipliers $p^\alpha, p^{\alpha\beta}, q$ corresponding to pointwise control constraints are C^1 functions.

Theorem (multitime maximum principle) *Suppose that the problem of maximizing the functional (2) constrained by (1) has an interior optimal solution $u^*(t)$, which determines* *the optimal evolution $x(t)$. Then there exists the costate functions $p^{\alpha}(t), p^{\alpha\beta}(t), q(t)$ such that*

$$(initial\,system) \qquad \frac{\partial x}{\partial t^{\alpha}} = \frac{\partial H}{\partial p^{\alpha}}, \ \frac{\partial H}{\partial p^{\alpha\beta}} = 0, \ \frac{\partial H}{\partial q} = 0$$

$$(adjoint\,or\,dual\,system) \qquad \frac{\partial H}{\partial x} + \frac{\partial p^{\alpha}}{\partial t^{\alpha}} = 0, \ \frac{\partial H}{\partial v_{\alpha}} - \frac{\partial p^{\alpha\beta}}{\partial t^{\beta}} = 0,$$

$$(critical\,point\,condition) \qquad \frac{\partial H}{\partial u} = 0$$

hold.

Proof. Firstly, we find the *infinitesimal deformation PDE system (3)*. We fix the control $u(t)$ and we variate the state $x(t)$ into $x(t,\varepsilon)$. Denote $\frac{\partial x}{\partial \varepsilon}(t,0) = y$, $\frac{\partial v_{\alpha}}{\partial \varepsilon}(t,0) = w_{\alpha}$. The infinitesimal deformation PDE system is

$$\frac{\partial y}{\partial t^{\alpha}} = w_{\alpha}, \ \frac{\partial w_{\alpha}}{\partial t^{\beta}} = \frac{\partial w_{\beta}}{\partial t^{\alpha}}$$

$$\frac{1}{2} h^{\alpha\beta}(t) \frac{\partial w_{\alpha}}{\partial t^{\beta}}(t) + h^{\alpha}(t) w_{\alpha}(t) + y(t) = 0.$$

Then, the *adjoint PDE system* is

$$y \frac{\partial p^{\alpha}}{\partial t^{\alpha}} = p^{\alpha} w_{\alpha}, \ w_{\alpha} \frac{\partial p^{\alpha\beta}}{\partial t^{\beta}} = p^{\alpha\beta} \frac{\partial w_{\alpha}}{\partial t^{\beta}}, \ \frac{1}{2} h^{\alpha\beta} w_{\alpha} \frac{\partial q}{\partial t^{\beta}} = -q(h^{\alpha} w_{\alpha} + y).$$

The sense of adjointness is

$$p^{\alpha} L_{\alpha} y - y M_{\alpha} p^{\alpha} = 0, \ w_{\alpha} P_{\beta} p^{\alpha\beta} - p^{\alpha\beta} Q_{\beta} w_{\alpha} = 0, \ q h^{\alpha\beta} R_{\beta} w_{\alpha} - h^{\alpha\beta} w_{\alpha} S_{\beta} q = 0,$$

where L_{α} and M_{α}, P_{α} and Q_{α}, R_{α} and S_{α} are linear second-order partial differential operators.

Secondly, let $u(t)$ be an optimal control. A variation $\hat{u} = u + \varepsilon h$ of the control determines the variation of the state $x = x(t;\varepsilon)$. The first variation of the Lagrangian $\mathscr{L}$ is

$$\frac{\partial \mathscr{L}}{\partial \varepsilon}|_{\varepsilon=0} = \frac{\partial L}{\partial x} x_{\varepsilon}(t,0) + \frac{\partial L}{\partial u} h + (p^{\alpha} + q h^{\alpha}) \frac{\partial v_{\alpha}}{\partial \varepsilon}(t,0)$$

$$-p^{\alpha} \frac{\partial x_{\varepsilon}}{\partial t^{\alpha}}(t,0) + (p^{\alpha\beta} - p^{\beta\alpha} + \frac{1}{2} q h^{\alpha\beta}) \frac{\partial v_{\alpha\varepsilon}}{\partial t^{\beta}}(t,0) + q x_{\varepsilon}(t,0) - qh.$$

For integration by parts (divergence formula), we use the identities

$$-p^{\alpha}(t) \frac{\partial x_{\varepsilon}}{\partial t^{\alpha}}(t,0) = \frac{\partial p^{\alpha}}{\partial t^{\alpha}}(t) x_{\varepsilon}(t,0) - \frac{\partial}{\partial t^{\alpha}}(p^{\alpha}(t) x_{\varepsilon}(t,0))$$

$$p^{\alpha\beta}(t) \frac{\partial v_{\alpha\varepsilon}}{\partial t^{\beta}}(t,0) = \frac{\partial}{\partial t^{\beta}}(p^{\alpha\beta}(t) v_{\alpha\varepsilon}(t,0)) - \frac{\partial p^{\alpha\beta}}{\partial t^{\beta}}(t) v_{\alpha\varepsilon}(t,0).$$

The condition $I'(0) = 0$ means vanishing of the integral

$$\int_{\Omega} \left(\frac{\partial L}{\partial x} + \frac{\partial p^{\alpha}}{\partial t^{\alpha}} + q \right) x_{\varepsilon}(t,0) + \left(p^{\alpha} + q h^{\alpha} - \frac{\partial p^{\alpha\beta}}{\partial t^{\beta}} \right) v_{\alpha\varepsilon}(t,0) + \left(\frac{\partial L}{\partial u} - q \right) h$$

$$-\int_{\partial\Omega} p^{\alpha}(t) n_{\alpha}(t) x_{\varepsilon}(t,0) + \int_{\partial\Omega} (p^{\alpha\beta}(t) - p^{\beta\alpha}(t)) v_{\alpha\varepsilon}(t) n_{\beta}(t),$$

taking into account the adjoint PDEs, the boundary conditions, and arbitrariness of h.

Using the adjoint PDE system, it follows

$$\frac{\partial H}{\partial x}+\frac{\partial p^{\alpha}}{\partial t^{\alpha}}=0,\ \frac{\partial H}{\partial v_{\alpha}}-\frac{\partial p^{\alpha\beta}}{\partial t^{\beta}}=0,\ p^{\alpha}|_{\partial\Omega_{t_0t_1}}=0,\ p^{\alpha\beta}|_{\partial\Omega_{t_0t_1}}=0$$

and hence

$$\frac{\partial H}{\partial u}=0.$$

Remark: If $u^*(t)$ is not an interior optimal control, then the critical point condition is replaced by

$$H(x^*(t),p^*(t),u^*(t))=\max_{u\in U}H(x^*(t),p^*(t),u),\ t\in\Omega_{t_0t_1}.$$

From the critical point condition, we understand that L cannot be independent of "u"; if L is independent of "u", then the foregoing theory must be changed - in fact the Hamiltonian is linear in control and we obtain a bang-bang solution.

4. Multitime maximum principle via second-order constraints

The key tool to get the necessary conditions for optimality works directly. Indeed, let us start with the generalized Lagrangian

$$\mathscr{L}=L+p(t)\left(-\frac{1}{2}h^{\alpha\beta}(t)\frac{\partial^2 x}{\partial t^{\alpha}\partial t^{\beta}}(t)-h^{\alpha}(t)\frac{\partial x}{\partial t^{\alpha}}(t)-x(t)+u(t)\right)$$

and its associated Hamiltonian

$$\mathscr{H}=L+p\left(-h^{\alpha}\frac{\partial x}{\partial t^{\alpha}}-x+u\right).$$

The Lagrange multipliers p corresponding to pointwise control constraints is a C^1 function.

Theorem (**multitime maximum principle**) *Suppose that the problem of maximizing the functional (2) constrained by (1) has an interior optimal solution $u^*(t)$, which determines the optimal evolution $x(t)$. Then there exists the costate functions $p(t)$ such that*

$$(initial\,PDE)\qquad \frac{1}{2}h^{\alpha\beta}(t)\frac{\partial^2 x}{\partial t^{\alpha}\partial t^{\beta}}(t)=\frac{\partial\mathscr{H}}{\partial p}$$

$$(adjoint\,or\,dual\,equation)\qquad \frac{1}{2}\frac{\partial^2(ph^{\alpha\beta})}{\partial t^{\alpha}\partial t^{\beta}}-\frac{\partial(ph^{\alpha})}{\partial t^{\alpha}}=\frac{\partial\mathscr{H}}{\partial x},$$

$$(critical\,point\,condition)\qquad \frac{\partial\mathscr{H}}{\partial u}=0$$

hold.

Proof. Firstly, we find the *infinitesimal deformation of PDE (1).* We fix the control $u(t)$ and we variate the state $x(t)$ into $x(t,\varepsilon)$. Denote $\frac{\partial x}{\partial\varepsilon}(t,0)=y(t)$. The infinitesimal deformation PDE is

$$\frac{1}{2}h^{\alpha\beta}(t)\frac{\partial^2 y}{\partial t^{\alpha}\partial t^{\beta}}(t)+h^{\alpha}(t)\frac{\partial y}{\partial t^{\alpha}}(t)+y(t)=0.$$

The *adjoint PDE* is

$$\frac{1}{2}\frac{\partial^2(h^{\alpha\beta}p)}{\partial t^\alpha \partial t^\beta}(t) - \frac{\partial(h^\alpha p)}{\partial t^\alpha}(t) + p(t) = 0.$$

The adjointness has the sense $pLy - yMp = 0$, where L and M are linear second-order partial differential operators.

The variation $\hat{u} = u + \varepsilon h$ of the control determines the variation of the state $x = x(t;\varepsilon)$. The first variation of the Lagrangian $\mathscr{L}$ is

$$\frac{\partial \mathscr{L}}{\partial \varepsilon}|_{\varepsilon=0} = \frac{\partial L}{\partial x}x_\varepsilon(t,0) + \frac{\partial L}{\partial u}h$$

$$p(t)\left(-\frac{1}{2}h^{\alpha\beta}(t)\frac{\partial^2 x_\varepsilon}{\partial t^\alpha \partial t^\beta}(t,0) - h^\alpha(t)\frac{\partial x_\varepsilon}{\partial t^\alpha}(t,0) - x_\varepsilon(t,0) + h(t)\right).$$

For integration by parts (divergence formula), we use the identities

$$p(t)h^\alpha(t)\frac{\partial x_\varepsilon}{\partial t^\alpha}(t,0) = \frac{\partial}{\partial t^\alpha}(p(t)h^\alpha(t)x_\varepsilon(t,0)) - \frac{\partial(ph^\alpha)}{\partial t^\alpha}(t)x_\varepsilon(t,0)$$

$$p(t)h^{\alpha\beta}(t)\frac{\partial^2 x_\varepsilon}{\partial t^\alpha \partial t^\beta}(t,0) = \frac{\partial}{\partial t^\alpha}\left(p(t)h^{\alpha\beta}(t)\frac{\partial x_\varepsilon}{\partial t^\beta}(t,0)\right) - \frac{\partial(ph^{\alpha\beta})}{\partial t^\alpha}(t)\frac{\partial x_\varepsilon}{\partial t^\beta}(t,0)$$

$$\frac{\partial(ph^{\alpha\beta})}{\partial t^\alpha}(t)\frac{\partial x_\varepsilon}{\partial t^\beta}(t,0) = \frac{\partial}{\partial t^\beta}\left(\frac{\partial(ph^{\alpha\beta})}{\partial t^\alpha}(t)x_\varepsilon(t,0)\right) - \frac{\partial^2(ph^{\alpha\beta})}{\partial t^\alpha \partial t^\beta}(t)x_\varepsilon(t,0).$$

The condition $I'(0) = 0$ means vanishing of the integral

$$\int_\Omega \left(\frac{\partial L}{\partial x} - \frac{1}{2}\frac{\partial^2(ph^{\alpha\beta})}{\partial t^\alpha \partial t^\beta} + \frac{\partial(ph^\alpha)}{\partial t^\alpha} - p\right)x_\varepsilon(t,0) + \left(\frac{\partial L}{\partial u} + p\right)h$$

$$-\frac{1}{2}\int_{\partial\Omega} ph^{\alpha\beta}n_\alpha \frac{\partial x_\varepsilon}{\partial t^\beta} + \frac{1}{2}\int_{\partial\Omega}\frac{\partial(ph^{\alpha\beta})}{\partial t^\alpha}n_\beta x_\varepsilon + \int_{\partial\Omega} ph^\alpha n_\alpha x_\varepsilon,$$

taking into account the adjoint PDE, the boundary conditions, and arbitrariness of h.

Using the adjoint PDE equation, it follows

$$\frac{\partial L}{\partial x} - \frac{1}{2}\frac{\partial^2(ph^{\alpha\beta})}{\partial t^\alpha \partial t^\beta} + \frac{\partial(ph^\alpha)}{\partial t^\alpha} - p = 0,\ p|_{\partial\Omega_{t_0t_1}} = 0$$

and hence

$$\frac{\partial L}{\partial u} + p = 0.$$

Remarks: (i) If $u^*(t)$ is not an interior optimal control, then the critical point condition is replaced by

$$\mathscr{H}(x^*(t), p^*(t), u^*(t)) = \max_{u\in U}\mathscr{H}(x^*(t), p^*(t), u),\ t \in \Omega_{t_0t_1}.$$

5. Examples

5.1. Let us formulate and solve a bi-temporal optimal problem with pointwise state constraints:

$$\min_{u(\cdot)} \frac{1}{2}\int_{\Omega_{t_0t_1}} (x(t)-\sin(2\pi t^1t^2))^2\,dt^1dt^2 + \frac{\alpha}{2}\int_{\Omega_{t_0t_1}} u^2(t)\,dt^1dt^2$$

subject to

$$-\Delta x(t) = u(t),\, t\in \Omega_{t_0t_1}; x(t)=0\,\mathrm{for}\, t\in \partial\Omega_{t_0t_1}.$$

To solve this problem we apply the two-time maximum principle via second-order constraints. For that, we introduce the Lagrangian

$$\mathscr{L} = -\frac{1}{2}(x(t)-\sin(2\pi t^1t^2))^2 - \frac{\alpha}{2}u^2(t) + p(t)(\Delta x(t)+u(t)).$$

Since $\frac{1}{2}h^{\alpha\beta} = -\delta^{\alpha\beta}$, $h^\alpha = 0$, $\frac{\partial L}{\partial x} = -(x(t)-\sin(2\pi t^1t^2))$, the adjoint PDE is

$$x(t)-\sin(2\pi t^1t^2)+\Delta p(t) = 0,\; p|_{\partial\Omega_{t_0t_1}} = 0.$$

On the other hand, the critical point condition gives $p(t) = \alpha u(t)$. Suppose $\alpha > 0$.

5.1.1. *Case of homogeneous PDE system.* The associated homogeneous system implies

$$\Delta^2 x(t) = \frac{1}{\alpha}\,x(t),\; \Delta^2 u(t) = \frac{1}{\alpha}\,u(t).$$

These PDEs show that we have to solve the eigenvalues problem

$$\Delta^2 u(t) = \frac{1}{\alpha}\,u(t),\; u(t)|_{\partial\Omega_{t_0t_1}} = 0.$$

Looking for the solution of the form

$$u(t) = u(t^1,t^2) = \sin\lambda_1(t^1-t_0^1)\,\sin\lambda_2(t^2-t_0^2),$$

we find the characteristic equation

$$\lambda_1^2+\lambda_2^2 = \pm\frac{1}{\sqrt{\alpha}}.$$

For

$$\lambda_1 = \frac{i}{\sqrt[4]{\alpha}}\cos\varphi,\; \lambda_2 = \frac{i}{\sqrt[4]{\alpha}}\sin\varphi$$

we get a first solution

$$u_1(t) = \sin\lambda_1(t^1-t_0^1)\,\sin\lambda_2(t^2-t_0^2),$$

of the PDE. For

$$\lambda_3 = \frac{1}{\sqrt[4]{\alpha}}\cos\varphi,\; \lambda_4 = \frac{1}{\sqrt[4]{\alpha}}\sin\varphi$$

we find the second solution

$$u_2(t) = \sin\lambda_3(t^1-t_0^1)\,\sin\lambda_4(t^2-t_0^2)$$

of the PDE. The general solution of the PDE is

$$u(t) = c_1u_1(t)+c_2u_2(t).$$

The boundary conditions imply

$$c_1 = 0, \lambda_1 = \frac{n\pi}{t_1^1 - t_0^1}, \lambda_2 = \frac{m\pi}{t_1^2 - t_0^2},$$

$$\alpha = \left[\left(\frac{n\pi}{t_1^1 - t_0^1}\right)^2 + \left(\frac{m\pi}{t_1^2 - t_0^2}\right)^2\right]^{-2}.$$

We fix the values n and m. A solution of the homogeneous PDE system $\Delta x = -u$, $\alpha\Delta u = -x$, with vanishing boundary conditions, is

$$x(t^1,t^2) = A \sin\frac{n\pi(t^1 - t_0^1)}{t_1^1 - t_0^1}\, \sin\frac{m\pi(t^2 - t_0^2)}{t_1^2 - t_0^2}$$

and

$$u(t^1,t^2) = \left[\left(\frac{n\pi}{t_1^1 - t_0^1}\right)^2 + \left(\frac{m\pi}{t_1^2 - t_0^2}\right)^2\right] x(t^1,t^2).$$

5.1.2. *Case of non-homogeneous PDE system.* The functions

$$\varphi^{nm}(t^1,t^2) = 2(t_1^1 - t_0^1)^{-\frac{1}{2}}(t_1^2 - t_0^2)^{-\frac{1}{2}} \sin\frac{n\pi(t^1 - t_0^1)}{t_1^1 - t_0^1}\, \sin\frac{m\pi(t^2 - t_0^2)}{t_1^2 - t_0^2}$$

are orthonormal eigenfunctions of the Laplacian operator corresponding to the eigenvalues

$$\lambda_{nm} = \left(\frac{n\pi}{t_1^1 - t_0^1}\right)^2 + \left(\frac{m\pi}{t_1^2 - t_0^2}\right)^2.$$

These eigenfunctions determine a complete system on $L^2(\Omega_{t_0t_1})$.

We look for the solutions of the PDE system

$$\Delta x = -u,\ \alpha\Delta u = -x + \sin(2\pi t^1 t^2)$$

in the form (Einstein convention of summation)

$$x(t^1,t^2) = \alpha_{nm}\varphi^{nm}(t^1,t^2),\ u(t^1,t^2) = \beta_{nm}\varphi^{nm}(t^1,t^2).$$

Replacing in the PDEs, we obtain the systems

$$\lambda_{nm}\alpha_{nm} + \beta_{nm} = 0,\ \alpha_{nm} + \alpha\lambda_{nm}\beta_{nm} = \gamma_{nm},$$

where

$$\sin(2\pi t^1 t^2) = \gamma_{nm}\varphi^{nm}(t^1,t^2),$$

with

$$\gamma_{nm} = 4(t_1^1 - t_0^1)^{-1}(t_1^2 - t_0^2)^{-1}\int_{t_0^1}^{t_1^1}\int_{t_0^2}^{t_1^2} \sin(2\pi t^1 t^2)\ \varphi^{nm}(t^1,t^2)dt^1dt^2.$$

The coefficients

$$\alpha_{nm} = \frac{-\gamma_{nm}}{\alpha\lambda_{nm}^2 - 1},\ \beta_{nm} = \frac{\gamma_{nm}\lambda_{nm}}{\alpha\lambda_{nm}^2 - 1}$$

determine the optimal control $u(t^1,t^2)$ and the optimal evolution $x(t^1,t^2)$.

5.1.3. *Laplace approach for non-homogeneous PDE system.* Let us find a solution of the non-homogeneous PDE system

$$\Delta x(t) = -u(t),\ \alpha\Delta u(t) = -x(t) + \sin(2\pi t^1 t^2),$$

with vanishing boundary conditions. We apply a bi-dimensional Laplace transform

$$X(p_1,p_2) = \int_0^\infty\int_0^\infty x(t^1,t^2)e^{-(p_1t^1+p_2t^2)}dt^1dt^2$$

$$U(p_1,p_2) = \int_0^\infty\int_0^\infty u(t^1,t^2)e^{-(p_1t^1+p_2t^2)}dt^1dt^2.$$

The non-homogeneous PDE system is transformed into

$$(p_1^2+p_2^2)X(p_1,p_2)+U(p_1,p_2)=0,$$

$$\alpha(p_1^2+p_2^2)U(p_1,p_2)+X(p_1,p_2) = -\frac{1}{2\pi}\left(Ci\,\Pi\cos\Pi + Si\,\Pi\sin\Pi\right),$$

where $\Pi = \frac{p_1p_2}{2\pi}$. It follows

$$X(p_1,p_2) = \frac{1}{2\pi}\frac{Ci\,\Pi\cos\Pi+Si\,\Pi\sin\Pi}{\alpha(p_1^2+p_2^2)^2-1}$$

$$U(p_1,p_2) = -(p_1^2+p_2^2)X(p_1,p_2).$$

The optimal control and the optimal evolution are obtained by

$$u(t^1,t^2) = -\frac{1}{4\pi^2}\int_{a-i\infty}^{a+i\infty}\int_{b-i\infty}^{b+i\infty}U(t^1,t^2)\,e^{p_1t^1+p_2t^2}dp_1dp_2$$

$$x(t^1,t^2) = -\frac{1}{4\pi^2}\int_{a-i\infty}^{a+i\infty}\int_{b-i\infty}^{b+i\infty}X(t^1,t^2)\,e^{p_1t^1+p_2t^2}dp_1dp_2.$$

5.2. The single-time case is well-known as optimal monetary policy. In this case, the PDE is replaced by the ODE

$$\frac{a^2}{2}\ddot{x}(t)+a\dot{x}(t)+x(t)=u(t),\ t\in[0,T] \tag{4}$$

$$x(0)=x_0, x(T)=x_T, \dot{x}(T)=0,$$

where x is the proportional rate of growth of money income, the control $u = m + b\dot{m}$, where $m = \frac{\dot{M}}{M}$ is a proportional rate of change of money supply $M(t)$, $b = const$ and a is a constant representing the length of the business cycle.

Some institutional and economic reasons ask $|u(t)| \leq 0.1$. The terminal value T means some future date. The conditions $x(T) = x_T$, $\dot{x}(T) = 0$ show that we can achieve a stable rate of growth of national income, using an optimal money supply police. This objective is equivalent to:

$$\min I(u(\cdot)) = \int_0^T dt \qquad \text{subject to (4)}.$$

To solve this problem we apply the maximum principle via second-order constraints. Having a linear optimal control with

$$\mathscr{L} = -1 + p(t)\left(\frac{a^2}{2}\ddot{x}(t)+a\dot{x}(t)+x(t)-u(t)\right)$$

$$= -1 + p(t)\left(\frac{a^2}{2}\ddot{x}(t) + a\dot{x}(t) + x(t)\right) - p(t)u(t),$$

the switching function is $\sigma = -p$, and the optimal control

$$u^*(t) = \begin{cases} 0.1 & \text{if} \quad p(t) > 0 \\ -0.1 & \text{if} \quad p(t) < 0 \end{cases}$$

is a bang-bang control. We add the adjoint boundary problem

$$\frac{a^2}{2}\ddot{p}(t) - a\dot{p}(t) + p(t) = 0,\ p(0) = p_0,\ p(T) = 0.$$

The general solution

$$p(t) = e^{at}(B_1 \cos at + B_2 \sin at)$$

and the boundary conditions give the optimal Lagrange multiplier

$$p(t) = e^{at}(\cos at - \frac{\cos aT}{\sin aT}\sin at)p_0.$$

For $u^*(t) = 0.1$, we obtain the general optimal evolution

$$x(t) = e^{-at}(C_1 \cos at + C_2 \sin at) + 0.05.$$

Imposing the boundary conditions, we must have

$$(x_T - 0.05)e^{aT} = C_1 \cos aT + C_2 \sin aT\,,$$
$$(C_2 - C_1)\cos aT - (C_2 + C_1)\sin aT = 0\,,$$

or

$$(x_T - 0.05)e^{aT} = C_1 \cos aT + C_2 \sin aT,\ (x_T - 0.05)e^{aT} = C_2 \cos aT - C_1 \sin aT\,.$$

It follows

$$C_1 = (x_T - 0.05)e^{aT}(\cos aT - \sin aT),\ C_2 = (x_T - 0.05)e^{aT}(\sin aT + \cos aT)\,.$$

Remark: If we want to find a solution by discretization, then we must have in mind that "discretized-then-optimize and optimize-then-discretized" are two different approaches. One is not universally better than the other.

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