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NON-LINEAR HIGHEST SEA WAVE GROUPS IN AN UNDISTURBED FIELD AND IN FRONT OF A VERTICAL WALL

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ABSTRACT. In this paper some non-linear effects for the mechanics of sea wave groups with large waves are investigated, either for waves in an undisturbed field or for waves in front of a vertical wall.
To the first-order in a Stokes expansion, Boccotti's quasi-determinism theory enables us to foresee the mechanics of wave groups, either in undisturbed or in diffracted fields, when a large wave occurs. The first formulation of this theory shows the random group mechanics when a large crest height occurs ('New wave'); the second theory formulation gives the random group mechanics when a large crest-to-trough wave height occurs.
The quasi-determinism theory in both formulations, for undisturbed fields, was extended recently to the second-order by the author. In this paper the procedure to derive the second-order solution is analyzed and is applied to random wave groups in front of a vertical wall. The non-linear effects are then investigated in space-time domain, and it is obtained a good agreement of analytical predictions with both field data and data from numerical simulation.

1. **Introduction**

The study of non-linear sea waves is a topic of interest for the comprehension of freak waves, which occur in the ocean, damaging ships and sea structures.
For the explanation of the occurrence of freak waves an easy way is to consider a strong current, opposite to the wave direction, which amplify the wave generating large waves. Really, most of the freak waves were recorded without any current, so that different approaches have to be considered [see, for example, Slunyaev et al. (2002)].

Here the mechanics of sea wave groups, to the first-order in a Stokes expansion, is firstly investigated applying the quasi-determinism (QD) theory (Boccotti, 1981, 1989, 1997, 2000). This theory, which enables us to predict the linear free surface displacement and velocity potential when a large wave occurs in a fixed time and location, may be applied either for waves in an undisturbed field or in the presence of a structure (see Boccotti, 2000 for a complete review). A verification was found during some small scale field experiments, both for progressive waves (Boccotti et al., 1993*a*) and for waves interacting with structures (Boccotti 1995, 1996, Boccotti et al., 1993*b*).

Phillips et al. (1993*a*, 1993*b*), proposed also an alternative approach for the derivation of the quasi-determinism theory and found a further field verification in the Atlantic Ocean.

The QD theory was given in two formulation: the first one (Boccotti, 1981, 1982) enables us to predict what happens in the space-time domain when a large crest occurs [the theory, in its first formulation and for time domain, was renamed as 'New wave' by Tromans et al. (1991)]; the second formulation enables us to predict what happens in the space-time domain when a large crest-to-trough wave height occurs.
In this paper the QD theory, in both the formulations, is extended to the second-order both for waves in an undisturbed field (see also Arena, 2005) and in front of a vertical wall (wave reflection). This result is obtained by solving the second-order system of differential equations governing an irrotational flow with a free surface.
The results for the second-order 'New wave' theory in an undisturbed field (that is the first formulation of the QD theory, for the highest wave crest), particularized for deep water, are identical to those derived by Fedele & Arena (2003) with a different approach (see also Arena & Fedele, 2005; Fedele & Arena, 2005).

Finally the analytical predictions, for waves in an undisturbed field, are compared with data of Montecarlo simulation of non-linear random waves and with field data given by Taylor and Williams (2002) from WACSIS dataset (Forristall et al, 2002).

2. **The quasi-determinism theory**

Boccotti developed the quasi-determinism (QD) theory, which is exact to the first-order in a Stokes expansion, in eighties, in two formulations.
The first formulation ('New wave') deals with the crest height, and shows that the space-time profile of highest crest is proportional to the autocovariance function (see Appendix B).
The second formulation of the theory deals with the crest-to-trough height; it was derived by obtaining firstly the probability density function of the surface displacement at point x_o+X, y_o+Y, at time t_o+T, given the condition

$$\eta(x_o, y_o, t_o) = \frac{1}{2}H, \quad \eta(x_o, y_o, t_o + T^*) = -\frac{1}{2}H \tag{1}$$

where t_o is an arbitrary time instant, (x_o, y_o) an arbitrary point, *H* the crest-to-trough wave height and T^* the abscissa of the absolute minimum of the autocovariance function (which is assumed to be also the first local minimum of this function on the positive domain: this condition is always verified for wind waves).
The theory shows then that, as $H/\sigma \to \infty$, condition (1) becomes both sufficient and necessary for the occurrence of a wave of given height *H* (for the formal derivation see Boccotti, 1989, 1997, 2000). Therefore, as $H/\sigma \to \infty$, the linear random function $\eta_1(x_o+X, y_o+Y, t_o+T)$ tends asymptotically to the deterministic function

$$\overline{\eta}_1(x_o+X, y_o+Y, t_o+T) = \frac{\Psi(X,Y,T) - \Psi(X,Y,T-T^*)}{\Psi(0,0,0) - \Psi(0,0,T^*)} \frac{H}{2}. \tag{2}$$

In words we have that "*if a wave with a given height H occurs at a fixed point* (x_o, y_o) *and H is very large with respect to the mean wave height at this point, we may expect the water surface near* (x_o, y_o) *to be very close to the deterministic form* (2)".
The linear velocity potential, when the large wave of height *H* occurs, is given by

$$\overline{\phi}_1(x_o+X,y_o+Y,z,t_o+T)=\frac{\Phi(X,Y,z,T)-\Phi(X,Y,z,T-T^*)}{\Psi(0,0,0)-\Psi(0,0,T^*)}\,\frac{H}{2}\,. \tag{3}$$

Note that in Equations (2) and (3) $X, Y,\ z$ and T are the independent variables.
The space-time covariances $\Psi(X,Y,T)$ and $\Phi(X,Y,z,T)$ are defined respectively as

$$\Psi(X,Y,T)\equiv<\eta(x_o,y_o,t)\eta(x_o+X,y_o+Y,t+T)>\,, \tag{4}$$

$$\Phi(X,Y,z,T)\equiv<\eta(x_o,y_o,t)\phi(x_o+X,y_o+Y,z,t+T)>\,. \tag{5}$$

Both free surface displacements (2) and velocity potential (3) may be rewritten as a function of the directional spectrum, for the more general condition of three-dimensional waves.
For long-crested random waves, in an undisturbed field, we have (Boccotti, 1989, 2000):

$$\overline{\eta}_1(y_o+Y,t_o+T)=$$
$$=\frac{H}{2}\frac{\int_0^\infty E(\omega)\left\{\cos\left[\varphi(\omega,Y,T)\right]-\cos\left[\varphi(\omega,Y,T)+\omega T^*\right]\right\}\mathrm{d}\omega}{\int_0^\infty E(\omega)\left[1-\cos\left(\omega T^*\right)\right]\mathrm{d}\omega} \tag{6}$$

$$\overline{\phi}_1(y_o+Y,z,t_o+T)=g\,\frac{H}{2}\cdot$$
$$\cdot\frac{\int_0^\infty \frac{E(\omega)}{\omega}\frac{\cosh\left[k\left(d+z\right)\right]}{\cosh\left[kd\right]}\left\{\sin\left[\varphi(\omega,Y,T)\right]-\sin\left[\varphi(\omega,Y,T)+\omega T^*\right]\right\}\mathrm{d}\omega}{\int_0^\infty E(\omega)\left[1-\cos\left(\omega T^*\right)\right]\mathrm{d}\omega} \tag{7}$$

where $E(\omega)$ is the frequency spectrum and

$$\varphi(\omega_j,Y,T)\equiv k_jY-\omega_jT\,, \tag{8}$$

with

$$k_j\tanh(k_jd)=\omega_j^2/g\,. \tag{9}$$

The second formulation of the QD theory [Equations (6) and (7)] shows then that the largest wave with height H is generated by a random (two-dimensional) wave group which reaches the apex stage of its development at (Y=0, T=0) (see Boccotti, 2000).

3. The derivation of the second-order quasi-determinism theory for random wave groups in an undisturbed field

The quasi-determinism theory, in both the formulations, is exact to the first-order, and satisfies the first-order Stokes equations (see Appendix A). In this paper the theory, in both the formulations, is extended to the second-order, by solving the second-order system of differential equations for an irrotational flow with a free surface.
The second-order solution for the ‘New wave’ (that is the first formulation of the quasi-determinism) theory is given in Appendix B.
Here the second-order solution for the second formulation of the quasi-determinism theory is obtained.

Following perturbation method, the second-order velocity potential $\overline{\phi}$ and free surface displacement $\overline{\eta}$ are given respectively by

$$\overline{\phi}(y_o+Y,z,t_o+T)=\overline{\phi}_1(y_o+Y,z,t_o+T)+\overline{\phi}_2(y_o+Y,z,t_o+T)+o(H^2) \tag{10}$$

$$\overline{\eta}(y_o+Y,t_o+T)=\overline{\eta}_1(y_o+Y,t_o+T)+\overline{\eta}_2(y_o+Y,t_o+T)+o(H^2) \tag{11}$$

being ($\overline{\eta}_1$, $\overline{\phi}_1$) the linear and ($\overline{\eta}_2$, $\overline{\phi}_2$) the second-order components.

The terms ($\overline{\eta}_2$, $\overline{\phi}_2$) are obtained solving the second-order system of differential equations for an irrotational flow with a free surface, which is given by (see Appendix A):

$$g\overline{\eta}_2+\left(\frac{\partial\overline{\phi}_2}{\partial T}\right)_{z=0}+\left(\frac{\partial^2\overline{\phi}_1}{\partial z\partial T}\right)_{z=0}\overline{\eta}_1+\frac{1}{2}\left(\frac{\partial\overline{\phi}_1}{\partial Y}\right)^2_{z=0}+\frac{1}{2}\left(\frac{\partial\overline{\phi}_1}{\partial z}\right)^2_{z=0}=0 \tag{12}$$

$$\left(\frac{\partial\overline{\phi}_2}{\partial z}\right)_{z=0}+\left(\frac{\partial^2\overline{\phi}_1}{\partial z^2}\right)_{z=0}\overline{\eta}_1-\left(\frac{\partial\overline{\phi}_1}{\partial Y}\right)_{z=0}\frac{\partial\overline{\eta}_1}{\partial Y}-\frac{\partial\overline{\eta}_2}{\partial T}=0 \tag{13}$$

$$\frac{\partial^2\overline{\phi}_2}{\partial Y^2}+\frac{\partial^2\overline{\phi}_2}{\partial z^2}=0 \tag{14}$$

$$\left(\frac{\partial\overline{\phi}_2}{\partial z}\right)_{z=-d}=0\,. \tag{15}$$

To solve the second-order equations, it is convenient to combine linearly Equations (12) and (13) so to cancel out $\overline{\eta}_2$ term. In particular we obtain:

$$\begin{aligned}\left(\frac{\partial^2\overline{\phi}_2}{\partial T^2}\right)_{z=0}+g\left(\frac{\partial\overline{\phi}_2}{\partial z}\right)_{z=0}=&-\frac{\partial}{\partial T}\left[\left(\frac{\partial^2\overline{\phi}_1}{\partial z\partial T}\right)_{z=0}\overline{\eta}_1\right]-\frac{1}{2}\frac{\partial}{\partial T}\left[\left(\frac{\partial\overline{\phi}_1}{\partial Y}\right)^2_{z=0}+\left(\frac{\partial\overline{\phi}_1}{\partial z}\right)^2_{z=0}\right]+\\&-g\left(\frac{\partial^2\overline{\phi}_1}{\partial z^2}\right)_{z=0}\overline{\eta}_1+g\left(\frac{\partial\overline{\phi}_1}{\partial Y}\right)_{z=0}\frac{\partial\overline{\eta}_1}{\partial Y}\end{aligned} \tag{16}$$

where the right hand side includes linear terms only.

3.1. The second-order $\overline{\eta}$ and $\overline{\phi}$ expressions when a large crest-to-trough wave height occurs

The linear free surface displacement $\overline{\eta}_1$, given by Eq. (6), may be rewritten in discrete form

$$\begin{aligned}&\overline{\eta}_1(y_o+Y,t_o+T)=\overline{\eta}_{1a}-\overline{\eta}_{1b}=\\&=\sum_{i=1}^{N}\alpha_i\cos\left[\varphi(\omega_i,Y,T)\right]-\sum_{i=1}^{N}\alpha_i\cos\left[\varphi(\omega_i,Y,T)+\omega_iT^*\right]\end{aligned} \tag{17}$$

where

$$\alpha_i = \frac{\frac{H}{2} E(\omega_i) \mathrm{d}\omega_i}{\int_0^\infty E(\omega)\left[1-\cos\left(\omega T^*\right)\right]\mathrm{d}\omega} \tag{18}$$

or in the equivalent form

$$\overline{\eta}_1(y_o + Y, t_o + T) = \sum_{i=1}^{2N} \alpha_i \cos\left[\lambda_i(\omega_i, Y, T)\right] \tag{19}$$

being

$$\begin{cases} \alpha_{i+N} = \alpha_i \\ \omega_{i+N} = \omega_i \end{cases} \qquad \text{for } i = 1, ..., N \tag{20}$$

$$\lambda_i(\omega_i, Y, T) = \begin{cases} \varphi(\omega_i, Y, T) & \text{for } i = 1, ..., N \\ \varphi(\omega_i, Y, T) + \omega_i T^* + \pi & \text{for } i = N+1, ..., 2N. \end{cases} \tag{21}$$

The velocity potential may be then rewritten as

$$\begin{aligned} \overline{\phi}_1(y_o + Y, t_o + T) = g\sum_{i=1}^{N} \alpha_i \omega_i^{-1} \frac{\cosh\left[k_i(d+z)\right]}{\cosh\left[k_i d\right]} \sin\left[\varphi(\omega_i, Y, T)\right] + \\ - g\sum_{i=1}^{N} \alpha_i \omega_i^{-1} \frac{\cosh\left[k_i(d+z)\right]}{\cosh\left[k_i d\right]} \sin\left[\varphi(\omega_i, Y, T) + \omega_i T^*\right] \end{aligned} \tag{22}$$

or as

$$\overline{\phi}_1(y_o + Y, t_o + T) = g\sum_{i=1}^{2N} \alpha_i \omega_i^{-1} \frac{\cosh\left[k_i(d+z)\right]}{\cosh\left[k_i d\right]} \sin\left[\lambda_i(\omega_i, Y, T)\right] \tag{23}$$

with definitions of Equations (20) and (21).
The solution of Eq. (16), including (14) and (15) Equations, gives the second-order velocity potential (for details see Longuet-Higgins, 1963 and Sharma & Dean, 1979):

$$\begin{aligned} \overline{\phi}_2(Y, z, T) = \frac{1}{4} g^2 \sum_{n=1}^{2N}\sum_{m=1}^{2N} \frac{\alpha_n}{\omega_n}\frac{\alpha_m}{\omega_m} \Bigg\{ \frac{\cosh[k_{nm}^-(d+z)]}{\cosh(k_{nm}^- d)} \frac{B_{nm}^-}{\omega_n - \omega_m} \sin\left(\lambda_n - \lambda_m\right) + \\ + \frac{\cosh[k_{nm}^+(d+z)]}{\cosh(k_{nm}^+ d)} \frac{B_{nm}^+}{\omega_n + \omega_m} \sin\left(\lambda_n + \lambda_m\right) \Bigg\} \end{aligned} \tag{24}$$

and, considering Equations (12) and (13), the free surface displacement is:

$$\overline{\eta}_2\left(Y, T\right) = \frac{1}{4}\sum_{n=1}^{2N}\sum_{m=1}^{2N} \alpha_n \alpha_m \left[A_{nm}^- \cos\left(\lambda_n - \lambda_m\right) + A_{nm}^+ \cos\left(\lambda_n + \lambda_m\right)\right] \tag{25}$$

being $\lambda_i = \lambda_i(\omega_i, Y, T)$,

$$A_{nm}^{\pm} = \frac{B_{nm}^{\pm} - k_n k_m \pm \rho_n \rho_m}{\sqrt{\rho_n \rho_m}} + \rho_n + \rho_m \tag{26}$$

$$B_{nm}^{\pm} = \\ = \frac{(\sqrt{\rho_n} \pm \sqrt{\rho_m})\left[\sqrt{\rho_m}\,(k_n^2 - \rho_n^2) \pm \sqrt{\rho_n}\,(k_m^2 - \rho_m^2)\right] + 2\left(\sqrt{\rho_n} \pm \sqrt{\rho_m}\right)^2 (k_n k_m \mp \rho_n \rho_m)}{\left(\sqrt{\rho_n} \pm \sqrt{\rho_m}\right)^2 - k_{nm}^{\pm} \tanh\left(k_{nm}^{\pm} d\right)} \tag{27}$$

$$k_{nm}^{\pm} = | k_n \pm k_m | \tag{28}$$

$$\rho_n = k_n \tanh(k_n d) = \omega_n^2 / g \ . \tag{29}$$

After some algebra, Equations (24) and (25) may be rewritten respectively as:

$$\begin{aligned}\overline{\phi}_2(Y,z,T) = \frac{1}{4} g^2 \sum_{n=1}^{N}\sum_{m=1}^{N} \frac{\alpha_n}{\omega_n}\frac{\alpha_m}{\omega_m} \Bigg\{ &\frac{\cosh[k_{nm}^{-}(d+z)]}{\cosh(k_{nm}^{-} d)} \frac{B_{nm}^{-}}{\omega_n - \omega_m} \Big[\big(1 - \cos(\omega_n T^*) + \\ &- \cos(\omega_m T^*) + \cos(\omega_n T^* - \omega_m T^*)\big)\sin\left(\varphi_n - \varphi_m\right) - \big(\sin(\omega_n T^*) - \sin(\omega_m T^*) + \\ &- \sin(\omega_n T^* - \omega_m T^*)\big)\cos\left(\varphi_n - \varphi_m\right)\Big] + \frac{\cosh[k_{nm}^{+}(d+z)]}{\cosh(k_{nm}^{+} d)} \frac{B_{nm}^{+}}{\omega_n + \omega_m} \Big[\big(1 - \cos(\omega_n T^*) + \\ &- \cos(\omega_m T^*) + \cos(\omega_n T^* + \omega_m T^*)\big)\sin\left(\varphi_n + \varphi_m\right) + \\ &- \big(\sin(\omega_n T^*) + \sin(\omega_m T^*) - \sin(\omega_n T^* + \omega_m T^*)\big)\cos\left(\varphi_n + \varphi_m\right)\Big]\Bigg\}\end{aligned} \tag{30}$$

$$\begin{aligned}\overline{\eta}_2(Y,T) = \frac{1}{4}\sum_{n=1}^{N}\sum_{m=1}^{N} \alpha_n \alpha_m \Big\{ &A_{nm}^{-}\Big[\big(1 - \cos(\omega_n T^*) - \cos(\omega_m T^*) + \cos(\omega_n T^* - \omega_m T^*)\big)\cdot \\ &\cdot \cos\left(\varphi_n - \varphi_m\right) + \big(\sin(\omega_n T^*) - \sin(\omega_m T^*) - \sin(\omega_n T^* - \omega_m T^*)\big)\sin\left(\varphi_n - \varphi_m\right)\Big] + \\ &+ A_{nm}^{+}\Big[\big(1 - \cos(\omega_n T^*) - \cos(\omega_m T^*) + \cos(\omega_n T^* + \omega_m T^*)\big)\cos\left(\varphi_n + \varphi_m\right) + \\ &+ \big(\sin(\omega_n T^*) + \sin(\omega_m T^*) - \sin(\omega_n T^* + \omega_m T^*)\big)\sin\left(\varphi_n + \varphi_m\right)\Big]\Big\}\end{aligned} \tag{31}$$

where

$$\varphi_j = \varphi(\omega_j, Y, T) \ . \tag{32}$$

Finally, from definition of Eq. (18), Equations (31) and (30) may be rewritten as a function of the frequency spectrum:

$$\begin{aligned}\overline{\eta}_2(Y,T) = \frac{H^2}{16}&\left\{\int_0^{\infty} E(\omega)\left[1 - \cos\left(\omega T^*\right)\right]\mathrm{d}\omega\right\}^{-2} \cdot \\ &\cdot \int_0^{\infty}\int_0^{\infty} E(\omega_n) E(\omega_m) \Big\{A_{nm}^{-}\Big[\big(1 - \cos(\omega_n T^*) - \cos(\omega_m T^*) + \\ &\cos(\omega_n T^* - \omega_m T^*)\big)\cos\left(\varphi_n - \varphi_m\right) + \big(\sin(\omega_n T^*) - \sin(\omega_m T^*) - \sin(\omega_n T^* - \omega_m T^*)\big)\cdot \\ &\cdot \sin\left(\varphi_n - \varphi_m\right)\Big] + A_{nm}^{+}\Big[\big(1 - \cos(\omega_n T^*) - \cos(\omega_m T^*) + \cos(\omega_n T^* + \omega_m T^*)\big)\cos\left(\varphi_n + \varphi_m\right) + \\ &+ \big(\sin(\omega_n T^*) + \sin(\omega_m T^*) - \sin(\omega_n T^* + \omega_m T^*)\big)\sin\left(\varphi_n + \varphi_m\right)\Big]\Big\}\mathrm{d}\omega_m \mathrm{d}\omega_n\end{aligned} \tag{33}$$

$$
\begin{aligned}
&\overline{\phi}_2(Y,z,T) = g^2 \frac{H^2}{16}\left\{\int_0^\infty E(\omega)\left[1-\cos\left(\omega T^*\right)\right]\mathrm{d}\omega\right\}^{-2} \int_0^\infty \int_0^\infty E(\omega_n) E(\omega_m) \frac{1}{\omega_n}\frac{1}{\omega_m} \cdot \\
&\cdot\left\{\frac{\cosh[k_{nm}^-(d+z)]}{\cosh(k_{nm}^- d)}\frac{B_{nm}^-}{\omega_n-\omega_m}\left[\left(1-\cos(\omega_n T^*)-\cos(\omega_m T^*)+\cos(\omega_n T^* + \right.\right.\right. \\
&\left.-\,\omega_m T^*)\right)\sin\left(\varphi_n-\varphi_m\right)-\left(\sin(\omega_n T^*)-\sin(\omega_m T^*)-\sin(\omega_n T^*-\omega_m T^*)\right)\cos\left(\varphi_n-\varphi_m\right)\Big]+ \qquad (34) \\
&+\frac{\cosh[k_{nm}^+(d+z)]}{\cosh(k_{nm}^+ d)}\frac{B_{nm}^+}{\omega_n+\omega_m}\left[\left(1-\cos(\omega_n T^*)-\cos(\omega_m T^*)+\cos(\omega_n T^*+\omega_m T^*)\right)\sin(\varphi_n + \right. \\
&\left.\left.+\varphi_m)-\left(\sin(\omega_n T^*)+\sin(\omega_m T^*)-\sin(\omega_n T^*+\omega_m T^*)\right)\cos\left(\varphi_n+\varphi_m\right)\right]\right\}\mathrm{d}\omega_m \mathrm{d}\omega_n \quad .
\end{aligned}
$$

Expressions (31) and (30) [or (33) and (34)] give respectively the second-order free surface displacements and velocity potential when a large crest-to-trough wave, with height H, occurs at Y=0, with the crest of largest wave at T=0.
Note that the solution procedure consider the linear group of long-crested waves with the large wave height H [given by second formulation of QD theory - Eq. (6)], as the superposition of two groups $\overline{\eta}_{1a}$ and $\overline{\eta}_{1b}$ [see Eq. (17)]. The first group $\overline{\eta}_{1a}$ has largest crest at ($Y=0, T=0$), with height $H_c = 0.5H/(1+\psi^*)$, where ψ^* is the narrow bandedness parameter defined as the absolute value of the quotient between absolute minimum and absolute maximum of the autocovariance function $\Psi(0,0,T)$ (Boccotti, 1989, 2000). The second group $\overline{\eta}_{1b}$ have the largest crest at ($Y=0, T=T^*$), with height H_c too (see Figure 1). The difference between these two wave groups [$\overline{\eta}_{1a}$ – $\overline{\eta}_{1b}$ - see Eq. (17)] gives the linear free surface displacements from the second formulation of the QD theory, that is a wave with both crest and trough amplitudes equal to H/2.

For the derivation of the second-order solution, each of the two linear groups has been decomposed in N components; the second-order solution is then obtained by considering all the interactions among the 2N wave components.
This approach may be also applied for other applications. For example for the derivation of the second-order wave groups in front of a vertical wall, when a very high crest occurs. This application is given in Appendix C.

4. **The non-linear wave groups in deep water**

In deep water, the long-crested $\overline{\eta}_2(Y,t)$ and $\overline{\phi}_2(Y,z,t)$ expressions [Equations (33) and (34) respectively] are slightly simplified. In particular we have

$$
\frac{\cosh[k_{nm}^{\pm}(d+z)]}{\cosh(k_{nm}^{\pm} d)} \to \exp(k_{nm}^{\pm} z) \quad \text{as} \quad d \to \infty\,, \tag{35}
$$

and from Eq. (29) it follows that:

$$
\rho_n = k_n = \omega_n^2 / g\ . \tag{36}
$$

Furthermore, in deep water, Equations (26) and (27) reduce themselves to:

$$A_{nm}^{-} = -|\,k_n - k_m\,|; \quad A_{nm}^{+} = k_n + k_m \tag{37}$$

and

$$B_{nm}^{+} = 0\,; \quad B_{nm}^{-} = \frac{4k_n k_m\left(\sqrt{k_n} - \sqrt{k_m}\right)^2}{\left(\sqrt{k_n} - \sqrt{k_m}\right)^2 - |\,k_n - k_m\,|}\,. \tag{38}$$

Finally, the second-order free surface displacement (31), which gives the second-order term $\overline{\eta}_2$ when a large wave height occurs, for long-crested waves in deep water, is simplified as:

$$\begin{aligned}
&\overline{\eta}_2(Y,T) = \frac{H^2}{16}\left\{\int_0^{\infty} E(\omega)\left[1-\cos\left(\omega T^*\right)\right]\mathrm{d}\omega\right\}^{-2}\cdot\\
&\cdot\int_0^{\infty}\int_0^{\infty} E(\omega_n)E(\omega_m)\left\{-|\,k_n-k_m\,|\left[\left(1-\cos(\omega_n T^*)-\cos(\omega_m T^*)+\cos(\omega_n T^*-\omega_m T^*)\right)\cdot\right.\right.\\
&\cos\left(\varphi_n-\varphi_m\right)+\left(\sin(\omega_n T^*)-\sin(\omega_m T^*)-\sin(\omega_n T^*-\omega_m T^*)\right)\sin\left(\varphi_n-\varphi_m\right)\Big]+\\
&+(k_n+k_m)\Big[\left(1-\cos(\omega_n T^*)-\cos(\omega_m T^*)+\cos(\omega_n T^*+\omega_m T^*)\right)\cos\left(\varphi_n+\varphi_m\right)+\\
&\left.\left.+\left(\sin(\omega_n T^*)+\sin(\omega_m T^*)-\sin(\omega_n T^*+\omega_m T^*)\right)\sin\left(\varphi_n+\varphi_m\right)\right]\right\}\mathrm{d}\omega_m\,\mathrm{d}\omega_n\;.
\end{aligned} \tag{39}$$

5. The computation of second-order long-crested free surface displacements and velocity potential

In the previous section the expressions of $\overline{\eta}_2$ and $\overline{\phi}_2$ have been obtained if a large wave height occurs (that is extended to the second-order the second formulation of the QD theory). In Appendix B both $\overline{\eta}_2$ and $\overline{\phi}_2$ have been obtained if a large crest occurs [that is extended to the second-order the first formulation of the QD theory (New wave)].

For the computation of $\overline{\eta}_2$ and $\overline{\phi}_2$ it is convenient to define the nondimensional frequency $w \equiv \omega/\omega_p$ and the nondimensional frequency spectrum $E_w(w)$. For a JONSWAP spectrum (Hasselmann et al, 1973), we have $E(\omega) = \alpha_P g^2 \omega_p^{-5} E_w(w)$, where

$$E_w(w) = w^{-5}\exp\left[-1.25 w^{-4}\right]\exp\left\{\ln\chi_1 \exp\left[-\frac{(w-1)^2}{2\chi_2^2}\right]\right\} \quad \text{with } w \equiv \omega/\omega_p \tag{40}$$

being α_P the Phillips parameter and $\omega_p (\equiv 2\pi/T_p)$ the peak frequency. Parameters χ_1 and χ_2 are equal respectively to 3.3 and 0.08 for the mean JONSWAP spectrum.

The wave number may be then written as $k_j = k_{wj} 2\pi/L_{p0}$ [where $L_{p0} \equiv gT_p^2/(2\pi)$], where the nondimensional wave number k_{wj} is obtained from equation

$$k_{wj}\tanh(k_{wj} 2\pi d/L_{p0}) = w_j^2\,. \tag{41}$$

In this case, Eq. (36) gives $\rho_j = w_j^2 (2\pi/L_{p0})$ and Equations (26) and (27) are rewritten as:

$$A_{nm}^{\pm} = A_{nm_w}^{\pm}(2\pi / L_{p0})$$
$$B_{nm}^{\pm} = B_{nm_w}^{\pm}(2\pi / L_{p0})^2 \tag{42}$$

where nondimensional coefficients $A_{nm_w}^{\pm}$ and $B_{nm_w}^{\pm}$ are

$$A_{nm_w}^{\pm} = \frac{B_{nm_w}^{\pm} - k_{wn}k_{wm} \pm w_n^2 w_m^2}{w_n w_m} + w_n^2 + w_m^2 \tag{43}$$

$$B_{nm_w}^{\pm} = \frac{(w_n \pm w_m)\left[w_m(k_{wn}^2 - w_n^4) \pm w_n(k_{wm}^2 - w_m^4)\right] + 2\left(w_n \pm w_m\right)^2 (k_{wn}k_{wm} \mp w_n^2 w_m^2)}{\left(w_n \pm w_m\right)^2 - |\,k_{wn} \pm k_{wm}\,| \tanh(|\,k_{wn} \pm k_{wm}\,| \,2\pi\, d / L_{p0})}. \tag{44}$$

Finally the total second-order free surface displacement $\overline{\eta}(Y,T) = \overline{\eta}_1(Y,T) + \overline{\eta}_2(Y,T)$ may be calculated as a function of nondimensional integrals; in particular $\overline{\eta}_1(Y,T)$ and $\overline{\eta}_2(Y,T)$ are given respectively by:

$$\overline{\eta}_1(Y,T) = \frac{H}{2}\int_0^\infty E_w(w)\left[\cos(\varphi) - \cos(\varphi + f_w)\right]\mathrm{d}w \left\{\int_0^\infty E_w(w)\left[1 - \cos\left(f_w\right)\right]\mathrm{d}w\right\}^{-1} \tag{45a}$$

$$\begin{aligned}
\overline{\eta}_2(Y,T) = {} & \frac{H^2}{16}\frac{2\pi}{L_{p0}}\left\{\int_0^\infty E_w(w)\left[1 - \cos\left(f_w\right)\right]\mathrm{d}w\right\}^{-2} \cdot \\
& \cdot \int_0^\infty \int_0^\infty E_w(w_n)E_w(w_m)\Big\{A_{nm_w}^{-}\Big[\Big(1 - \cos(f_{w_n}) - \cos(f_{w_m}) + \\
& \cos(f_{w_n} - f_{w_m})\Big)\cos\left(\varphi_n - \varphi_m\right) + \Big(\sin(f_{w_n}) - \sin(f_{w_m}) - \sin(f_{w_n} - f_{w_m})\Big) \cdot \\
& \cdot \sin\left(\varphi_n - \varphi_m\right)\Big] + A_{nm_w}^{+}\Big[\Big(1 - \cos(f_{w_n}) - \cos(f_{w_m}) + \cos(f_{w_n} + f_{w_m})\Big)\cos\left(\varphi_n + \varphi_m\right) + \\
& + \Big(\sin(f_{w_n}) + \sin(f_{w_m}) - \sin(f_{w_n} + f_{w_m})\Big)\sin\left(\varphi_n + \varphi_m\right)\Big]\Big\}\mathrm{d}w_m \mathrm{d}w_n
\end{aligned} \tag{45b}$$

where $f_{w_j} \equiv 2\pi w_j T^*/T_p$ and $\varphi_j \equiv k_{wj} 2\pi Y / L_{p0} - w_j 2\pi T / T_p$ [see Eq. (8)].

6. Applications: The highest sea wave groups in time domain

6.1. The second-order wave groups when a large crest-to-trough wave height occurs

The quasi-determinism theory gives the linear free surface displacement and velocity potential when a large crest-to-trough wave height occurs. Figure 1 shows the linear wave group $\overline{\eta}_1$, at point ($Y = 0$) when a large wave of height H occurs, for a mean JONSWAP spectrum. Figure 1 shows also the two groups $\overline{\eta}_{1a}$ and $\overline{\eta}_{1b}$ in which is decomposed $\overline{\eta}_1$ ($\equiv \overline{\eta}_{1a} - \overline{\eta}_{1b}$). Note that both $\overline{\eta}_{1a}$ and $\overline{\eta}_{1b}$ have largest crest with amplitude $H_c = 0.29H$, being for mean JONSWAP spectrum $\psi^* = 0.73$ [note that $H_c \equiv 0.5H/(1+\psi^*)$].

Figure 2 shows then the second-order effects. In particular, if the wave with height H occurs at $Y = 0$, with $H/\sigma \to \infty$, it shows the time domain linear wave group $\overline{\eta}_1$, the second-order term $\overline{\eta}_2$ and the total second-order free surface displacements $\overline{\eta} = \overline{\eta}_1 + \overline{\eta}_2$, at point $Y = 0$. It is assumed the mean JONSWAP spectrum in deep water.

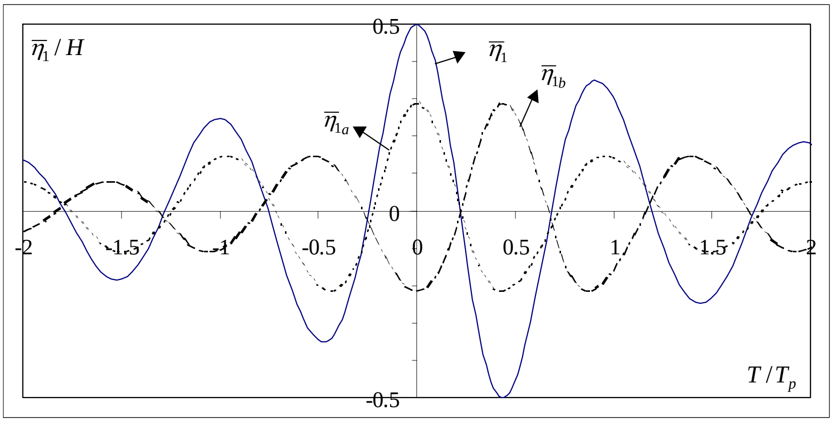

Figure 1. The linear wave group $\overline{\eta}_1$ when a large wave of height *H* occurs at ($Y = 0$) and the two groups $\overline{\eta}_{1a}$ and $\overline{\eta}_{1b}$ in which is decomposed $\overline{\eta}_1$ ($\equiv \overline{\eta}_{1a} - \overline{\eta}_{1b}$).

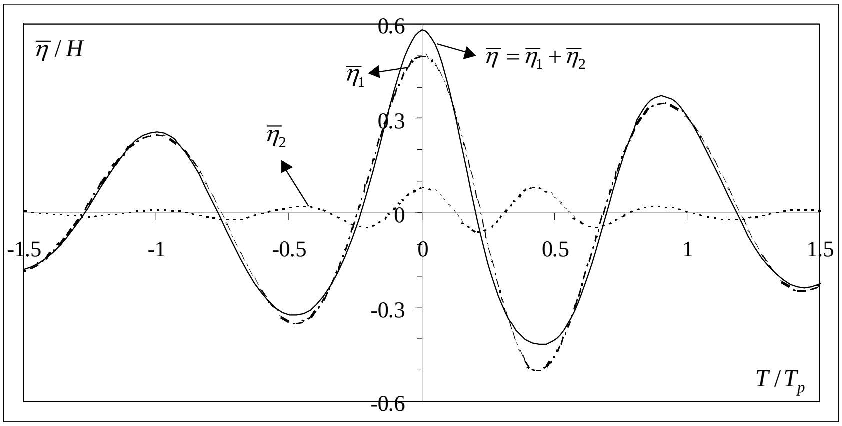

Figure 2. Let us assume that a wave with height *H* occurs at ($Y = 0$), with $H / \sigma \to \infty$: the linear wave group $\overline{\eta}_1$, the second-order term $\overline{\eta}_2$ and the total second-order free surface displacements $\overline{\eta} = \overline{\eta}_1 + \overline{\eta}_2$. The spectrum is the mean JONSWAP and the water is deep.

Finally Figure 3 shows a particular of Figure 2. The effects of second-order, for the highest wave, increased the crest height by 16% (it is equal to 0.58*H*), and decreased the trough depth by 16% (the trough amplitude is equal to 0.42*H*).
As for the period T_h of highest wave, from linear QD theory we obtain that it is slightly smaller than T_p (Boccotti, 2000). For example, for the mean JONSWAP spectrum T_h=0.92T_p. As we can see the second-order effects do not modify T_h. A slightly difference may be appreciated for the crest and trough duration's, which are respectively equal to 0.43T_p and 0.49T_p (from the linear QD theory they are both equal to 0.46T_p).

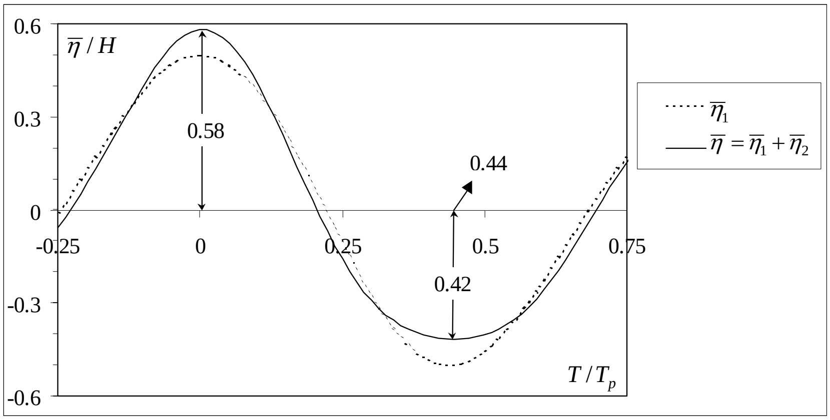

Figure 3. Particular of figure 2: the linear $\overline{\eta}_1$ and the total second-order free surface displacements $\overline{\eta} = \overline{\eta}_1 + \overline{\eta}_2$.

6.2. The second-order wave groups when a high crest occurs ('New wave')

..1 The undisturbed wave field

Let us suppose that a large crest of height H_C occurs at $Y = 0$, $T = 0$, in an undisturbed wave field. Following the first formulation of the quasi-determinism theory ('New wave') we have that a random wave group, at the apex of its development, generates this wave. To the second-order in the Stokes expansion, the wave group when a large crest occurs is given by Eq. (B10).

The first and second-order wave group, at point Y=0, is shown in Figure 4*a*.

..2 The wave field in front of the vertical wall

The wave group in front of a vertical wall, when a large wave occurs, is obtained in Appendix C. Therefore, if a large crest of height H_C occurs at wall ($Y = 0$), at time $T = 0$, the second-order free surface displacements is given by Eq. (C18). Both first and second-order free surface displacements are given in Figure 4*b*.

The comparison between Figures 4*a* and 4*b* (both are obtained for the mean JONSWAP spectrum) shows that non-linear effects at a vertical wall are greater than in an undisturbed field. For example the second-order highest crest is equal to 1.11H_C in an undisturbed field and to 1.26H_C on the vertical wall.

Finally we have that, in time domain, both non-linear wave groups are symmetric with respect to Y-axis.

6.3. Comparison with data

To validate our analytical predictions, Monte Carlo simulations of second-order sea states with the mean JONSWAP spectrum have been carried out, by generating 40000 waves. The

data have been then processed to analyze the structure of time domain wave groups, if either a large crest height (section 6.3.1), or a large crest-to-trough wave height, occurs (section 6.3.2).
The wave groups with a large crest, in time domain, have been also compared with average wave time histories given by Taylor and Williams (2002).

6.3.1 The largest crest heights from numerical simulation and field data

Figure 5 shows the highest crest obtained processing data of numerical simulation: the crest amplitude C is equal to 1.32 times the significant wave height; the crest duration is equal to Δt_C=0.40T_p. Troughs, front and back the highest crest, have amplitudes 0.54C and 0.68C and duration 1.34 Δt_C and 1.27 Δt_C respectively. Linear profile is given by broken line (it is easily obtained from numerical simulation, by considering the η_1 term only).

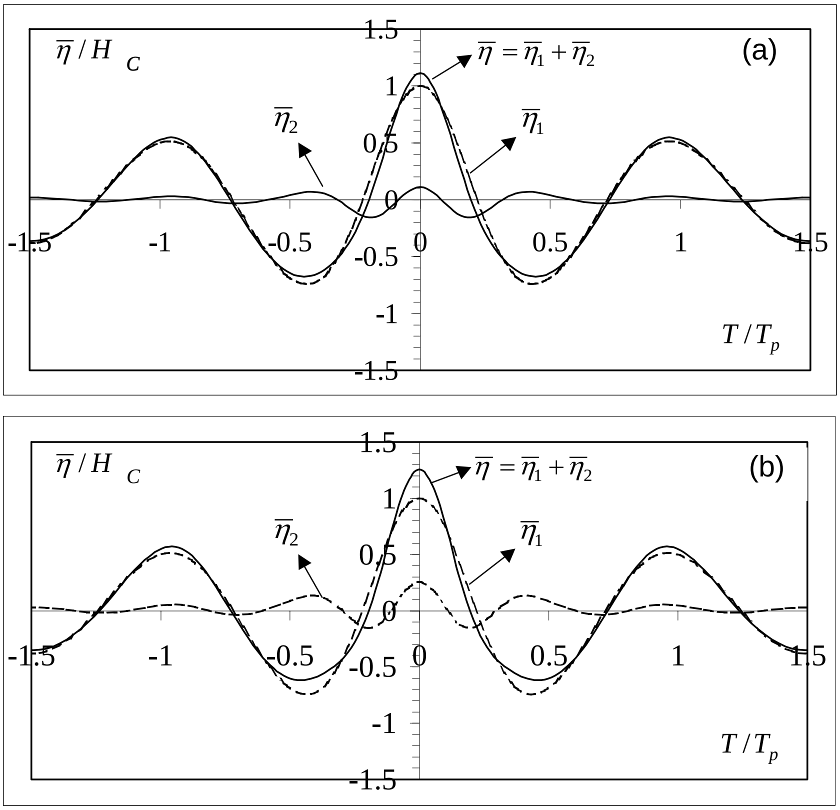

Figure 4. Let us assume that a crest of height H_C occurs at ($Y=0$), with $H_C/\sigma\to\infty$: the linear wave group $\overline{\eta}_1$, the second-order term $\overline{\eta}_2$ and the total second-order free surface displacements $\overline{\eta}=\overline{\eta}_1+\overline{\eta}_2$. The spectrum is the mean JONSWAP and the water is deep. Upper panel: the undisturbed wave field. Lower panel: the wave group at a vertical wall.

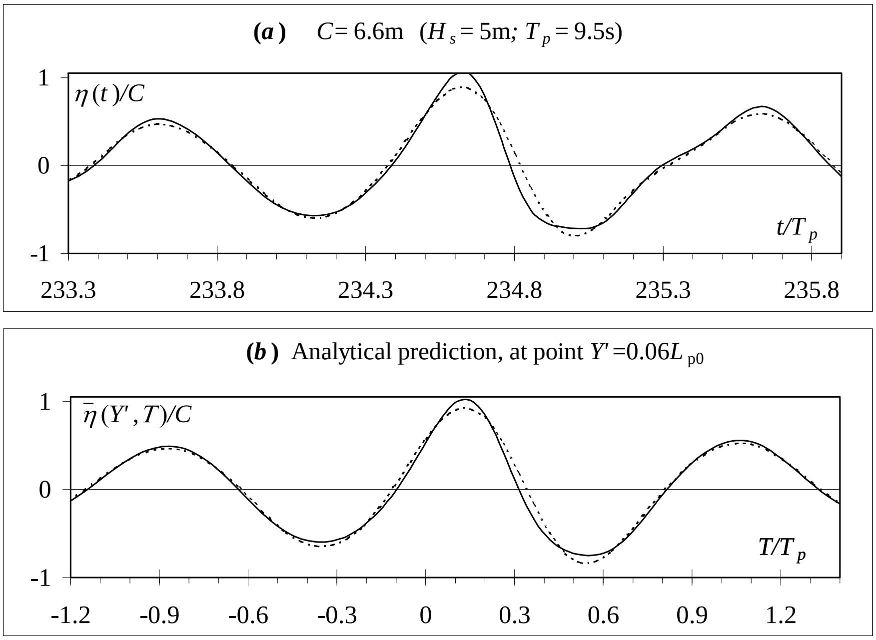

Figure 5. Comparison between largest crest obtained from Monte Carlo simulation of 40000 non-linear waves with mean JONSWAP spectrum, and analytical prediction. Upper panel shows the numerical data: dotted line gives the linear free surface displacement and continuous line gives the second-order one. Lower panel gives the theoretical prediction $\overline{\eta}(Y',T)$ obtained from Eq. (B10), at point $Y'/L_{p0} = 0.06$.

Theoretical prediction, given in lower panel, is obtained from Equations (B3) and (B10). In particular, because the $\eta(t)$ profile of Figure 5 is non-symmetric, the $\overline{\eta}(Y',T)$ is calculated at fixed points Y' close to 0. The value of Y' is obtained with an iterative procedure: we choose the value $Y = Y'$ which maximizes the coefficient of correlation between time series $\eta(t)$ of upper panel and $\overline{\eta}(Y,T)$ [let us note that a time shift is included to have largest crests of $\eta(t)$ and $\overline{\eta}(Y,T)$ at equal time instant]. As we can see theoretical $\overline{\eta}(Y',T)$ profile of Fig. 5*b* well agree with data.
Theoretical linear profile, obtained from quasi-determinism theory [Eq. (B3)], is given by broken line.

Finally theoretical prediction is compared with the average wave time histories when a high wave crest (or a deep trough – see Note) occurs, obtained by Taylor and Williams (2002) by processing field data with significant wave height $H_s \cong 4\,\mathrm{m}$, from WACSIS dataset (Forristall et al, 2002).
The use of the average profile in time domain is useful because the individual actual waves are both irregular and non-linear. Figure 6 shows then the average shape of the largest crest, which was obtained by considering the surface displacement time series around the 10% of largest crest (Taylor and Williams, 2002). Each crest was time shifted, so to have the

maximum value of free surface displacement at time T=0. Figure 6 shows also the average shape of largest troughs (see Note), which are taken positive. The peak period, for a Pierson-Moskowitz spectrum, is calculated as (Boccotti, 2000):

$$T_p = 4.7\pi\sqrt{H_s / g}\ . \tag{40}$$

In detail, from the average shape of the largest crest we obtain that: the largest crest has amplitude C and duration equal to Δt_C=3.4s; troughs, front and back the highest crest, have amplitudes 0.46C and 0.49C and duration 1.42 Δt_C and 1.49 Δt_C respectively. From the average shape of the largest trough we obtain that: the largest trough has amplitude 0.78C and duration equal to 1.15Δt_C; crests, front and back the largest trough, have amplitudes 0.54C and 0.52C and duration 1.27 Δt_C and 1.28 Δt_C respectively.

In the lower panel we find our analytical prediction for the Pierson-Moskowitz spectrum: the deterministic wave profiles $\overline{\eta} = \overline{\eta}_1 + \overline{\eta}_2$ when a large crest occurs and $-\overline{\eta} = \overline{\eta}_1 - \overline{\eta}_2$ (see Note) when a large trough occurs are plotted.

In detail, from $\overline{\eta}$ non-linear theoretical profile (crest) we obtain that the largest crest has amplitude C and duration Δt_C'=0.36T_p; the front/back trough has amplitude 0.54C and duration equal to 1.40 Δt_C'. From $\overline{\eta}'$ non-linear theoretical profile (trough) we obtain that the largest trough has amplitude 0.78C and duration 1.26Δt_C'; the front/back crest has amplitude 0.66C and duration equal to 1.27 Δt_C'.

We find then a good agreement between data and analytical prediction, both for the crest and the trough profiles: the crests are taller and spikier, and the troughs are smaller and broader.

7. Applications: The highest sea wave groups in space domain

The quasi-determinism theory, in both the formulations, enables us to obtain both the free surface displacement and velocity potential in space-time domain. In section 6 the wave groups with large waves have been analyzed in time domain, at a fixed point. Here we consider the wave groups in space domain, at fixed time instant.

In all the applications have been assumed the mean JONSWAP spectrum and the water deep.

7.1. The second-order wave groups when a large crest-to-trough wave height occurs

Figure 9 shows the non-linear wave group in space domain, when a large crest-to-trough wave height occurs, at some fixed time instant. The highest wave occurs at T=0, at the apex stage of group development. As we can see, a well-defined wave group moves along the y-axis. The wave group (individual wave) propagation speed is nearly equal to the group (wave) celerity for a periodic wave with period equal to the peak period T_p; in deep water, group celerity is then equal to one half the wave celerity. Because propagation speed for individual waves is greater than propagation speed of the wave group, each wave 'runs along the envelope from the tail where it is born to the head where it dies' (Boccotti, 2000).

The wave group shows also firstly a development stage, during which the height of the largest wave (at that fixed instant) increases; therefore at time t_o we have the apex of the group development: at this time the wave crest at point x_o reaches its maximum. After t_o we have the group decay stage.

All these results on the space time evolution of non-linear wave groups are in full agreement to those of Boccotti's book (2000), from the first-order quasi-determinism theory.

As for the second-order effects, they increase the crest amplitude and decrease the trough amplitude.

7.2. The space-domain second-order wave groups in front of a vertical wall, when a large crest occurs

Figure 10 shows the non-linear wave group in space domain, when a large crest of height H_C occurs at a vertical wall. The highest crest occurs at T=0, on the wall (Y=0).

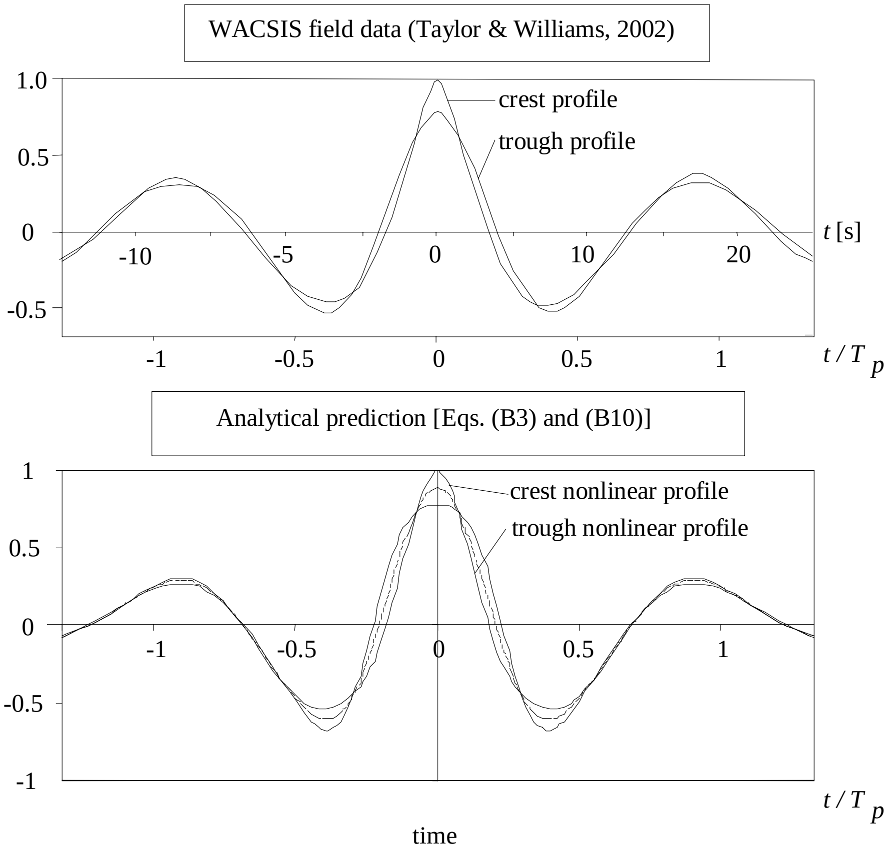

Figure 6. Upper panel: the average shape of the 10% of largest crest and trough amplitudes from WACSIS dataset (from Taylor and Williams, 2002).
Lower panel: analytical prediction for the Pierson-Moskowitz spectrum [Eqs. (B3) and (B10)]; in particular we have the deterministic wave profiles $\overline{\eta} = \overline{\eta}_1 + \overline{\eta}_2$ when a large crest occurs and $-\overline{\eta}' = \overline{\eta}_1 - \overline{\eta}_2$ when a large trough occurs.

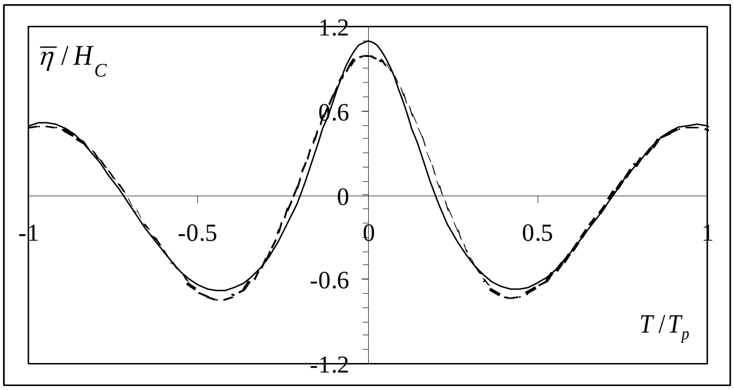

Figure 7. The average shape of the 1% of *largest crest amplitudes*, from second-order numerical simulation of 40000 waves. Dotted line gives the average linear free surface displacement (obtained processing only linear component from numerical simulation). Continuous line gives the average second-order free surface displacement.

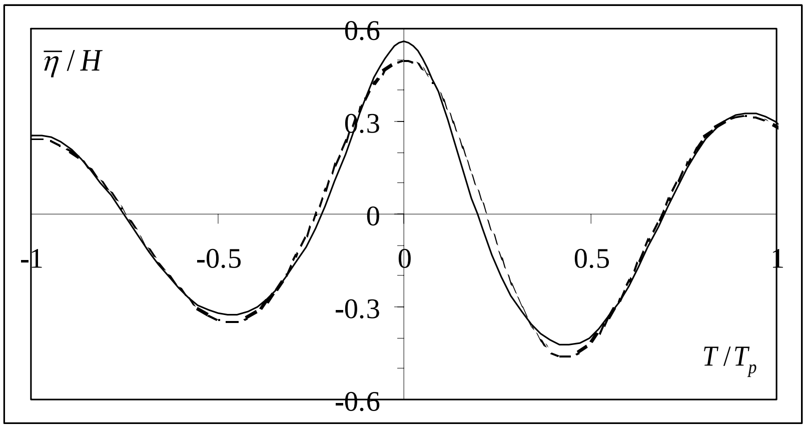

Figure 8. The average shape of the 1% of *largest crest-to-trough wave height*, from second-order numerical simulation of 40000 waves. Dotted line gives the average linear free surface displacement (obtained processing only linear component from numerical simulation). Continuous line gives the average second-order free surface displacement.

Note that in front of the vertical wall we have standing waves, as we may appreciate from Figure 11, which shows the non-linear free surface displacement at fixed time instant $(T-t_o)/T_p$, between –1 and 0, when largest crest occurs at wall (Y=0) at time instant t_o].
In particular Figure 11 may be compared with Figure 12, which shows the same wave group [that is the non-linear free surface displacement at fixed time instant $(T-t_o)/T_p$, between –1 and 0] in an undisturbed field, when a large crest occurs at Y=0, T=t_o.

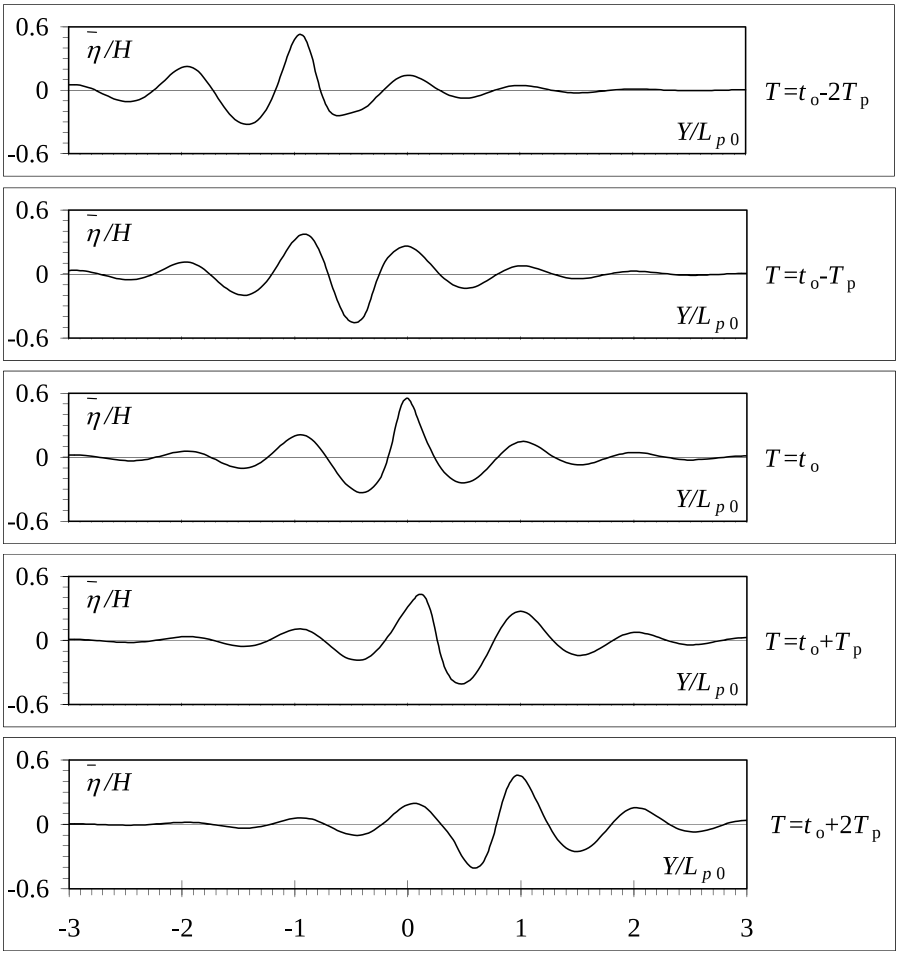

Figure 9. The second-order free surface displacements in space domain (in an undisturbed field), at some fixed time instant, when a very large crest-to-trough wave with height H occurs at (Y=0, T=t_o).

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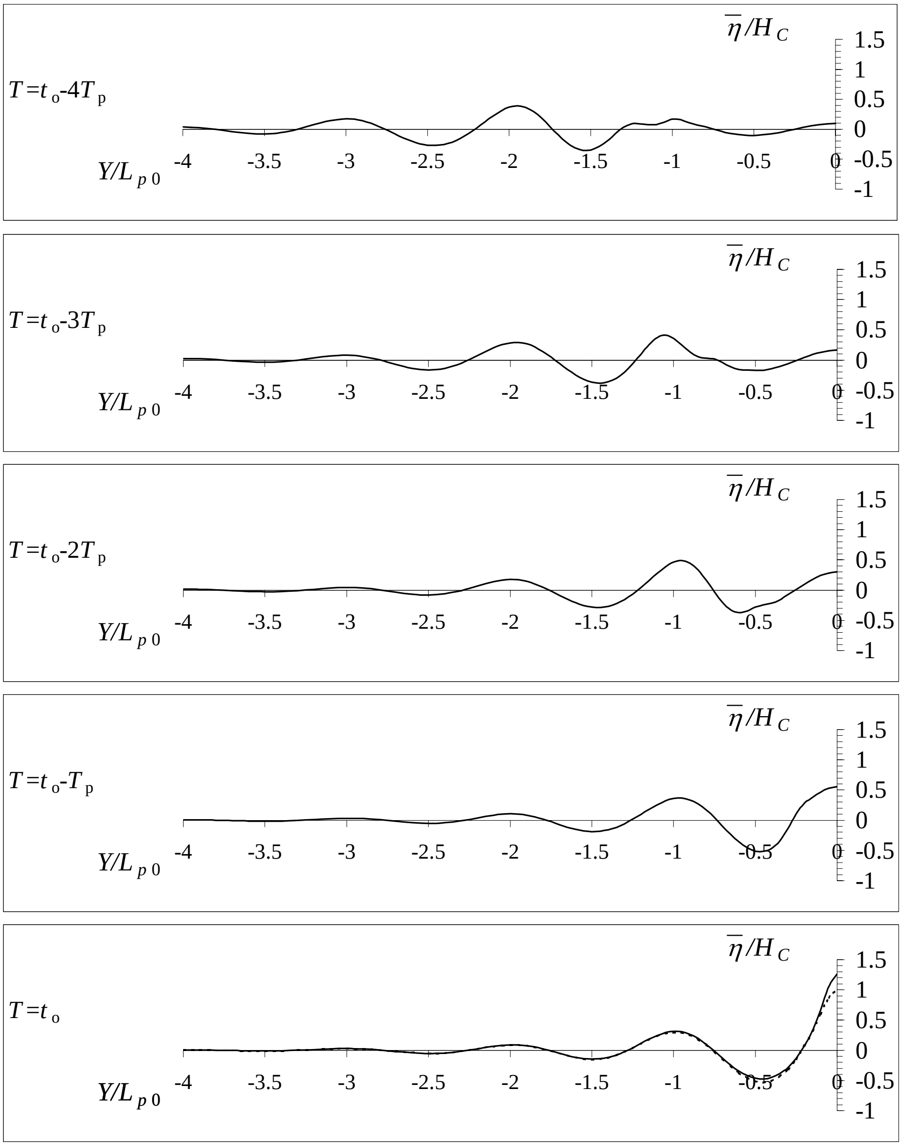

Figure 10*a*. The second-order free surface displacements in front of a vertical wall (in space domain), at some fixed time instant, when a very high crest with height H_C occurs at (Y=0, T=t_o). Note that dotted line at lower panel (T=t_o) given linear free surface displacement (from QD theory).

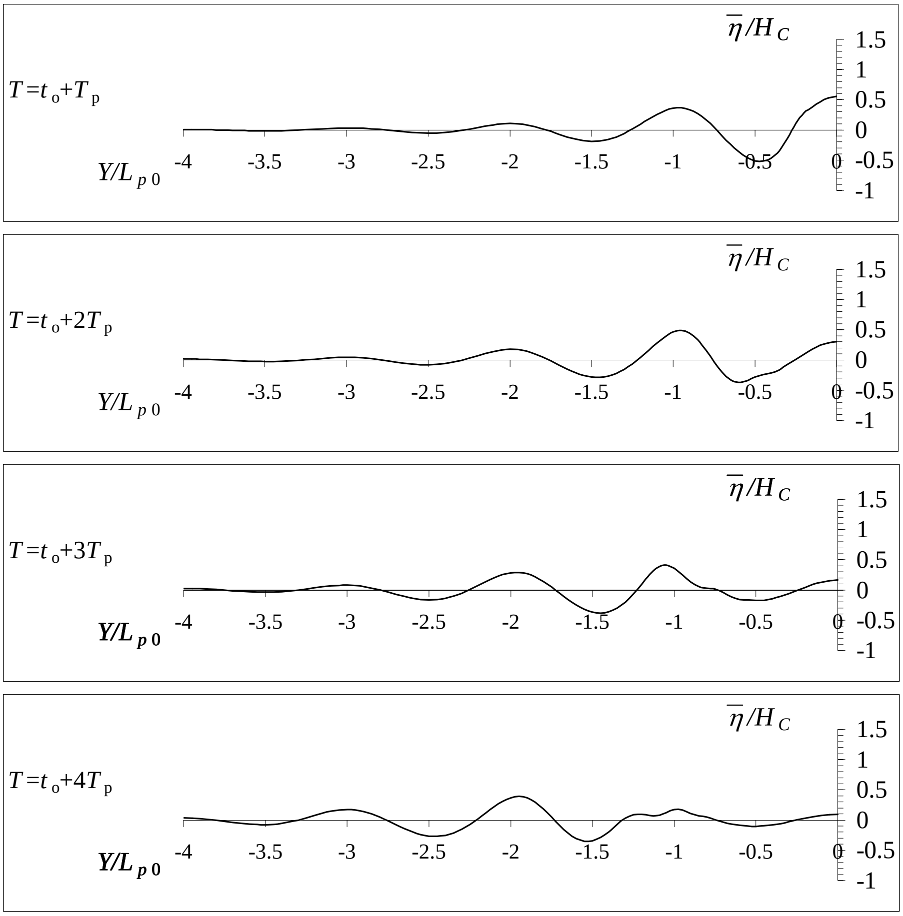

Figure 10*b*. See caption of Figure 10*a*.

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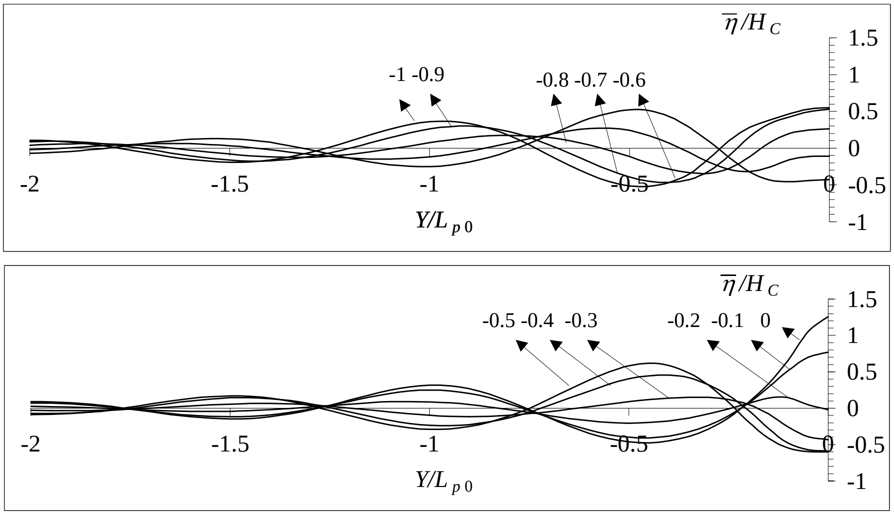

Figure 11. The second-order free surface displacements in front of a vertical wall (in space domain), at some fixed time instant, when a very high crest with height H_C occurs at (Y=0, T=t_o). To each line is associated the time instant defined as $(T\text{-}t_o)/T_p$, which ranges between –1 and 0.

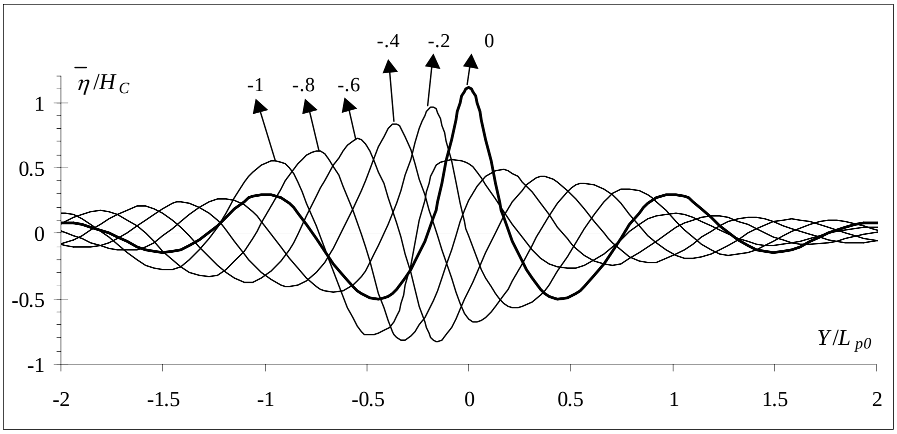

Figure 12. The second-order free surface displacements in an undisturbed field in space domain, at some fixed time instant, when a very high crest with height H_C occurs at (Y=0, T=t_o). To each line is associated the time instant defined as $(T\text{-}t_o)/T_p$, which ranges between –1 and 0.

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APPENDIX A.
THE SYSTEM OF DIFFERENTIAL EQUATIONS
FOR AN IRROTATIONAL FLOW WITH A FREE SURFACE

Let us consider an irrotational flow with a free surface, at a constant depth *d*. Let the vertical free surface displacement be $\eta(x,y,t)$, and the velocity potential $\phi(x,y,z,t)$ where the vertical *z* axis, positive upwards, has origin at the mean water level and (*x*,*y*) denote the horizontal plane in Cartesian co-ordinates.

Both the velocity potential ϕ and the free surface displacement η have to satisfy the system of differential equations, for the irrotational flow with a free surface. This system includes:

i) the Bernoulli equation

$$g\eta + \left(\frac{\partial\phi}{\partial t}\right)_{z=\eta} + \frac{1}{2}\left[\left(\frac{\partial\phi}{\partial y}\right)^2 + \left(\frac{\partial\phi}{\partial z}\right)^2\right]_{z=\eta} = 0 \tag{A1}$$

ii) the free surface general equation

$$\left(\frac{\partial\phi}{\partial z}\right)_{z=\eta} = \left(\frac{\partial\phi}{\partial y}\right)_{z=\eta}\frac{\partial\eta}{\partial y} + \frac{\partial\eta}{\partial t} \tag{A2}$$

iii) the continuity equation

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 0 \tag{A3}$$

iv) the solid boundary condition at the horizontal bottom

$$\left(\frac{\partial\phi}{\partial z}\right)_{z=-d} = 0\,. \tag{A4}$$

To solve the first two equations, a Taylor series expansion is usually taken at *z*=0 (mean water level) (Longuet-Higgins, 1963). For example, the second term in Eq. (A1) gives:

$$\left(\frac{\partial\phi}{\partial t}\right)_{z=\eta} = \left(\frac{\partial\phi}{\partial t}\right)_{z=0} + \left(\frac{\partial^2\phi}{\partial z\partial t}\right)_{z=0}\eta + \frac{1}{2}\left(\frac{\partial^3\phi}{\partial z^2\partial t}\right)_{z=0}\eta^2 + \dots\,. \tag{A5}$$

Furthermore perturbation method may be applied, to obtain the solution: the free surface displacement and the velocity potential are written as

$$\eta \equiv \eta_1 + \eta_2 + \mathrm{o}(H^2), \quad \phi \equiv \phi_1 + \phi_2 + \mathrm{o}(H^2) \tag{A6}$$

(where o(H^2) define terms of order smaller than H^2, and η_i and ϕ_i have order H^i).
Finally, from Equation (A1-A4) we may derive the linear and the second-order system.
To the first-order, the system (A1-A4) gives:

$$\left(\frac{\partial\phi_1}{\partial t}\right)_{z=0} = -g\eta_1\,;\; \left(\frac{\partial\phi_1}{\partial z}\right)_{z=0} = \frac{\partial\eta_1}{\partial t}\,;\; \frac{\partial^2\phi_1}{\partial x^2} + \frac{\partial^2\phi_1}{\partial y^2} + \frac{\partial^2\phi_1}{\partial z^2} = 0\,;\; \left(\frac{\partial\phi_1}{\partial z}\right)_{z=-d} = 0\,. \tag{A7}$$

The second-order system, particularized for long-crested waves (two-dimensional) is given by Equations (12-15); note that it is given as a function of the independent variables Y, z, T, which are defined as $Y = y - y_o$ (direction of wave propagation), $T = t - t_o$, with y_o an arbitrary point and t_o an arbitrary time instant.

APPENDIX B.
WHAT HAPPENS WHEN A LARGE CREST HEIGHT OCCURS: THE LINEAR QUASI-DETERMINISM THEORY AND THE EXTENSION TO THE SECOND-ORDER

B.1 THE FIRST FORMULATION OF THE QUASI-DETERMINISM THEORY

The first formulation of the quasi-determinism theory by Boccotti (1981, 1982, 2000), for linear high wave crest (‘New wave’), enables us to predict the space-time evolution of the free surface displacement and of the velocity potential when a very high crest occurs at some fixed time and location.

The theory shows that, if a local wave maximum of given elevation H_C occurs at time t_o at a fixed point (x_o , y_o), and if $H_C/\sigma \to \infty$ (σ being the standard deviation of the free surface displacement), with probability approaching 1 the surface displacement at point ($x_o + X, y_o + Y$) and the velocity potential at point ($x_o + X, y_o + Y$, z), at time $t_o + T$, are respectively equal to the deterministic form

$$\overline{\eta}_1(x_o + X, y_o + Y, t_o + T) = \frac{\Psi(X,Y,T)}{\Psi(0,0,0)} H_C , \tag{B1}$$

$$\overline{\phi}_1(x_o + X, y_o + Y, z, t_o + T) = \frac{\Phi(X,Y,z,T)}{\Psi(0,0,0)} H_C , \tag{B2}$$

where the space-time covariances $\Psi(X,Y,T)$ and $\Phi(X,Y,z,T)$ are defined by Equations (4) and (5).

Let us note that both the expressions (B1) and (B2) are exact for $H_C/\sigma \to \infty$, that is for the crest height very large with respect to the mean crest height.

B1.1) The linear deterministic wave group for long-crested waves in an undisturbed field

For long-crested waves in an undisturbed wave field the free surface displacement [see Eq. (B1)] of the wave group at point $y_o + Y$ at time $t_o + T$, when an exceptional crest height of given elevation H_C occurs at time t_o at fixed point y_o , may be rewritten as a function of the frequency spectrum $E(\omega)$:

$$\overline{\eta}_1(Y,t) = H_C \frac{\int_0^\infty E(\omega)\cos\varphi \mathrm{d}\omega}{\int_0^\infty E(\omega)\mathrm{d}\omega} . \tag{B3}$$

The velocity potential at a fixed point (y_o +Y, z), when the very large crest occurs at point y_o , is then given by:

$$\overline{\phi}_1(y_o+Y,z,t_o+T)=gH_C\frac{\int_0^\infty E(\omega)\frac{1}{\omega}\frac{\cosh[k(d+z)]}{\cosh(kd)}\sin\varphi\mathrm{d}\omega}{\int_0^\infty E(\omega)\mathrm{d}\omega} \tag{B4}$$

where the wave number k is obtained from equation $k\tanh(kd)=\omega^2/g$.

B.2 THE SECOND-ORDER $\overline{\eta}$ AND $\overline{\phi}$ EXPRESSIONS WHEN A LARGE CREST OCCURS

The second-order $\overline{\eta}$ and $\overline{\phi}$, when a large crest occurs, are easily obtained: in this case the linear free surface displacement $\overline{\eta}_1$ [Eq. (B1)] and velocity potential $\overline{\phi}_1$ [Eq. (B2)] give a wave group, with largest crest with amplitude H_C at (Y=0,T=0). Therefore they may be rewritten in discrete form as a summation of N terms:

$$\overline{\eta}_1(y_o+Y,t_o+T)=\sum_{i=1}^{N}\alpha_i^{'}\cos[\varphi(\omega_i,Y,T)] \tag{B5}$$

$$\overline{\phi}_1(y_o+Y,z,t_o+T)=g\sum_{i=1}^{N}\alpha_i^{'}\omega_i^{-1}\frac{\cosh[k_i(d+z)]}{\cosh[k_i d]}\sin[\varphi(\omega_i,Y,T)] \tag{B6}$$

where

$$\alpha_i^{'}=H_C\frac{E(\omega_i)\mathrm{d}\omega_i}{\int_0^\infty E(\omega)\mathrm{d}\omega} . \tag{B7}$$

By solving the second-order system given by Equations (12-15), following procedure given by Sharma & Dean (1979), we obtain:

$$\overline{\eta}_2(Y,T)=\frac{1}{4}\sum_{n=1}^{N}\sum_{m=1}^{N}\alpha_n^{'}\alpha_m^{'}\left\{A_{nm}^{-}\cos(\varphi_n-\varphi_m)+A_{nm}^{+}\cos(\varphi_n+\varphi_m)\right\} \tag{B8}$$

$$\begin{aligned}\overline{\phi}_2(Y,z,T)=g^2\sum_{n=1}^{N}\sum_{m=1}^{N}\frac{\alpha_n^{'}}{\omega^n}\frac{\alpha_m^{'}}{\omega^m}\Bigg\{&\frac{\cosh[k_{nm}^{-}(d+z)]}{\cosh(k_{nm}^{-}d)}\frac{B_{nm}^{-}}{\omega_n-\omega_m}\sin(\varphi_n-\varphi_m)+\\&+\frac{\cosh[k_{nm}^{+}(d+z)]}{\cosh(k_{nm}^{+}d)}\frac{B_{nm}^{+}}{\omega_n+\omega_m}\sin(\varphi_n+\varphi_m)\Bigg\}\end{aligned} \tag{B9}$$

or, as a function of the frequency spectrum:

$$\begin{aligned}\overline{\eta}_2(Y,t)=\frac{H_C^2}{4}\left\{\int_0^\infty E(\omega)\mathrm{d}\omega\right\}^{-2}\int_0^\infty\int_0^\infty E(\omega_n)\cdot\\ \cdot E(\omega_m)\left\{A_{nm}^{-}\cos(\varphi_n-\varphi_m)+A_{nm}^{+}\cos(\varphi_n+\varphi_m)\right\}\mathrm{d}\omega_m\mathrm{d}\omega_n\end{aligned} \tag{B10}$$

$$
\begin{aligned}
\overline{\phi}_2(Y,z,T) &= g^2 \frac{H_C^2}{4}\left\{\int_0^\infty E(\omega)\mathrm{d}\omega\right\}^{-2} \int_0^\infty \int_0^\infty E(\omega_n)E(\omega_m)\cdot \\
&\cdot\frac{1}{\omega_n}\frac{1}{\omega_m}\left\{\frac{\cosh[k_{nm}^-(d+z)]}{\cosh(k_{nm}^- d)}\frac{B_{nm}^-}{\omega_n-\omega_m}\sin(\varphi_n-\varphi_m)+\right. \\
&\left.+\frac{\cosh[k_{nm}^+(d+z)]}{\cosh(k_{nm}^+ d)}\frac{B_{nm}^+}{\omega_n+\omega_m}\sin(\varphi_n+\varphi_m)\right\}\mathrm{d}\omega_m\mathrm{d}\omega_n .
\end{aligned}
\tag{B11}
$$

The second-order free surface displacement and velocity potential, when the large crest of amplitude H_C occurs, are then $\overline{\eta} = \overline{\eta}_1 + \overline{\eta}_2$ and $\overline{\phi} = \overline{\phi}_1 + \overline{\phi}_2$ respectively.

Note that expressions (B10) and (B11), for the special case of long-crested waves in deep water (see section 4), were derived by Arena & Fedele (2003) and Fedele & Arena (2003) with a different approach.

APPENDIX C.
REFLECTION OF NON-LINEAR WAVE GROUPS: THE EXTENSION TO THE SECOND-ORDER OF THE QUASI-DETERMINISM THEORY

The reflection deals with the interaction of waves with a vertical wall.

Here the reflection of long-crested waves is considered; in particular we analyze what happens when a large crest height occurs on a vertical wall. The linear analytical solution for the quasi-determinism theory (‘New wave’), for waves on a vertical wall (Boccotti, 1989, 2000), is then extended to the second-order.

Let the linear incident free surface displacement $\overline{\eta}_{1i}$ and velocity potential $\overline{\phi}_{1i}$ be given by Equations (B10) and (B11) respectively.

In front of the vertical wall we have the reflected waves, such to satisfy the boundary condition at the wall. Being the wall located at $Y = 0$, this condition yields:

$$
\left(\frac{\partial \overline{\phi}_1}{\partial Y}\right)_{Y=0} = 0 \,, \tag{C1}
$$

where the total linear velocity potential is given by $\overline{\phi}_1 = \overline{\phi}_{1i} + \overline{\phi}_{1r}$, with $\overline{\phi}_{1r}$ the reflected term.

In detail, the reflected random waves which satisfy the condition (C1) are derived by considering that the *j*th component of incident wave, has amplitude $\alpha_j^{'}$, frequency ω_j and direction $\theta_j = 0$ ($\forall j$, because we have long crested waves), where we now define

$$
\varphi'(\omega,\theta,Y,T) \equiv kY\cos\theta - \omega T \,, \tag{C2}
$$

in place of $\varphi(\omega,Y,T)$.

We find that (see Boccotti, 2000) the corresponding jth reflected wave component has amplitude $\alpha'_{r\,j} = \alpha'_j$, frequency $\omega_{r\,j} = \omega_j$ and direction $\theta_{r\,j} = \pi$.

Therefore the reflected waves are given by

$$\overline{\eta}_{1r}(y_o + Y, t_o + T) = \sum_{j=1}^{N} \alpha'_j \cos\left[\varphi'(\omega_j, \theta_{r\,j}, Y, T)\right]. \tag{C3}$$

The total linear free surface displacement in front of the vertical wall, which is defined as

$$\overline{\eta}_{1R}(y_o + Y, t_o + T) = \overline{\eta}_{1i} + \overline{\eta}_{1r}, \tag{C4}$$

is then

$$\overline{\eta}_{1R}(y_o + Y, t_o + T) = \sum_{j=1}^{2N} \alpha'_j \cos\left[\varphi'(\omega_j, \theta_j, Y, T)\right] \tag{C5}$$

where

$$\begin{cases} \alpha'_j = \alpha'_{j+N} \\ \omega_j = \omega_{j+N} \\ \theta_j = 0 \\ \theta_{j+N} = \pi \end{cases} \qquad \text{for } j = 1, \ldots, N \tag{C6}$$

The corresponding total velocity potential, to the first-order, in front of a vertical wall is then:

$$\overline{\phi}_{1R}(y_o + Y, t_o + T) = g \sum_{j=1}^{2N} \alpha'_j \omega_j^{-1} \frac{\cosh\left[k_j(d+z)\right]}{\cosh(k_j d)} \sin\left[\varphi'(\omega_j, \theta_j, Y, T)\right] \tag{C7}$$

with definitions (C2) and (C6).
Note that Equations (C5) gives the wave group when a large crest height, of amplitude $2H_C$, occurs at a point close to (or on) the vertical wall.
The second-order $\overline{\eta}_{2R}$ and $\overline{\phi}_{2R}$ are then obtained following the procedure solution of section 3. In this case we have a slightly different structure of $A^{\pm}_{nm}$ and $B^{\pm}_{nm}$ coefficients. In detail, $\overline{\eta}_{2R}$ and $\overline{\phi}_{2R}$, for long-crested waves, are given respectively by:

$$\overline{\eta}_{2R}(Y,T) = \frac{1}{4} \sum_{n=1}^{2N} \sum_{m=1}^{2N} \alpha_n \alpha_m \left[A'^{-}_{nm} \cos(\varphi'_n - \varphi'_m) + A'^{+}_{nm} \cos(\varphi'_n + \varphi'_m)\right] \tag{C8}$$

$$\begin{aligned} \overline{\phi}_{2R}(Y,z,T) = \frac{1}{4} g^2 \sum_{n=1}^{2N} \sum_{m=1}^{2N} \frac{\alpha_n}{\omega_n} \frac{\alpha_m}{\omega_m} \Bigg\{ & \frac{\cosh[k^-_{nm}(d+z)]}{\cosh(k^-_{nm} d)} \frac{B'^{-}_{nm}}{\omega_n - \omega_m} \sin(\varphi'_n - \varphi'_m) + \\ & + \frac{\cosh[k^+_{nm}(d+z)]}{\cosh(k^+_{nm} d)} \frac{B'^{+}_{nm}}{\omega_n + \omega_m} \sin(\varphi'_n + \varphi'_m) \Bigg\} \end{aligned} \tag{C9}$$

where

$$A'^{\pm}_{nm} = \frac{B'^{\pm}_{nm} - \vec{k}_n \cdot \vec{k}_m \pm \rho_n \rho_m}{\sqrt{\rho_n \rho_m}} + \rho_n + \rho_m \tag{C10}$$

$$B'^{\pm}_{nm} = \frac{(\sqrt{\rho_n} \pm \sqrt{\rho_m})\left[\sqrt{\rho_m}(k_n^2 - \rho_n^2) \pm \sqrt{\rho_n}(k_m^2 - \rho_m^2)\right] + 2\left(\sqrt{\rho_n} \pm \sqrt{\rho_m}\right)^2 (\vec{k}_n \cdot \vec{k}_m \mp \rho_n \rho_m)}{\left(\sqrt{\rho_n} \pm \sqrt{\rho_m}\right)^2 - k'^{\pm}_{nm} \tanh\left(k'^{\pm}_{nm} d\right)} \tag{C11}$$

$$k'^{\pm}_{nm} = |\vec{k}_n \pm \vec{k}_m| \,. \tag{C12}$$

Note that either for ($n = 1,...N, m = 1,..., M$) or for ($n = N+1,...,2N, m = M+1,...,2M$) we have $\vec{k}_n \cdot \vec{k}_m = k_n k_m$ and $|\vec{k}_n \pm \vec{k}_m| = |k_n \pm k_m|$. We have instead $\vec{k}_n \cdot \vec{k}_m = -k_n k_m$ and $|\vec{k}_n \pm \vec{k}_m| = |k_n \mp k_m|$ if either ($n = 1,...N, m = M+1,...,2M$) or ($n = N+1,...,2N, m = 1,..., M$). Therefore we may write:

$$(A'^{\pm}_{nm}, B'^{\pm}_{nm}) = \begin{cases} (A^{\pm}_{nm}, B^{\pm}_{nm}) \quad \text{either for} \begin{cases} (n = 1,...N, m = 1,..., M) \\ \text{or for } (n = N+1,...,2N, m = M+1,...,2M) \end{cases} \\ (\tilde{A}^{\pm}_{nm}, \tilde{B}^{\pm}_{nm}) \quad \text{either for} \begin{cases} (n = 1,...N, m = M+1,...,2M) \\ \text{or for } (n = N+1,...,2N, m = 1,..., M) \end{cases} \end{cases} \tag{C13}$$

being

$$\tilde{A}^{\pm}_{nm} = \frac{B^{\pm}_{nm} + k_n k_m \pm \rho_n \rho_m}{\sqrt{\rho_n \rho_m}} + \rho_n + \rho_m \tag{C14}$$

$$\tilde{B}^{\pm}_{nm} = \frac{(\sqrt{\rho_n} \pm \sqrt{\rho_m})\left[\sqrt{\rho_m}(k_n^2 - \rho_n^2) \pm \sqrt{\rho_n}(k_m^2 - \rho_m^2)\right] + 2\left(\sqrt{\rho_n} \pm \sqrt{\rho_m}\right)^2 (-k_n k_m \mp \rho_n \rho_m)}{\left(\sqrt{\rho_n} \pm \sqrt{\rho_m}\right)^2 - k^{\mp}_{nm} \tanh\left(k^{\mp}_{nm} d\right)} . \tag{C15}$$

After some algebra, the free surface displacements $\overline{\eta}_{2R}$ is given by:

$$\begin{aligned} \overline{\eta}_{2R}(Y,T) = \frac{1}{2}\sum_{n=1}^{N}\sum_{m=1}^{N} \alpha_n \alpha_m \Big\{ & \left[A^-_{nm}\cos(k_n Y - k_m Y) + \tilde{A}^-_{nm}\cos(k_n Y + k_m Y)\right] \cdot \\ & \cdot \cos\left(\omega_n T - \omega_m T\right) + \left[A^+_{nm}\cos(k_n Y + k_m Y) + \tilde{A}^+_{nm}\cos(k_n Y - k_m Y)\right]\cos\left(\omega_n T + \omega_m T\right)\Big\} \end{aligned} \tag{C16}$$

or, as a function of the frequency spectrum:

$$\begin{aligned} \overline{\eta}_{2R}(Y,T) = \frac{H_C^2}{2}\left\{\int_0^\infty E(\omega)\mathrm{d}\omega\right\}^{-2} \int_0^\infty \int_0^\infty E(\omega_n)E(\omega_m)\Big\{ & \left[A^-_{nm}\cos(k_n Y - k_m Y) + \right. \\ & \left. \tilde{A}^-_{nm}\cos(k_n Y + k_m Y)\right]\cos\left(\omega_n T - \omega_m T\right) + \left[A^+_{nm}\cos(k_n Y + k_m Y) + \right. \\ & \left. \tilde{A}^+_{nm}\cos(k_n Y - k_m Y)\right]\cos\left(\omega_n T + \omega_m T\right)\Big\} \mathrm{d}\omega_m \mathrm{d}\omega_n \end{aligned} \tag{C17}$$

The amplitude of linear highest wave at wall, given by Eq. (C5), is equal to $2H_C$; Eq. (C17) gives then the second-order free surface displacement when the largest crest of amplitude equal to $2H_C$ occurs.

Furthermore, if we have that a large crest height of amplitude H_C occurs at a fixed point Y in front of the wall, at time T=0, the total second-order free surface displacements $\overline{\eta}_R$ ($\equiv \overline{\eta}_{1R} + \overline{\eta}_{2R}$) is given by:

$$
\begin{aligned}
&\overline{\eta}_R(Y,T) = H_C \left\{ \int_0^\infty E(\omega)\mathrm{d}\omega \right\}^{-1} \int_0^\infty E(\omega)\cos(\omega T)\cos(kY)\mathrm{d}\omega + \\
&+ \frac{H_C^2}{8} \left\{ \int_0^\infty E(\omega)\mathrm{d}\omega \right\}^{-2} \int_0^\infty \int_0^\infty E(\omega_n)E(\omega_m) \Big\{ \Big[A_{nm}^- \cos(k_n Y - k_m Y) + \\
&\tilde{A}_{nm}^- \cos(k_n Y + k_m Y) \Big] \cos\big(\omega_n T - \omega_m T\big) + \Big[A_{nm}^+ \cos(k_n Y + k_m Y) + \\
&\tilde{A}_{nm}^+ \cos(k_n Y - k_m Y) \Big] \cos\big(\omega_n T + \omega_m T\big) \Big\} \mathrm{d}\omega_m \mathrm{d}\omega_n
\end{aligned}
\tag{C18}
$$

and the corresponding total second-order velocity potential $\overline{\phi}_R$ ($\equiv \overline{\phi}_{1R} + \overline{\phi}_{2R}$) is

$$
\begin{aligned}
&\overline{\phi}_R(Y,z,T) = -gH_C \left\{ \int_0^\infty E(\omega)\mathrm{d}\omega \right\}^{-1} \int_0^\infty E(\omega)\omega^{-1} \frac{\cosh[k(d+z)]}{\cosh(kd)} \sin(\omega T)\cos(kY)\mathrm{d}\omega + \\
&- g\frac{H_C^2}{8} \left\{ \int_0^\infty E(\omega)\mathrm{d}\omega \right\}^{-2} \int_0^\infty \int_0^\infty E(\omega_n)E(\omega_m)\omega_n^{-1}\omega_m^{-1} \Big\{ 2\Big[C_{nm}^- \cos(k_n Y - k_m Y) + \\
&\tilde{C}_{nm}^- \cos(k_n Y + k_m Y) \Big] \sin\big(\omega_n T - \omega_m T\big) + 2\Big[C_{nm}^+ \cos(k_n Y + k_m Y) + \\
&\tilde{C}_{nm}^+ \cos(k_n Y - k_m Y) \Big] \sin\big(\omega_n T + \omega_m T\big) \Big\} \mathrm{d}\omega_m \mathrm{d}\omega_n
\end{aligned}
\tag{C19}
$$

where

$$
\begin{aligned}
&C_{nm}^- = B_{nm}^- \frac{1}{\omega_n - \omega_m} \frac{\cosh[|k_n - k_m|(d+z)]}{\cosh(|k_n - k_m| d)}; \quad \tilde{C}_{nm}^- = \tilde{B}_{nm}^- \frac{1}{\omega_n - \omega_m} \frac{\cosh[|k_n + k_m|(d+z)]}{\cosh(|k_n + k_m| d)}; \\
&C_{nm}^+ = B_{nm}^+ \frac{1}{\omega_n + \omega_m} \frac{\cosh[|k_n + k_m|(d+z)]}{\cosh(|k_n + k_m| d)}; \quad \tilde{C}_{nm}^+ = \tilde{B}_{nm}^+ \frac{1}{\omega_n + \omega_m} \frac{\cosh[|k_n - k_m|(d+z)]}{\cosh(|k_n - k_m| d)}.
\end{aligned}
\tag{C20}
$$

Note that this second-order extension is exact if the large crest occurs on the vertical wall, that is if y_0=0. In this case Yaxis coincides with $y$axis. The general solution, which is valid for the largest waves occurring at any fixed point in front of the wall, may be derived with a similar procedure.

Nomenclature

E	frequency spectrum
E_w	nondimensional frequency spectrum
g	acceleration of gravity
H	crest-to-trough wave height
H_C	crest amplitude
H_s	significant wave height
k	wave number
L	wavelength
T_h	period of highest waves
T_p	peak period
T	time
t_o	fixed time instant
w	nondimensional frequency
x	horizontal coordinate axis
x_o	fixed point of the x-axis
X	horizontal axis with origin at point x_o
y	horizontal coordinate axis
Y_o	fixed point of the y-axis
Y	horizontal axis with origin at point y_o
z	vertical coordinate axis, with the origin at the mean water level
α	wave amplitude
α_P	Phillips's parameter
η	surface displacement
$\overline{\eta}$	surface displacement of deterministic wave groups
ρ	water density
σ	r.m.s. surface displacement of a sea state
ϕ	velocity potential
$\overline{\phi}$	velocity potential of deterministic wave groups
Φ	covariance of surface displacement and velocity potential
χ_1, χ_2	shape parameters of the JONSWAP spectrum
ψ^*	narrow bandedness parameter
Ψ	covariance of surface displacement
θ	angle between wave direction and y axis
ω	angular frequency
ω_p	peak frequency

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