DOI:10.1478/C1A0702003

Atti dell'Accademia Peloritana dei Pericolanti
Classe di Scienze Fisiche, Matematiche e Naturali
Vol. LXXXV, C1A0702003 (2007)
Adunanza del 28 novembre 2006

COMPLETE INTERSECTIONS IN $\mathbf{P}^2$ THROUGH SEPARATING SEQUENCES

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(Nota presentata dal Socio Ordinario Gaetana Restuccia)

ABSTRACT. In this paper we characterize the finite sets of points in $\mathbf{P}^2$, arising as a complete intersection of two curves, by means of their realizable sequences. Actually, we show that a reduced 0-dimensional scheme in $\mathbf{P}^2$ is a complete intersection of type (a, b) iff all its realizable sequences can be obtained by means of direct transpositions from a special one M_{ab}.

0. Introduction

We are considering a very old problem, going back to mathematicians as Euler, Cramer, Bezout, Maclaurin and Cayley: the characterization of a finite set of points in $\mathbf{P}^2$ which is the complete intersection of two curves. More recently, Griffith and Harris [1] suggested the idea of using the "Cayley-Bacharach property", generalizing the "nine points theorem" for the intersection of two cubics. Such a suggestion was largely followed: Davis, Geramita, Maroscia, Orecchia, Sauer are mathematicians who contributed to the complete solution of the problem for a 0-dimensional scheme in $\mathbf{P}^2$, extending also the investigation to more general situations ([2],[3],[4],[5]).

In this paper we characterize the finite sets of points in $\mathbf{P}^2$, arising as a complete intersection of two curves, by means of their realizable sequences([6], [7]). More precisely, we show that a reduced 0-dimensional scheme in $\mathbf{P}^2$ is a complete intersection (briefly c.i.) of type $(a, b), a \leq b$, iff all its realizable sequences can be obtained by means of direct transpositions (see $\S 1, \mathbf{1.B}_3$) from a special one M_{ab}; the sequence M_{ab} is realized just by the intersection of a curve C of degree b with a curve C' split into a lines. As a consequence, we find again the well known description of a c.i. given by its Castelnuovo function structure and the Cayley-Bacharach condition; it is also immediate to restate the Cayley- Bacharach theorem for the colength 1 subschemes of a c.i. ([8],[4],[1],[2]).

Our first definition of a realizable sequence is given and studied in [6] and [7] only for a reduced scheme $\mathbf{X}$; so, this paper deals with that special situation. However, the definition of a realizable sequence can be extended to any, non necessarily reduced, 0-dimensional scheme (see [9]); in a next paper, we will investigate more deeply some properties that still hold in the non reduced case; as a consequence, we will prove that our characterization of a c.i. in $\mathbf{P}^2$ is still valid also in the non reduced case.

This paper was written while both authors were members of I.N.D.A.M.G.S.A.G.A.

1. Recalls and Notation

A The Hilbert function.

Let us recall some general definitions and results, that will be applied to the special case of 0-dimensional schemes in $\mathbf{P}^2$.

1.A$_1$ To any projective scheme $\mathbf{X}$ in $\mathbf{P}^r = \mathbf{P}^r_k$ (k algebraically closed) we associate a saturated homogeneous ideal $I = \oplus_{i\in\mathbf{N}} I_t$ of $R = k[X_0, ..., X_r]$ and a coordinate ring $A = R/I$, of Krull dimension $dim\mathbf{X} + 1$.

1.A$_2$ The Hilbert function $H_{\mathbf{X}}(\cdot)$ of $\mathbf{X}$ is the Hilbert function of its standard graded algebra $A = R/I = \oplus_{i\geq 0} A_i$, that is the function $H(A, \cdot) : \mathbf{N} \longrightarrow \mathbf{N}$ defined as follows: $H(A, i) = dim_k A_i$.

When $dim\mathbf{X} = 0$, it is interesting to consider the first difference of $H_{\mathbf{X}}$, that is the function $\Gamma_{\mathbf{X}}(\cdot) = \Gamma(A, \cdot) : \mathbf{N} \longrightarrow \mathbf{N}$, defined as $\Gamma(A, i) = H(A, i) - H(A, i-1), i > 0,\ \Gamma(A, 0) = 1$. $\Gamma_{\mathbf{X}}$ is called "Castelnuovo function "of $\mathbf{X}$ ([2], §2) and it is zero for $t >> 0$. In this situation the degree of $\mathbf{X}$ is, by definition, $e(\mathbf{X}) = e(A) = \Sigma_i \Gamma(A, i)$; $e(\mathbf{X})$ coincides with $H_{\mathbf{X}}(t), t >> 0$ ([3], §0).

1.A$_3$ The functions $f : \mathbf{N} \longrightarrow \mathbf{N}$ that arise as the Castelnuovo function of a projective scheme are described by Macaulay's theorem ([10]). In the special case of a 0-dimensional projective scheme in $\mathbf{P}^2$, such a theorem says what follows ([2],§2).

Let: $\alpha = \alpha(\mathbf{X}) = \alpha(A) = \min\{t : I_t \neq (0)\}$,
$\beta = \beta(\mathbf{X}) = \beta(A) = \min\{t : \text{the elements of } \mathrm{I_t} \text{ do not have a common factor}\}$,
$\tau = \tau(\mathbf{X}) = \tau(A) = \max\{t : \Gamma_{\mathbf{X}}(t) \neq 0\}$.
Then:
$\Gamma_{\mathbf{X}}(t) \geq 0,\ \Gamma_{\mathbf{X}}(t) \neq 0 \Leftrightarrow 0 \leq t \leq \tau$
For $t \geq 0 : \Gamma_{\mathbf{X}}(t) \leq t + 1$ and $\Gamma_X(t) = t + 1$ iff $t \leq \alpha - 1$
For $t \geq \alpha : \Gamma_{\mathbf{X}}(t) \leq \Gamma_{\mathbf{X}}(t-1)$
For $\tau + 1 \geq t \geq \beta : \Gamma_{\mathbf{X}}(t) < \Gamma_{\mathbf{X}}(t-1)$.

1.A$_4$ If a scheme $\mathbf{X}$ of points in $\mathbf{P}^2$ is a complete intersection of two curves of degrees a and b, $a \leq b$, that is a $c.i$ of type (a, b) (briefly: $CI(a, b)$) , then its Castelnuovo function has a well described shape. More precisely, if we denote such a function as $\Gamma(a, b; \cdot)$, then:

$$\Gamma(a, b; t) = \begin{cases} t+1, & 0 \leq t \leq a-1 \\ a, & a-1 \leq t \leq b-1 \\ a+b-t-1, & b-1 \leq t \leq a+b-1 \\ 0, & t \geq a+b-1 \end{cases}$$

It is well known that the Castelnuovo function $\Gamma(a, b; \cdot)$ is not sufficient to characterize a complete intersection $\mathbf{X}$. The further condition to be added is the Cayley -Bacharah condition, which says that all the points of $\mathbf{X}$ must have the same degree of separation, or, in other words, that all the realizable sequences of $\mathbf{X}$ must end with the same number; this

number is necessarily $a+b-2$, as every scheme realizes the non decreasing sequence C_Γ (see [6], Def.2.4 and Prop.2.3), which ends with $a+b-2$.

B Separating sequences.

The following definitions and results can be found in [6], [7],[9]; we recall them here, for reader's convenience.

1.B$_1$ A hypersurface of $\mathbf{P}^r$ separates a point P from a set of points $\{P_1, ..., P_h\}$ iff it contains $\{P_1, ..., P_h\}$, but does not contain P.

Let $\mathbf{X}$ be a reduced projective scheme supported at $\{P_1, ..., P_n\}$. To any ordering $(P_1, ..., P_n)$ of the points of $\mathbf{X}$ we associate a sequence of natural numbers $S = S_{P_1,...,P_n} = (s_1, ..., s_n)$, where $s_1 = 0$ and $s_k, k > 1$, is the least degree of a hypersurface separating P_k from $P_1, ..., P_{k-1}$. The number s_k is the degree of separation of P_k with respect to $P_1, ..., P_{k-1}$. We say that $\mathbf{X}$ realizes S or that S is $\mathbf{X}$-realizable. More generally, a sequence S is called realizable iff there exists a scheme $\mathbf{X}$ realizing it.

1.B$_2$ To any sequence of natural numbers $S = (s_1, ..., s_n)$ let us associate a function $\gamma_S : \mathbf{N} \longrightarrow \mathbf{N}$ defined as follows: $\gamma_S(t) = card\ \{i : s_i = t\}$, where $card\ J$ denotes the cardinality of the set J.

Theorem (see [6],Cor.1.4) If $\mathbf{X}$ is any reduced 0-dimensional projective scheme and S is a sequence realized by $\mathbf{X}$, then $\gamma_S(\cdot) = \Gamma_{\mathbf{X}}(\cdot)$.

Notation-Definition 1. *Let S be a sequence realized by $\mathbf{X}$. In this situation, $\gamma_S(\cdot) = \Gamma_{\mathbf{X}}(\cdot)$ will be denoted also $\Gamma(S, \cdot)$ and called "the Castelnuovo function of S".*

1.B$_3$ A permutation of a sequence $(s_1, ..., s_n)$ of natural numbers, obtained by interchanging two elements s_i, s_{i+1}, will be called a direct transposition (briefly $d.t.$) if $s_{i+1} < s_i$.

Let us recall the following statement (see [6], Cor.2.2):

Proposition If $S_1 = (s_1, ..., s_i, s_{i+1}, ..., s_n)$ is realizable and $s_i > s_{i+1}$, then also $S_2 = (s_1, ..., s_{i+1}, s_i, ..., s_n)$ is realizable and it is $\mathbf{X}$-realizable, for every $\mathbf{X}$ realizing S_1.

1.B$_4$ In this paper we will simply call "segment" any "1-segment" (see [7],§2), that is any finite increasing sequence of consecutive natural numbers.

1.B$_5$ We denote $\widetilde{S}_\Gamma$ the set of all sequences realized by a scheme $\mathbf{X}$ varying among all schemes with Castelnuovo function Γ; let $M_\Gamma = sup\ \widetilde{S}_\Gamma$, where the maximum is computed with respect to the lexicographic order arising from the natural ordering of $\mathbf{N}$. M_Γ turns out to be a sequence of α segments, $\alpha = inf\{t : \Gamma(t) \neq t+1\}$. More precisely:

$$
\begin{array}{rcllllllll}
M_\Gamma & = & (0, & 1, & 2, & ..., & ..., & ..., & ..., & n_0, \\
 & & & & 1, & 2, & ..., & ..., & ..., & n_1, \\
 & & & & & ... & ... & ... & ... & ..., \\
 & & & & & & ... & ... & ... & ... \\
 & & & & & & i, & ..., & ..., & n_i \\
 & & & & & & & ... & ... & ... \\
 & & & & & & & (\alpha-1), & ..., & n_{\alpha-1}) \ ,
\end{array}
$$

where $n_{i+1} \leq n_i, \quad i = 0, ..., \alpha - 2$.

In particular, the description of the Castelnuovo function $\Gamma(a, b; \cdot)$ of a c.i. recalled in $1.\mathbf{A}_4$ is equivalent to say that $M_{\Gamma(a,b;\cdot)}$ has the following shape:

$$
M_{\Gamma(a,b;\cdot)} =
$$

$$
\begin{array}{lllllllllll}
(0, & 1, & 2, & ..., & (a-1), & ..., & ..., & ..., & (a+b-3), & (a+b-2), \\
 & 1, & 2, & ..., & (a-1), & ..., & ..., & ..., & (a+b-3), & \\
 & & ... & ... & ... & ... & ... & ... & & \\
 & & & (a-2), & (a-1), & ..., & ..., & b, & & \\
 & & & & (a-1), & ..., & (b-1)) \ . & & &
\end{array}
$$

We recall the following:

<u>Theorem</u> (see [7],Cor.6.3,Th.7.3) M_Γ is a realizable sequence. Every realizable sequence S with Castelnuovo function Γ can be obtained from M_Γ with a finite number of direct transpositions.

2. Realizable sequences characterizing a $CI(a, b)$.

Our aim is to characterize a 0-dimensional reduced scheme, which is a $c.i$ in $\mathbf{P}^2$, by means of its set of realizable sequences. More precisely, we prove the following

Theorem 1. *A reduced 0-dimensional scheme $\mathbf{X}$ of $\mathbf{P}^2 = \mathbf{P}^2_k$, k algebraically closed, is a $CI(a, b)$ iff every realizable sequence S of $\mathbf{X}$ can be obtained, by $d.t.$, from the sequence*

$$
\begin{array}{lllllllll}
M_{ab} = (0, & 1, & 2, & ..., & ..., & (b-1), & & & \\
 & 1, & 2, & ..., & ..., & (b-1), & b, & & \\
 & ... & ... & ... & ... & ... & ... & & \\
 & & & (a-1), & ..., & (b-1), & ..., & (a+b-2)) \ .
\end{array}
$$

Remark 1. M_{ab} is a sequence of segments, each of which contains b elements; it is easy to check that it can be obtained from $M_{\Gamma(a,b)}$ by $d.t.$. So, if S arises from M_{ab} by $d.t.$, then it comes by $d.t.$ also from $M_{\Gamma(a,b)}$ (see $\mathbf{1}.\mathbf{B}_5$).

Notation-Definition 2. *If $\mathbf{Y}$ is any 0-dimensional reduced scheme in $\mathbf{P}^2$, and $P \in \mathbf{Y}$, let us denote $d_{\mathbf{Y}}(P)$ the degree of separation of P with respect to $\mathbf{Y} - \{P\}$, that is the least degree of a curve containing $\mathbf{Y} - \{P\}$ and not containing P.*

Remark 2. If $\mathbf{X}$ is a $c.i.$ of type (a, b), the Cayley- Bacharach condition says that $d_{\mathbf{X}}(P) = a + b - 2$, for every point $P \in \mathbf{X}$. Theorem 1 gives information on the possible degrees of separation of a point of $\mathbf{X}$ with respect to any subscheme not containing it.

In the sequel we denote $f_d, g_d, ...$ a homogeneous polynomial of degree d.

We need the following preliminary result (see also [2],Th.3.1):

Proposition 1. *I is the homogeneous ideal of a $c.i$ of type (a, b) iff the following conditions are satisfied:*
$i)$ $\Gamma(R/I, \cdot) = \Gamma(a, b; \cdot)$
$ii)$ For every $h_t, t < a$, the ideal $J = (h_t, I)$ satisfies the condition: $e(R/J) \leq tb$.

<u>Proof</u> Let us suppose $I = (f_a, g_b)$ to be the ideal of a $c.i.$ Then condition $i)$ is clearly satisfied $(1.A_4)$. Moreover, the regularity of the sequence (f_a, g_b) implies: $depthJ \geq 2$; so also the sequence (h_t, g_b) is regular; this implies $e(R/(h_t, g_b)) = tb$ and the inclusion $(h_t, g_b) \subseteq J$ gives $e(J) \leq tb$.

Viceversa, let us suppose $i)$ and $ii)$ verified. We first recall that condition $i)$ implies I to have two generators f_a and g_b and all the other generators (if they exist) to be of degree greater than b. So, I is a $c.i$ iff f_a, g_b is a regular sequence: in fact, if such a sequence is regular, its Γ 's structure says that there are no more generators in degree greater than b ([11]) . Let us suppose $G.C.D(f_a, g_b) = h_{\bar{t}},\ 0 < \bar{t} < a$, and prove that $J = (h_{\bar{t}}, I)$ satisfies the relation $e(R/J) > \bar{t}b$, against $ii)$. To this aim, let us denote c the minimal degree in which a third generator of I appears; then c is a degree of maximal growth, so that we can apply the "splitting theorem" 3.1 of [12]. We get that $\Gamma(R/J, \cdot)$ is the truncation of $\Gamma(R/I, \cdot)$ at level $\bar{t} = \Gamma(R/I, c - 1)$, $c = a + b - \bar{t}$, that is:

$$\Gamma(R/J, u) = \begin{cases} u + 1, & u \leq \bar{t} - 1 \\ \bar{t}, & \bar{t} \leq u \leq a + b - \bar{t} - 1 \\ \Gamma(I, u), & u \geq a + b - \bar{t}\ . \end{cases}$$

As a consequence, for $u >> 0$, $H(R/J, u) = (a + b - \bar{t})\bar{t} = e(R/J) > \bar{t}b$.

Now we are ready to prove the first part of Theorem 1 .

Theorem 2. *Let $\mathbf{X}$ be a 0-dimensional reduced scheme in $\mathbf{P}^2$ such that all its realizable sequences can be obtained from M_{ab} by a finite number of $d.t.$. Then $\mathbf{X}$ is a $CI(a, b)$.*

<u>Proof</u> It is enough to prove that, if I is the homogeneous ideal of $\mathbf{X}$, then it satisfies condition $ii)$ of Proposition 1, as the equality $\Gamma(M_{ab}, \cdot) = \Gamma(a, b; \cdot)$ guaranties condition $i)$. Let us suppose that condition $ii)$ is not satisfied, so that there exists a form $h_t,\ t < a$, such that $e(R/J) = e(R/J^{sat}) = m > tb$, with $J = (h_t, I)$. As I and, as a consequence J^{sat}, are ideals of reduced schemes, this hypothesis means that there are just m distinct

points $P_1, ..., P_m$ of $\mathbf{X}$ lying on a curve of degree t. So any ordering of $\mathbf{X}$ starting with $P_1, ..., P_m$, gives rise to a realizable sequence $S = (s_1, ..., s_m, u, ...)$, where $u \leq t$, as any other point of $\mathbf{X}$ is separated from $P_1, ..., P_m$ by a curve of degree $\leq t$. Now, it is enough to observe that such a sequence cannot be obtained by $d.t.$ from M_{ab}, because in M_{ab} the number of the elements appearing before the last $u, u < a$ is ub and a sequence of $d.t.$ cannot increase such a number.

To prove the other implication of Theorem 1, we need some preliminary considerations.

It is immediate to prove the following :

Proposition 2. *Let $S = (s_1, ..., s_n)$ be a sequence of natural numbers.*
$i)$ If $s_i < s_{i+h}$, no product of d.t. can shift s_i after s_{i+h}.
$ii)$ If $s_i \geq s_{i+h}$, there exists a product of d.t. shifting s_i after s_{i+h}.
$iii)$ There exists a product of d.t. taking s_i to the last place in the sequence iff : $k > i \Rightarrow s_k \leq s_i$.

Definition 1. *The sequence $S_h = (s_1, ..., s_h)$ will be called "truncation of $S = (s_1, ..., s_n)$ at s_h"; it may be useful to set $S = S_n$.*

Let us remark that $\Gamma(S_h, t)$ is the number of occurrences of t in S before s_{h+1}. We will denote Σ_n the group of all permutations on $\{1, ..., n\}$.

Let $M = (m_1, ..., m_n)$ and $S = (s_1, ..., s_n)$ be sequences with the same function Γ. To the couple (S, M) we can associate all the elements $\psi \in \Sigma_n$ satisfying the condition

$$m_{\psi(h)} = s_h, \ h = 1, ..., n.$$

Each ψ determines a bijection $\Psi : S \longrightarrow M$, defined by $\Psi(s_h) = m_{\psi(h)}, \ h = 1, ..., n$.

We point our attention on a special ψ, uniquely associated to (S, M), as follows.

Definition 2. *Let us denote ϕ_{SM} (or, briefly, ϕ) the element of Σ_n satisfying the following conditions:*

$$m_{\phi(h)} = s_h, \qquad \Gamma(M_{\phi(h)}, m_{\phi(h)}) = \Gamma(S_h, s_h).$$

In other words, if $u = s_h$ is the r-th u appearing in S, then $m_{\phi(h)}$ is the r-th u appearing in M.

We will denote $\Phi_{SM} : S \longrightarrow M$ (or, briefly, Φ) the bijection: $\Phi(s_h) = m_{\phi(h)}$.

Remark 3. It is easy to check that:
$a)$ $\Phi_{SM} \circ \Phi_{S'S} = \Phi_{S'M}$, for every S, S' such that $\Gamma(S, \cdot) = \Gamma(S', \cdot) = \Gamma(M, \cdot)$.
$b)$ If $\psi_{i,i+1}$ is a cycle, of order two, exchanging i and $i+1$ and $s_i \neq s_{i+1}$, the corresponding $\Psi_{i,i+1}$, acting on S, is a Φ, that we will denote $\Phi_{i,i+1}$.

Proposition 3. *The following facts are equivalent:*
$i)$ Φ^{-1} is a product of d.t.
$ii)$ S can be obtained from M by d.t.

Proof $i) \Rightarrow ii)$ Obvious, as Φ^{-1} sends M to S.
$ii) \Rightarrow i)$ Let $\Theta = \prod_{h=1}^{r} \Theta_{i,i+1}^{(h)}$ be the product of d.t. sending M to S. Then $\Theta^{-1} = \prod_{h=r}^{1} (\Theta_{i,i+1}^{(h)})^{-1}$, where, according to Remark $b)$ to Definition 2, $(\Theta_{i,i+1}^{h})^{-1} = \Phi_{i,i+1}$. As a consequence, thanks to $a)$ of the same Remark, Θ^{-1} is a Φ.

Proposition 4. *The following facts are equivalent:*
$i)$ $\forall h,\ S_h$ can be obtained from $\Phi(S_h)$ by d.t..
$ii)$ S can be obtained from M by d.t..

Proof $i) \Rightarrow ii)$ Obvious, as $S = S_n$.
$ii) \Rightarrow i)$ According to Proposition 3, Φ^{-1} is a product of d.t.; to prove that so is its restriction to S_h, it is enough to verify that $(\Phi/S_{n-1})^{-1}$ is a product of d.t. and then use induction. Let us consider the decomposition $\Phi = \prod \Phi_{i,i+1}$, where $\Phi_{i,i+1}^{-1}$ is a d.t. from M to S. It is enough to observe that the restriction of Φ to S_{n-1} can be obtained from that product by deleting all the elements acting non trivially on s_n.

Now we enter the heart of the problem with the following

Proposition 5. *The following facts are equivalent:*
$i)$ S can be obtained from M by d.t..
$ii)$ $h \in \{1, ..., n\},\ t > \phi(h),\ m_t \in \Phi(S_h) \Rightarrow m_t < m_{\phi(h)}$.
In other words, all the elements of $\Phi(S_h)$, following $\Phi(s_h) = m_{\phi(h)}$ in M, must be strictly less than s_h.
$iii)$ $h \in \{1, ..., n\}, u > s_h \Rightarrow \Gamma(S_{h-1}, u) = \Gamma(S_h, u) \leq \Gamma(M_{\phi(h)}, u)$.

Proof $i) \Rightarrow ii)$ Proposition 4 says that S_h can be obtained from $\Phi(S_h)$ by d.t.; as a consequence, Proposition 2 $iii)$ states that $\Phi(s_h)$ must be followed, in $\Phi(S_h)$, by elements less than it.
$ii) \Rightarrow i)$ Let us prove that S_h can be obtained by d.t. from $\Phi(S_h)$, for every h: the value $h = n$ will give the statement. We use induction on h. The case $h = 1$ gives $S_1 = \Phi(S_1)$, so that there is nothing to prove. Let us suppose the statement true for $h - 1$ and prove it for h. It is enough to prove that $\Phi(s_h)$ can be brought to the last place by d.t. and this is true, thank to Proposition 2 $iii)$.

To prove the equivalence between $ii)$ and $iii)$, we have better to rewrite condition $ii)$ in the following equivalent form:

$$ii')\ h \in \{1, ..., n\}, u > s_h \Rightarrow \Gamma(S_h, u) = \Gamma(\Phi(S_h)_{m_{\phi(h)}}, u),$$

where we define: $\Phi(S_h)_{m_{\phi(h)}} = M_{\phi(h)} \bigcap \Phi(S_h)$. $ii') \Rightarrow iii)$ It is enough to use the obvious relation: $\Gamma(\Phi(S_h)_{m_{\phi(h)}}, u) \leq \Gamma(M_{\phi(h)}, u)$.
$iii) \Rightarrow ii)$ For any $h \in \{1, ..., n\}$ and $u > s_h$, we must prove the following implication:

$$\Gamma(S_h, u) \leq \Gamma(M_{\phi(h)}, u) \Rightarrow \Gamma(S_h, u) = \Gamma(\Phi(S_h)_{m_{\phi(h)}}, u).$$

Without any hypothesis on u, we have:

$\Gamma(S_h, u) = \Gamma(\Phi(S_h), u) \geq \Gamma(\Phi(S_h)_{m_{\phi(h)}}, u)$,

so that the equality to be proved can be substituted by the inequality

$\Gamma(S_h, u) \leq \Gamma(\Phi(S_h)_{m_{\phi(h)}}, u)$.

If such a relation is not true, there exists $u = s_t,\ t < h$, such that $\phi(t) > \phi(h)$. The definition of ϕ holds: $\Gamma(M_{\phi(t)}, u) = \Gamma(S_t, u)$.

On the other hand, we have:

$\Gamma(M_{\phi(h)}, u) < \Gamma(M_{\phi(t)}, u)$,

$\Gamma(S_t, u) \leq \Gamma(S_h, u)$.

Hence we get: $\Gamma(M_{\phi(h)}, u) < \Gamma(S_h, u)$, a contradiction.

Now let us point our attention on the case $M = M_{ab}$. In this situation, Proposition 5 $iii)$ gives rise to the following

Proposition 6. *A realizable S can be obtained from M_{ab} by d.t. iff, for every $h = 1, ..., n$, the following relations hold:*

$i)$ $u > s_h \geq b-1,\ \Gamma(S_{h-1}, u) \neq 0 \Rightarrow \Gamma(S_{h-1}, u) < \Gamma(S_{h-1}, u-1)$

$ii)$ $s_h < b-1,\ 0 \leq \tau \leq \Gamma(S_{h-1}, s_h) \Rightarrow \Gamma(S_{h-1}, b-1+\tau) \leq \Gamma(S_{h-1}, s_h) - \tau$.

Proof Let us write condition $iii)$ of Proposition 5 in the hypothesis $M = M_{ab}$.

If $s_h \geq b-1,\ 0 < t \leq \Gamma(S_{h-1}, s_h)$, we have:

$$\Gamma(M_{\phi(h)}, s_h + t) = \Gamma(M_{\phi(h)}, s_h) - t - 1 = \Gamma(S_{h-1}, s_h) - t, \tag{1}$$

so that condition $iii)$ of s:

$$\Gamma(S_{h-1}, s_h + t) \leq \Gamma(S_{h-1}, s_h) - t,\ s_h \geq b-1,\ 0 < t \leq \Gamma(S_{h-1}, s_h). \tag{2}$$

If $s_h < b-1$, there are two different situations, according to the value at which we compute $\Gamma(M_{\phi(h-1)}, \cdot)$:

$$\Gamma(M_{\phi(h)}, s_h + t) = \Gamma(M_{\phi(h)}, s_h) - 1 = \Gamma(S_{h-1}, s_h),\ \text{if } s_h + t \leq b-1; \tag{3}$$

if $s_h + t \geq b-1$, let us set $s_h + t = b - 1 + \tau,\ \tau \leq \Gamma(S_{h-1}, s_h)$, so getting

$$\Gamma(M_{\phi(h)}, b-1+\tau) = \Gamma(M_{\phi(h)}, b-1) - \tau - 1 = \Gamma(M_{\phi(h-1)}, s_h) - \tau = \Gamma(S_{h-1}, s_h) - \tau. \tag{4}$$

Condition (3) allows to translate $iii)$ of Proposition 5 into:

$$\Gamma(S_{h-1}, s_h + t) \leq \Gamma(S_{h-1}, s_h),\ s_h + t \leq b-1 \tag{5}$$

Condition (5) is always verified, because every scheme realizing S_{h-1} is contained in a curve of degree s_h, so that $\alpha(S_h) \leq s_h$ and that implies $\Gamma(S_h, \cdot)$ not increasing, from s_h ahead.

Analogously, (4) allows us to translate $iii)$ into:

$$\Gamma(S_{h-1}, b-1+\tau) \leq \Gamma(S_{h-1}, s_h) - \tau,\ 0 \leq \tau \leq \Gamma(S_{h-1}, s_h),\ s_h < b-1, \tag{6}$$

which is condition (ii) of the statement.

So, we just have to prove that (2) is equivalent to (i).

$(2) \Rightarrow i)$ If we choose $t = 1$, (2) becomes:

$$\Gamma(S_{h-1}, s_h + 1) \leq \Gamma(S_{h-1}, s_h) - 1,\ \text{if}\ \Gamma(S_{h-1}, s_h) \geq 1, \tag{7}$$

which is condition $i)$ when we chose $u = s_h + 1$, for any h.

Now let us suppose the existence of $\bar{h}$ and $\bar{u}$ such that $i)$ fails and get a contradiction. As $\bar{u} > b-1 \geq a-1 \geq \alpha(S_h) - 1$, if $i)$ is not true we must have the equality:

$$\Gamma(S_{\bar{h}-1}, \bar{u}) = \Gamma(S_{\bar{h}-1}, \bar{u}-1) \neq 0. \tag{8}$$

However, $\bar{u} > b-1$ implies $\Gamma(S, \bar{u}) < \Gamma(S, \bar{u}-1)$, so that there is at least an element $\bar{u}-1$ appearing in the sequence S after $s_{\bar{h}}$; let $s_{\bar{h}+w} = \bar{u}-1$ be the first $\bar{u}$ following $s_{\bar{h}}$. Clearly no $\bar{u}$ can appear in the sequence S with index between $\bar{h}$ and $\bar{h}+w$, because, otherwise, we should have: $\Gamma(S_{\bar{h}+w}, \bar{u}) > \Gamma(S_{\bar{h}+w}, \bar{u}-1)$. So, (8) implies: $\Gamma(S_{\bar{h}+w-1}, \bar{u}) = \Gamma(S_{\bar{h}+w-1}, \bar{u}-1)$, contrary to (7), with h replaced by $\bar{h}+w$ and $\bar{u}-1 = s_{\bar{h}+w}$.

$i) \Rightarrow 2)$ It is enough to write $i)$ with $u = s_h + 1, ..., s_h + t$ and add each side of the inequality.

Now, we are ready to prove the remaining implication of Theorem 2.1, that is:

Theorem 3. *Any sequence S, realized by a c.i. scheme of type (a,b), can be obtained from M_{ab} by d.t..*

<u>Proof</u> Let us prove that a sequence S not verifying the conditions of Proposition 6 cannot be realized by a c.i.scheme. First of all we prove that, if the conditions of Proposition 6 are not satisfied by S, there exists a couple of integers $(\bar{h}, \bar{u})$, $\bar{u} > sup\ (s_{\bar{h}}, b-1)$, such that

$$\Gamma(S_{\bar{h}-1}, \bar{u}) = \Gamma(S_{\bar{h}-1}, \bar{u}-1) \neq 0. \tag{9}$$

In fact,let $(s_{\bar{h}}, \bar{u})$ be a couple not satisfying Proposition 6. There are two possibilities. If $s_{\bar{h}} \geq b-1$, the relation $\bar{u} \geq \alpha(S_{\bar{h}-1}) - 1$ implies that $\Gamma(S_{\bar{h}-1}, \cdot)$ cannot be strictly increasing from $\bar{u}$ to $\bar{u}+1$, so that equality (9) holds.

If $s_{\bar{h}} < b-1$, set $\bar{u} = b-1+\bar{t}$, where $\bar{t}$ is the first τ for which $ii)$ fails. Then the inequalities :

$$\Gamma(S_{\bar{h}-1}, (b-1)+(\bar{t}-1)) \leq \Gamma(S_{\bar{h}-1}, s_{\bar{h}}) - \bar{t} + 1, \qquad \Gamma(S_{\bar{h}-1}, b-1+\bar{t}) > \Gamma(S_{\bar{h}}, s_{\bar{h}-1}) - \bar{t}$$

imply $\Gamma(S_{\bar{h}-1} + (b-1) + (\bar{t}-1)) \leq \Gamma(S_{\bar{h}-1}, b-1+\bar{t})$. As the other inequality always holds, we have: $\Gamma(S_{\bar{h}-1}, b-1+\bar{t}) = \Gamma(S_{\bar{h}-1}, (b-1)+(\bar{t}-1))$.

So (9) is proved and we just observe that a sequence with a truncation $S_{\bar{h}-1}$ for which (9) holds cannot be realized by a c.i.. In fact for a subscheme Y of a c.i. of type (a, b) we cannot have $\Gamma(Y, t) = \Gamma(Y, t+1), t \geq b$ ([13]).

We can easily characterize the $CI(a, b)$-schemes realizing M_{ab}.

Proposition 7. *A reduced scheme $\mathbf{X}$ is the c.i. of a curve C_b with a curve C_a, splitting into a distinct lines, iff $\mathbf{X}$ realizes exactly M_{ab} and the sequences obtained from M_{ab} by d.t..*

Proof Let us suppose the sequences realized by $\mathbf{X}$ to be M_{ab} and the ones obtained from it by d.t.. We prove the statement by induction on a. If $a = 1$, we just have b points on a line and $M_{1b} = M_{\Gamma_{\mathbf{X}}} = (0, ..., b-1)$. So, let us suppose the statement true until $a-1$ and prove it for a. The subsequence $M_{(a-1)b} \subset M_{ab}$ is realized by a subscheme $\mathbf{Y}$ of $\mathbf{X}$. The hypothesis that all the realizable sequences of $\mathbf{X}$ arise from M_{ab} by d.t. implies the analogous property for the couple $(\mathbf{Y}, M_{(a-1)b})$. In fact, every sequence S' realized by $\mathbf{Y}$ can be considered as a truncation of a sequence realized by $\mathbf{X}$ and $\Phi(S) = M_{ab},\ \Phi(S') = M_{(a-1)b}$; now we can apply Proposition 4. Using induction, we conclude that $\mathbf{Y}$ is the c.i. of a curve C'_b with a $C'_{a-1} = L_1 \cup ... \cup L_{a-1}$, split into $a-1$ lines. As C'_b is defined uniquely, apart from a multiple of C'_{a-1}, and any curve containing $\mathbf{X}$ contains also $\mathbf{Y}$, we can choose $C'_b = C_b$. Now we prove that $\mathbf{X} - \mathbf{Y} = \{P_1, ..., P_b\}$ lies on a line. To this aim, let us first observe that the separating degrees of $P_1, ..., P_b$ are respectively $a-1, a, ..., a+b-2$; the first point $P_j,\ j > 2$, not on the line $L = (P_1, P_2)$, could be separated from the previous ones by $C'_{a-1} \cup L$, so that its degree of separation should be $a < a + j - 2$, a contradiction.

Viceversa, let us suppose $\mathbf{X} = (L_1 \cup ... \cup L_a) \cap C_b$. Thanks to Theorem 1, it is enough to prove that $\mathbf{X}$ realizes M_{ab}. We use induction on a. If $a = 1$, the statement is obvious. So, let us suppose $\mathbf{X} = \mathbf{Y} \cup (L_a \cap C_b)$, where $\mathbf{Y} = (L_1 \cup ... \cup L_{a-1}) \cap C_b$ realizes $M_{(a-1)b}$. If $L_a \cap C_b = (P_1, ..., P_b)$, then P_i is separated from the previous points of $\mathbf{X}$ by a curve $C_{a-1} \cup C_{i-1}$, so that its degree of separation is $\leq a + i - 2$; as the set of numbers corresponding to $\{P_1, ..., P_b\}$ is exactly $\{(a-1), a, ..., (a+b-2)\}$, we get that P_i is associated to $a + i - 2$.

3. Some Applications

Our proof of Theorem 3 lies on a rewriting of condition $iii)$ in Proposition 5, when $M = M_{ab}$. It turns out to be of interest also to see what condition $ii)$ becomes in the case $M = M_{ab}$. To this aim, let us prove the following

Proposition 8. *If $M = M_{ab}$, condition $ii)$ in Proposition 5 is equivalent to the following description of $\Phi(S_h), h = 1, ..., n, n = ab$ (see Definition 2):*

1) $\Phi(S_h) = (\Sigma_0^h, \Sigma_1^h, ..., \Sigma_{a-1}^h)$, *where $\Sigma_j^h = (j, j+1, ..., \sigma_j^h)$ is a (possibly empty) truncation of the j-th segment Σ_j appearing in M_{ab}.*

2) *Let us denote j_k the first j_u for which $\Sigma^h_{j_u} \neq \Sigma_{j_u}$. Then $j \geq j_k$ implies $\sigma^h_j \geq \sigma^h_{j+1}$.*

Proof Let us suppose condition $ii)$ of Proposition 5 verified and prove 1) and 2) for h decreasing from n to 1.

If $h = n$, we have $S_n = S, \Phi(S_n) = M_{ab}$ and 1) and 2) are trivially verified.

Now let us suppose 1) and 2) verified for $h = k+1$ and prove them for $h = k$. As condition $ii)$ is supposed true, in particular, for $\Phi(s_{k+1}) = m_{\phi(k+1)} \in \Phi(S_{k+1})$, we have necessarily:

$$\Phi(s_{k+1}) = \sigma^{k+1}_j, \tag{10}$$

where $\sigma^{k+1}_j > \sigma^{k+1}_{j+1}$, for some $j = 1, ..., n$. As a consequence, 1) and 2) are verified also for $\Phi(S_k)$, obtained from $\Phi(S_{k+1})$ by deleting $\Phi(s_{k+1})$.

Now, let us suppose 1) and 2) true and prove $ii)$. Let us consider $S_k \subset S_{k+1}$ and their images $\Phi(S_k) \subset \Phi(S_{k+1})$. As both $\Phi(S_k)$ and $\Phi(S_{k+1})$ satisfy 1) and 2), $\Phi(s_{k+1})$ must be the last element of a segment of $\Phi(S_{k+1})$, and the required inequality $\sigma^{k+1}_j \geq \sigma^{k+1}_{j+1}$ must be strict; in fact, the equality $\sigma^{k+1}_j = \sigma^{k+1}_{j+1}$ would imply that condition 2) fails for $\Phi(S_k)$. So, condition (10) is verified and, as a consequence, all the elements following $\Phi(s_k)$ in $\Phi(S_{k+1})$ are less than it.

As a consequence of Theorem 1 and Proposition 8, we can enrich a well known characterization of the 0-dimensional schemes in $\mathbf{P}^2$, with Castelnuovo function $\Gamma(a, b)$, which are c.i..

Theorem 4. *([8],[4],[14]) Let $\mathbf{X}$ be a zero dimensional reduced scheme in $\mathbf{P}^2$, with $\Gamma(\cdot) = \Gamma(a, b; \cdot)$. The following facts are equivalent:*

$i)$ $\mathbf{X}$ is a c.i..

$ii)$ Every realizable sequence of $\mathbf{X}$ ends with $a + b - 2$.

$iii)$ Every realizable sequence of $\mathbf{X}$ arises from M_{ab} by d.t..

Proof Theorem 1 states the equivalence between $i)$ and $iii)$. The implication $iii) \Rightarrow ii)$ is obvious, as $a + b - 2$ is the last and the greatest element of M_{ab}. So, we just need to prove $ii) \Rightarrow iii)$. To this aim, let us suppose that $\mathbf{X}$ realizes a sequence S which cannot arise by d.t. from M_{ab}; we will produce a sequence S', realized by $\mathbf{X}$, ending with a number strictly less than $a + b - 2$.

Our hypothesis on S says that there exists an integer k such that the conditions of Proposition 8 are satisfied for $h = n, n-1, ..., k+1$, but they are not satisfied at $h = k$. This is equivalent to say that $\Phi(s_{k+1})$ is not between the elements σ^{k+1}_j, satisfying the condition $\sigma^{k+1}_j > \sigma^{k+1}_{j+1}$.

Let us denote b_k the first integer t satisfying the relation: $\Gamma(S_{k+1}, t) < \Gamma(S_{k+1}, t-1)$. The shape of $\Phi(S_{k+1})$, described in Proposition 8, says that $b_k \leq b$. If $s_{k+1} < b - 1$, Macaulay's theorem $\mathbf{1.A}_3$, applied to $\Phi(S_k)$, says that $\Phi(s_{k+1})$ cannot be different from σ^{k+1}_j, the end of the segment Σ^{k+1}_j to which it belongs. Moreover, the property $\Gamma(\Phi(S_{k+1}), \Phi(s_{k+1})) = \Gamma(S_{k+1}, s_{k+1})$ guaranties that $\Phi(s_{k+1}) = \sigma^{k+1}_j$ implies $\sigma^{k+1}_j > \sigma^{k+1}_{j+1}$, as $\Phi(s_{k+1})$ is the last element of $\Phi(S_{k+1})$, which is equal to s_{k+1}. So, if condition

1) or condition 2) is not satisfied, we necessarily have: $s_{k+1} \geq b-1$. In this situation, $\Phi(s_{k+1}) \neq \sigma_j^{k+1}$ implies

$$\Gamma(\Phi(S_k), s_{k+1}) = \Gamma(\Phi(S_k), s_{k+1}+1).$$

So, we can use the splitting theorem of [12], applied to the subscheme $\mathbf{X}_k$ corresponding to the sequence S_k. It says that there is a subscheme $\mathbf{Y}$ of $\mathbf{X}_k$ contained on a curve C_t, of degree $t \geq 1$ and every curve of degree $\leq s_{k+1}+1$ containing $\mathbf{Y}$ has C_t as a component. As a consequence, the curves of degree a and b, say C_a and C_b, containing $\mathbf{X}$, must have C_t as a component, as $s_{k+1} \geq b-1$. Let us denote C_u the maximal common component of C_a and C_b, so that $C_a = C'_{a-u} \cup C_u,\ C_b = C'_{b-u} \cup C_u$. The points of $\mathbf{X}$ not lying on C_u must be in $C'_{a-u} \cap C'_{b-u}$. The maximal degree of separation of the points in $C'_{a-u} \cap C'_{b-u}$ is $(a-u)+(b-u)-2$. So, if we order the points of $\mathbf{X}$ taking before the ones lying on C_u, followed by the ones on $X - C_u$, we easily see that the degree of separation of the last ones cannot exceed $(a+b-u-2) < a+b-2$. In such a way, we produce an ordering of $\mathbf{X}$ giving rise to a realizable sequence S' ending with a number less than $a+b-2$, a contradiction.

As condition $ii)$ is equivalent to the Cayley-Bacharach property, a direct proof of the equivalence between $i)$ and $ii)$ is given in [3](4.21).

Remark 4. Proposition 8 gives an algorithm to produce all the sequences realized by some CI(a,b). In fact it says that any such sequence ends with $a+b-2$ and, for $k = n, ..., 1$ it can be built, step by step, choosing $s_k = \Phi(s_k)$ as one of the σ_j^k's satisfying the condition $\sigma_j^k > \sigma_{j+1}^k$. The number of possibilities increases quickly, but the beginning is the following:

$$\begin{array}{llll} ... & ..., & (a+b-4), & (a+b-3), \\ & & & \\ & & & \left.\right\rangle\ (a+b-3), \quad (a+b-2)\ . \\ & & & \\ ... & ..., & (a+b-3), & \\ & & & \left.\right\rangle\ (a+b-4), \\ ... & ..., & (a+b-5), & \end{array}$$

Let us observe that the fact that every realizable sequence ends with $(..., a+b-3,\ a+b-2)$ is a restatement of the Cayley-Bacharach theorem, stating that, if P, Q is any couple of points of $\mathbf{X}$, then $H_{\mathbf{X}}(t) = H_{\mathbf{X}-\{P,Q\}}(t),\ t < a+b-3$ ([8],[4],[1]).

Proposition 8 can be used to give a condition characterizing a CI(a,b) in terms of the Castelnuovo function of its subschemes. To this aim, let us observe what follows. Theorem 1 and Proposition 8 give a characterization of $\mathbf{X}$ in terms of a family of subsequences $\Phi(S_h)$ of M_{ab}, each of which is linked to a truncation S_h of a sequence S realized by $\mathbf{X}$.

It is convenient to link the subsequence $\Phi(S_h)$ to a subset of $\mathbf{X}$, more then to a subsequence of S. In fact, let $(P_1, ..., P_h, P_{h+1}, ..., P_n)$ be an ordering of $\mathbf{X}$ realizing S; then $(P_1, ..., P_h)$ realizes S_h and any other sequence S'_h realized by $\{P_1, ..., P_h\} = \mathbf{X}_h$ is such that $\Phi(S'_h) = \Phi(S_h)$. In fact S'_h can be completed to a sequence S' realizing $\mathbf{X}$ such that $s'_t = s_t,\ t \geq h$; moreover, $\Phi(S_h)$ and $\Phi(S'_h)$ are obtained from M_{ab} by deleting the elements $\Phi(s_t),\ t \geq h$. So, it is meaningful to introduce the following

Notation Let $\mathbf{Y} = \{P_1, ..., P_h\}$ be a subset of $\mathbf{X}$ and S_h any sequence realized by it. We set:

$$\Phi(S_h) = \Phi(\mathbf{Y}).$$

Remark 5. $\Phi(\mathbf{X}) = M_{ab}$ is more fit to describe a c.i. scheme than its Castelnuovo function $\Gamma_{\mathbf{X}}(\cdot)$ (or, equivalently, $M_{\Gamma_{\mathbf{X}}(\cdot)}$). Analogously, the sequence $\Phi(\mathbf{Y})$, with the order inherited by M_{ab}, gives more information than $\Gamma_{\mathbf{Y}}(\cdot)$ (or, equivalently, $M_{\Gamma_{\mathbf{Y}}(\cdot)}$), that can be easily built from it; in fact, all the sequences realized by $\mathbf{Y}$ can be obtained by d.t. from $\Phi(\mathbf{Y})$, which is, itself, built by d.t. from $M_{\Gamma_{\mathbf{Y}}(\cdot)}$.

Let us denote $\widetilde{S_{\mathbf{X}}}$ the set of all sequences realized by $\mathbf{X}$. When $\mathbf{Y}$ spans the set of all subsets of $\mathbf{X}$, $\Phi(\mathbf{Y})$ spans the set of all subsequences $\Phi(S_h) \subset M_{ab}$, for $S \in \widetilde{S_{\mathbf{X}}},\ h = 1, ..., n$. So, using Proposition 8, Theorem 1 gives rise to the following:

Proposition 9. *A reduced 0-dimensional scheme $\mathbf{X}$ of $\mathbf{P}^2$, such that $\Gamma_{\mathbf{X}}(\cdot) = \Gamma(a, b; \cdot)$, is a $CI(a, b)$ iff, for every subset $\mathbf{Y}$ of $\mathbf{X}$, $\Phi(\mathbf{Y})$ satisfies the following conditions:*
$i)$ $\Phi(\mathbf{Y}) = (\Sigma_0^{\mathbf{Y}}, \Sigma_1^{\mathbf{Y}}, ..., \Sigma_{a-1}^{\mathbf{Y}})$, where $\Sigma_j^{\mathbf{Y}} = (j, j+1, ..., \sigma_j^{\mathbf{Y}})$ is a (possibly empty) truncation of the j-th segment Σ_j appearing in M_{ab}.
$ii)$ If j_k is the first j_u for which $\Sigma_{j_u}^{\mathbf{Y}} \neq \Sigma_{j_u}$, then $j \geq j_k$ implies $\sigma_j^{\mathbf{Y}} \geq \sigma_{j+1}^{\mathbf{Y}}$.

It is immediate to verify that conditions $i)$ and $ii)$ can be read on $\Gamma_{\mathbf{Y}}(\cdot)$, giving rise to the following

Proposition 10. *A reduced 0-dimensional scheme $\mathbf{X}$ of $\mathbf{P}^2$ is a $CI(a, b)$ iff*
a) $\Gamma_{\mathbf{X}}(\cdot) = \Gamma(a, b; \cdot)$;
b) for every subset $\mathbf{Y}$ of $\mathbf{X}$, $\Gamma_{\mathbf{Y}}(\cdot)$ satisfies the condition: $\Gamma_{\mathbf{Y}}(t) > \Gamma_{\mathbf{Y}}(t+1),\ t \geq b-1,\ \Gamma_{\mathbf{Y}}(t) \neq 0$.

Example Let us suppose $\Gamma_{\mathbf{X}}(\cdot) = \Gamma(3, 4; \cdot)$ and X not a c.i., so that there exists a subscheme $\mathbf{Y} \subset \mathbf{X}$ not satisfying one of the two conditions $i)$ and $ii)$ of Proposition 9. By Maculay's theorem ([10]), it is enough to consider subschemes $\mathbf{Y}$ such that $\Phi(\mathbf{Y})$ comes from M_{ab} by deleting elements bigger than $b-1$. In this case the only possible $\Phi(\mathbf{Y})$ turns out to be the following:

$$\Phi(\mathbf{Y}) = (0 \quad \begin{array}{cccc} 1 & 2 & & \\ 1 & 2 & 3 & 4 \\ & 2 & 3 & \quad 5 \end{array} \quad).$$

So, $\Gamma(\mathbf{Y}) = (1, 2, 3, 2, 1, 1)$; hence $\mathbf{Y}$ splits into two subschemes ([12]), one of which consists of six points on a line. As a consequence, six points of $\mathbf{X}$ lie on a line and the others are the c.i. of a conic and a cubic.

Let us observe that the $d_{\mathbf{Y}}(P)$'s, for some $P \in \mathbf{Y}$, are exactly the last elements of some realizable sequence of $\mathbf{Y}$, that is, the numbers appearing in $\Phi(\mathbf{Y})$ followed by smaller integers. So, using the notation of 9, we can state the following

Proposition 11. *If $\mathbf{X}$ is a $CI(a, b)$, then:*
i) $Sup\{\sigma_j^{\mathbf{Y}}\} = d_{\mathbf{Y}}(P)$, for some $P \in \mathbf{Y}$.
ii) For any $P \in \mathbf{Y}$, $d_{\mathbf{Y}}(P) \in \{\sigma_j^{\mathbf{Y}},\ j \geq j_{k-1}\}$.

Remark 6. Taking into account the link between $\Gamma_{\mathbf{Y}}$ and $\Phi(\mathbf{Y})$, we can restate Proposition 11 by saying that, if $\nu = d_{\mathbf{Y}}(P)$, for some $P \in \mathbf{Y}$, then:
i) $\Gamma_{\mathbf{Y}}(\nu) > \Gamma_{\mathbf{Y}}(\nu + 1)$, if $\nu < b - 1$;
ii) $\Gamma_{\mathbf{Y}}(\nu) > \Gamma_{\mathbf{Y}}(\nu + 1) + 1$, if $\nu \geq b - 1, \Gamma_{\mathbf{Y}}(\nu + 1) \neq 0$.

Remark 7. If $\mathbf{Y} = \mathbf{X}$, Proposition 11 says the well known fact that all the points of a $CI(a, b)$ have degree of separation $a + b - 2$ (Cayley-Bacharah condition).

If $card\,\mathbf{Y} = card\,\mathbf{X} - 1$, Proposition 11 says that all the subschemes obtained from a $CI(a, b)$ by deleting one point satisfy the Cayley-Bacharah condition and the degree of separation is $a + b - 3$ (C.B. Theorem).

If $card\,\mathbf{Y} = card\,\mathbf{X} - 2$, Proposition 11 says that every subset $\mathbf{Y}$, obtained from a $C(a, b)$ by deleting 2 points, satisfies the condition: $d_{\mathbf{Y}}(P) = a + b - 3$ or $d_{\mathbf{Y}}(P) = a + b - 4$.(Let us remark that this result could also be obtained by using the C.B. condition and Corollary 2.2 of [6].). More precisely, there is always some point with degree of separation $a + b - 3$, while a point with degree $a + b - 4$ may exist or not; for instance, if the scheme $\mathbf{X}$ realizes M_{ab} (see Proposition 7), there are points of $\mathbf{Y}$ with degree of separation $a + b - 4$ (see Proposition 12), while a scheme in uniform position realizes only the sequence C_Γ (see Corollary 2.7 of [6]), and, as a consequence, all its subschemes satisfy the C.B. condition.

Proposition 12. *If $\mathbf{X}$ is a $CI(a, b)$ realizing M_{ab}, then any subsequence9 is of the type $\Phi(\mathbf{Y})$, for some $\mathbf{Y} \subset \mathbf{X}$.*

<u>Proof</u> We know, (see Proposition 7) that such a scheme is the intersection of a curve of degree b with a lines.

If we consider the ordering of $\mathbf{X}$ realizing M_{ab}, we can verify that the subscheme $\mathbf{Y}$ corresponding to S realizes S. To this aim, we observe that the sequence S, realized by $\mathbf{Y}$, either coincides with S or is obtained from it by replacing some elements with smaller ones. Let us prove that such a replacement cannot occur. If we reorder $\mathbf{X}$ starting with $\mathbf{Y}$, S' can be completed to a sequence $\bar{S}'$ such that $\Gamma(\bar{S}', \cdot) = \Gamma(a, b; \cdot)$. We now work by induction on a. If $a = 1$, the equality $S = S'$ is obvious. So, let us suppose it true until $a - 1$ and prove it for a. We denote X_1 the subscheme of $\mathbf{X}$ realizing $M_{(a-1)b}$, S_1 the subsequence of S contained in $M_{(a-1)b}$ and $\mathbf{Y}_1$ the scheme $\mathbf{Y} \cap \mathbf{X}_1$. By induction, $\mathbf{Y}_1$ realizes S_1. The numbers corresponding to $\mathbf{Y} - \mathbf{Y}_1$ cannot be less than the corresponding ones in $S - S_1$: in fact the hypothesis on S guaranties that they are smaller than the elements of $M_{(a-1)b} - S_1$. As a consequence, S' must coincide with S.

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Presented: November 28, 2006
Published on line: October 8, 2007