

## MICROSCOPIC STRUCTURES FROM REDUCTION OF CONTINUUM NONLINEAR PROBLEMS

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**ABSTRACT.** We present an application of the Amann–Zehnder exact finite reduction to a class of nonlinear perturbations of elliptic elasto–static problems. We propose the existence of minmax solutions by applying Ljusternik–Schnirelmann theory to a finite dimensional variational formulation of the problem, based on a suitable spectral cut–off. As a by–product, with a choice of fit variables, we establish a variational equivalence between the above spectral finite description and a discrete mechanical model. By doing so, we decrypt the abstract information encoded in the AZ reduction and give rise to a concrete and finite description of the continuous problem.

*This paper is dedicated to Professor Giuseppe Grioli,  
 Master of Continuum Mechanics,  
 in occasion of his 100th birthday*

### 1. Introduction

In this paper we revisit some aspects of a rather classical analytical model occurring in many physical settings, the following stationary nonlinear problem:

$$\begin{cases} \frac{\partial u}{\partial t} + N(u) = 0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \\ \text{with } N(u) := -Lu - \lambda u - \varepsilon F(u), \end{cases} \quad (1)$$

where  $\Omega$  is an open domain in  $\mathbb{R}^n$  with piecewise smooth boundary,  $u \in H := H_0^1(\Omega; \mathbb{R})$ ,  $-Lu$  is elliptic and  $F(u)$  is a nonlinear operator, with  $\varepsilon > 0$  a perturbation parameter. Here  $-\lambda u$  represents a sort of linear soft device, possibly breaking, for  $\lambda > 0$  large enough, the elliptic character of  $-Lu - \lambda u$ . Stationary elastic membranes, reaction–diffusion, Ginzburg–Landau, among others, can be settled into this format.

First at all, we recall that if  $N'(u)$  is symmetric with respect to the  $L^2$  scalar product, ( $F'(u)$  actually) Volterra–Vainberg theorem applies (see [23], [4]), and we can write the energy functional:

$$E(u) = \int_0^1 \langle N(tu), u \rangle_{L^2} dt, \quad (2)$$

and obtain a variational formulation for (1):  $dE(u) = 0$ . Then, we apply to  $E(u)$  a version of Amann–Zehnder (AZ) exact finite reduction, which is a global Lyapunov–Schmidt technique, introduced in Hamiltonian dynamics in [2] and [10], and in fields theory in [1]

(see also [5, 6, 18]), (1). In such a way, we reduce the original variational problem to a finite dimensional system of ‘algebraic’ equations: the energy functional (2) is shrunk to a  $m$  variables function  $\tilde{E} : \mathbb{R}^m \rightarrow \mathbb{R}$ , whose critical points, i.e.,  $d\tilde{E}(x) = 0$ , correspond exactly to the solutions of (1).

We will introduce some appropriate conditions on  $F$  in order to prove that the reduced functional  $\tilde{E}(x)$  is a *generating function quadratic at infinity* (GFQI), i.e. it coincides, or corresponds in a suitable sense (see details in Section 4), with a non-degenerate quadratic form  $x^\top Qx$  outside a compact set.<sup>1</sup> This is crucial for our aim: indeed, proving that the sublevel sets

$$Q^c = \{x : x^\top Qx \leq c\}, \quad \tilde{E}^c = \{x : \tilde{E}(x) \leq c\},$$

are diffeomorphic provided  $|c|$  being suitably large, we can apply the ‘deformation lemma’ and claim the existence of a critical point of  $\tilde{E}$  corresponding to the critical point of  $Q$  (0, of course), the so-called *minmax* solution, which is in general a saddle. This is precisely the case when the linear soft device is strong enough, so the original energy functional is not coercive anymore. Actually,  $\tilde{E}$  can be very different from  $Q$  in a compact subset, where many other critical points may arise. Nevertheless, the topological complexity of such a compact perturbation  $\tilde{E}$  of  $Q$  cannot wipe out the minmax critical point inherited from the 0 of  $Q$ .

Contemporary literature displays a plenty of existence, multiplicity, bifurcation results for nonlinear PDEs like (1); the functional setting can be very general, and diverse topological techniques<sup>2</sup> have been applied. Even though the above abstract settings allow for very general results of existence and multiplicity, here we propose a fairly self contained introduction to AZ and we adopt some stronger hypotheses on  $F$ , transforming the variational functional (2) to a finite GFQI, which finally is well arranged towards numerical applications, as e.g. already implemented in [6].

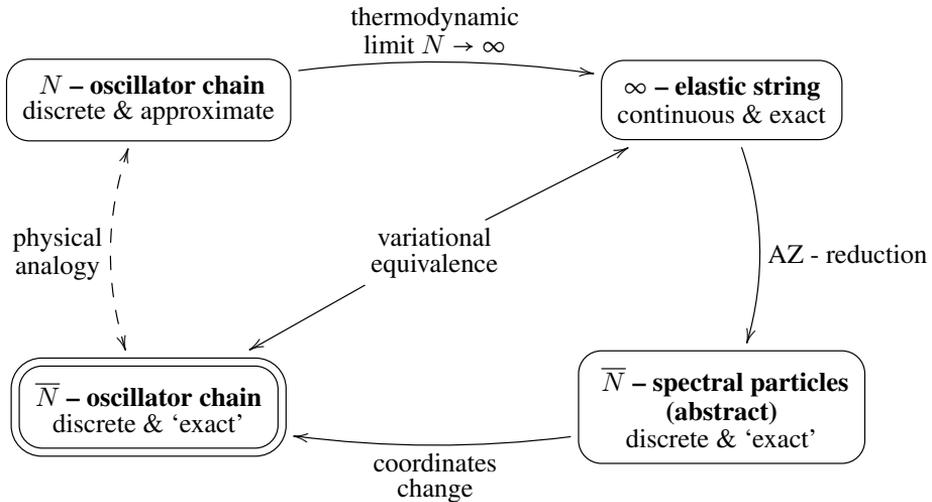
In the last part of the present work (Section 6), we point out the interplay among discrete/continuous and approximate/exact aspects in problem (1), as synthetically illustrated in the following diagram:

As a starting point, we recall that, for instance, the elastic string can be modeled by means of a chain of  $N$  beads connected by springs of length  $h$  and sending  $N \rightarrow \infty$  and  $h \rightarrow 0$  keeping mass and strength densities constant (see [13, Chapter 4.]). Here, the continuous system modeled by (1) can be thought as the thermodynamic limit of a large collection of interconnected small particles. Then, by applying AZ, we get back to a finite dimensional problem where approximations are absent, in the sense that the full nonlinear complexity of the set of equilibria is unaltered. However, as described in Section 2, the reduced system is written in spectral variables  $x$ , therefore most of the mechanical sense of the starting problem is encrypted in an abstract representation.

In order to tailor a physically meaningful interpretation to  $\tilde{E}(x)$ , we consider its leading term, the quadratic part. Surely, it is possible to find a plenty of finite linear physical

<sup>1</sup>Generating functions quadratic at infinity were originally developed by Chaperon, Sikorav and Viterbo studying Hamiltonian dynamics (see [7, 8, 15, 20, 21, 22, 24]), leading to major results in dynamical systems and a new powerful format in symplectic topology.

<sup>2</sup>Conley index[9], Leray–Schauder degree theory, Morse theory [17, 12, 11], Mountain Pass Lemma and Linking theorems [3].



**Figure 1.** In this picture ‘exact’ means equivalent from the variational point of view of the elastic string.

systems whose potential energy corresponds precisely to this quadratic part. In Section 6.2, we suggest a suitable linear coordinate<sup>3</sup> transformation  $x \mapsto y = \Upsilon x$ , selecting, among many variationally equivalent systems, the one mimicking most faithfully the linear part of the original mechanical problem. More precisely, by choosing the above variables  $y$ , we pick out the fittest elastic chain from the thermodynamic limit sequence converging to the linear part of (1).

By doing so we provide a further motivation for the AZ reduction coming from the microphysics of the matter. This excursion gives in a sense an accomplishment to a line of thought started in [19].

**2. The Amann–Zehnder finite reduction for fields theory**

Let  $L$  be an elliptic operator, i.e.,

$$Lu := \sum_i \frac{\partial}{\partial x_i} \left( \sum_j a_{ij}(x) \frac{\partial u}{\partial x_j} \right) = \nabla \cdot (a \nabla u).$$

The scalar product on  $H$ , defined as

$$\begin{aligned} \langle\langle \cdot, \cdot \rangle\rangle : H \times H &\longrightarrow \mathbb{R}, \\ (u, v) &\longmapsto \langle\langle u, v \rangle\rangle := \int_{\Omega} -Lu \cdot v dx. \end{aligned} \tag{3}$$

<sup>3</sup>Actually, we will use the symbol  $\mu$  for the spectral coordinates instead of  $x$ , see (30).

is positive definite, indeed:

$$\begin{aligned} \langle\langle u, u \rangle\rangle &:= \int_{\Omega} (-Lu)u dx = - \int_{\Omega} \nabla \cdot (ua \nabla u) + \int_{\Omega} \nabla u^{\top} a \nabla u \\ &= - \int_{\partial\Omega} \underbrace{u}_{=0 \text{ Dirichlet}} \underbrace{a \nabla u \cdot \mathbf{n}}_{=0 \text{ Neumann}} + \int_{\Omega} \nabla u^{\top} a \nabla u \geq 0. \end{aligned} \quad (4)$$

As a result, we will take  $\|u\|^2 := \langle\langle u, u \rangle\rangle$  as the norm on  $H$ .

The eigenfunctions associated to the eigenvalues of the elliptic operator  $L$  form a basis for the space  $H$ ,

$$\begin{aligned} -L\hat{u}_j &= \lambda_j \hat{u}_j, \\ 0 &= \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots, \\ \|\hat{u}_j\| &= 1. \end{aligned}$$

For every  $v \in H$  we can write

$$v = \sum_{j=1}^{\infty} \langle\langle v, \hat{u}_j \rangle\rangle \hat{u}_j = \sum_{j=1}^{\infty} v_j \hat{u}_j.$$

We are able now to define the Green operator of  $L$ ,  $g : H \rightarrow H$ ,  $g = (-L)^{-1}$ ,

$$gv = g \left( \sum_{j=1}^{\infty} v_j \hat{u}_j \right) = \sum_{j=1}^{\infty} \frac{v_j}{\lambda_j} \hat{u}_j.$$

It is clear that

$$-Lgv = g(-L)v = v, \quad \forall v \in H.$$

For every fixed  $m \in \mathbb{N}$ , we can consider the following decomposition of  $H$ :

$$\begin{aligned} H &= \mathbb{P}_m H \oplus \mathbb{Q}_m H, \\ v &= \mathbb{P}_m v + \mathbb{Q}_m v = \sum_{j=1}^m v_j \hat{u}_j + \sum_{j=m+1}^{\infty} v_j \hat{u}_j. \end{aligned}$$

We will briefly write

$$v = \mu + \eta = \sum_{j=1}^m \mu_j \hat{u}_j + \sum_{j=m+1}^{\infty} \eta_j \hat{u}_j$$

and we will refer to  $\mu \in \mathbb{P}_m H$  as to the *finite* head of  $v$ , whereas to  $\eta \in \mathbb{Q}_m H$  as to the *infinite* tail of  $v$ .

If we substitute the relation  $u = gv$  in the Dirichlet problem (1):

$$-Lu - \lambda u = \varepsilon F(u), \quad u|_{\partial\Omega} = 0 \quad (5)$$

we obtain

$$\begin{aligned} v - \lambda gv &= \varepsilon F(gv), \\ v &= (\mathbb{I} - \lambda g)^{-1} \varepsilon F(gv), \end{aligned}$$

which solutions are the fixed points of the map:

$$\begin{aligned}
 H &\longrightarrow H, \\
 v &\longmapsto (\mathbb{I} - \lambda g)^{-1} F(gv).
 \end{aligned}$$

The map  $(\mathbb{I} - \lambda g)^{-1}$  is well defined for  $\lambda \neq \lambda_j, j = 0, 1, \dots$  and can be written as

$$(\mathbb{I} - \lambda g)^{-1} v = (\mathbb{I} - \lambda g)^{-1} \left( \{v_j\}_{j=1}^\infty \right) = \left\{ \frac{\lambda_j}{\lambda_j - \lambda} v_j \right\}_{j=1}^\infty.$$

The cut-off decomposition splits the problem into a finite and an infinite part:

$$v = \mathbb{P}_m v + \mathbb{Q}_m v = \mu + \eta, \tag{6}$$

$$\mu = \mathbb{P}_m (\mathbb{I} - \lambda g)^{-1} \varepsilon F(g(\mu + \eta)), \tag{7}$$

$$\eta = \mathbb{Q}_m (\mathbb{I} - \lambda g)^{-1} \varepsilon F(g(\mu + \eta)). \tag{8}$$

These equations can also be equivalently rewritten as follows,

$$\mathbb{P}_m N(g(\mu + \eta)) = 0, \quad (\text{finite}) \tag{9}$$

$$\mathbb{Q}_m N(g(\mu + \eta)) = 0. \quad (\text{infinite}) \tag{10}$$

Under appropriate assumptions on  $F$ , (10) is uniquely solved for  $\eta$ . Indeed, we can prove that

**PROPOSITION 1.** *Let  $F : H \rightarrow H$  be Lipschitz, then for every fixed  $\varepsilon > 0$  and  $\mu \in \mathbb{P}_m H$  there exists  $m \in \mathbb{N}$  such that*

$$\eta \longmapsto (\mathbb{I} - \lambda g)^{-1} \varepsilon \mathbb{Q}_m F(g(\mu + \eta)), \tag{11}$$

*is a contraction.*

*Proof.* First, we notice that  $\eta \mapsto g(\mu + \eta)$  is a Lipschitz map with constant  $\frac{1}{\lambda_{m+1}}$ . We deduce that  $(\mathbb{I} - \lambda g)^{-1}$  is bounded, indeed

$$\|(\mathbb{I} - \lambda g)^{-1} v\| = \left\| \left( \frac{\lambda_j}{\lambda_j - \lambda} v_j \right) \right\| \leq \sup_{j \in \mathbb{N}} \left| \frac{\lambda_j}{\lambda_j - \lambda} \right| \|v\|,$$

and clearly  $\sup_j \left| \frac{\lambda_j}{\lambda_j - \lambda} \right| := C_1 < +\infty$ , because the sequence  $\frac{\lambda_j}{\lambda_j - \lambda} \rightarrow 1$ . By assumption,  $F$  is Lipschitz with constant  $C_2$ , therefore also the map (11) is Lipschitz:

$$\begin{aligned}
 \|(\mathbb{I} - \lambda g)^{-1} \varepsilon \mathbb{Q}_m F(g(\mu + \eta_1)) - (\mathbb{I} - \lambda g)^{-1} \varepsilon \mathbb{Q}_m F(g(\mu + \eta_2))\| &\leq \\
 &\leq \frac{\varepsilon C_1 C_2}{\lambda_{m+1}} \|\eta_1 - \eta_2\|.
 \end{aligned}$$

For sufficiently large  $m$ , we conclude that  $\frac{\varepsilon C_1 C_2}{\lambda_{m+1}} < 1$ . □

Substituting the fixed point  $\eta = \tilde{\eta}(\mu)$  of map (11) into equation (7), we obtain

$$\mu = (\mathbb{I} - \lambda g)^{-1} \varepsilon \mathbb{P}_m F(g(\mu + \tilde{\eta}(\mu))), \quad \mu = (\mu_1, \dots, \mu_m) \in \mathbb{R}^m, \tag{12}$$

which is a finite dimensional equivalent formulation for problem (1). Indeed, for every solution  $\tilde{\mu}$  of this last equation we can write a solution of (1) by means of the formula:

$$\tilde{u} = g(\tilde{\mu} + \tilde{\eta}(\tilde{\mu}))$$

Conversely, to every solution  $\tilde{u}$  of (1), corresponds a solution  $\tilde{\mu}$  of (12) by

$$\tilde{\mu} = \mathbb{P}_m(-L)\tilde{u}.$$

**3. Infinite dimensional and reduced finite dimensional variational formulation.**

If we assume the Gateaux derivative  $N'(u)$  of the non linear operator considered in (1) to be *symmetric* with respect to the  $L^2$ -scalar product, *i.e.*

$$\begin{aligned} \langle N'(u)h, k \rangle &= \langle N'(u)k, h \rangle, \quad \forall h, k \in H, \\ \langle u, v \rangle &:= \langle u, v \rangle_{L^2} = \int_{\Omega} uv dx, \end{aligned}$$

the Volterra–Vainberg theorem allows us to write an energy functional,

$$\begin{aligned} E : H &\longrightarrow \mathbb{R}, \\ u &\longmapsto E(u) := \int_{t=0}^{t=1} \langle N(tu), u \rangle dt, \end{aligned} \tag{13}$$

and the corresponding variational principle  $dE(u) = 0$  is equivalent to the original Dirichlet problem (1). More precisely,

**THEOREM 2.** *Every critical point of  $E$  is a solution of (1), and viceversa, i.e.*

$$dE(u)h = 0 \quad \forall h \in H \quad \Leftrightarrow \quad \begin{cases} N(u) = 0, \\ u|_{\partial\Omega} = 0. \end{cases} \tag{14}$$

*Proof.*

$$\begin{aligned} dE[u] \cdot h &= \frac{d}{d\lambda} E[u + \lambda h] \Big|_{\lambda=0} = \int_0^1 \langle N'(tu)th, u \rangle + \langle N(tu), h \rangle dt = \\ &= \int_0^1 \langle N'(tu)u, th \rangle + \langle N(tu), h \rangle dt = \int_0^1 \frac{d}{dt} \langle N(tu), th \rangle dt = \langle N(u), h \rangle. \end{aligned}$$

So  $dE[u] = 0$  if and only if  $N(u) = 0$  as claimed. □

The finite parameters reduction of the preceeding section can be applied to the functional  $E(u)$  to obtain a finite parameters variational principle, which is equivalent to the infinite dimensional one, *i.e.* every critical point of  $E$  is also a critical point for

$$\begin{aligned} \tilde{E} : \mathbb{R}^m &\longrightarrow \mathbb{R}, \\ \tilde{E}(\mu) &:= E(g(\mu + \tilde{\eta}(\mu))). \end{aligned} \tag{15}$$

This equivalence between the finite parameters variational principle  $\frac{d}{d\mu} \tilde{E}(\mu) = 0$  and the original Dirichlet problem (1) is resumed and proved in the following

**THEOREM 3.** *There is a one-to-one correspondence between the critical points  $\tilde{\mu}$  of  $\tilde{E}(\mu)$  and the solutions  $u$  of the Dirichlet problem (1).*

*Proof.* Plainly,

$$\begin{aligned} d\tilde{E}(\mu) &= dE[u] \cdot \frac{d}{d\mu} (g(\mu + \tilde{\eta}(\mu))) d\mu = \\ &= \left\langle N(u) \Big|_{u=g(\mu + \tilde{\eta}(\mu))}, g(d\mu + \tilde{\eta}'(\mu)d\mu) \right\rangle = \\ &= \left\langle \underbrace{\mathbb{P}_m N(u) + \mathbb{Q}_m N(u)}_{=0 \text{ by (10)}} \Big|_{u=g(\mu + \tilde{\eta}(\mu))}, \underbrace{g(d\mu)}_{\in \mathbb{P}_m H} + \underbrace{g(\tilde{\eta}'(\mu)d\mu)}_{\in \mathbb{Q}_m H} \right\rangle = \\ &= \langle \mathbb{P}_m N(g(\mu + \tilde{\eta}(\mu))), g(d\mu) \rangle. \end{aligned}$$

We used the facts that (i) the diagonal isomorphism  $g$  maps  $\mathbb{P}_m H$  and  $\mathbb{Q}_m H$  into themselves and that (ii)  $\mathbb{P}_m H$  and  $\mathbb{Q}_m H$  are orthogonal also with respect to the  $L^2$  scalar product:

$$\langle \hat{u}_i, \hat{u}_j \rangle = \frac{1}{\lambda_i} \langle -L\hat{u}_i, \hat{u}_j \rangle = \frac{1}{\lambda_i} \langle \langle \hat{u}_i, \hat{u}_j \rangle \rangle = 0, \quad i \neq j.$$

We can conclude that

$$d\tilde{E}(\mu) = 0 \quad \iff \quad \mathbb{P}_m N(g(\mu + \tilde{\eta}(\mu))) = 0.$$

□

#### 4. Quasi-quadratic functions and their properties

**4.1. The deformation Lemma.** Standard cohomology on a differentiable manifold  $M$ ,

$$H^k(M) := \frac{\{\text{closed } k\text{-forms on } M\}}{\{\text{exact } k\text{-forms on } M\}},$$

is generalized to the *relative cohomology* by

$$H^k(M, N) := \left\{ \alpha \in H^k(M) \mid \iota^* \alpha = 0 \right\},$$

where  $N$  is a submanifold (possibly with boundary) of  $M$  and  $\iota : N \hookrightarrow M$  is the inclusion map.

It is well known that the compactness hypothesis for the classical calculus of variations can be weakened by adopting the Palais–Smale condition on the pair  $(M, f)$ :

**DEFINITION 4 (Palais–Smale).** We will say  $f : M \rightarrow \mathbb{R}$  is Palais–Smale, if every sequence  $\{x_i\}$  such that  $|f(x_i)|$  is bounded and  $|f'(x_i)| \rightarrow 0$ , admits a converging subsequence.

Under this condition the critical levels are still compact, even if it could not be the case for  $M$ . It is possible to associate a critical value  $\gamma(\alpha, f)$  to every non zero cohomology class  $\alpha$ , i.e.,

$$\forall \alpha \in H^*(M) \setminus \{0\} \quad \exists x \in M : \quad f(x) = \gamma(\alpha, f), \quad df(x) = 0.$$

Let  $f^\lambda := \{x \in M \mid f(x) \leq \lambda\}$  denote the *sublevel set* corresponding to  $\lambda \in \mathbb{R}$ .

**THEOREM 5 (minmax).** Let  $f : M \rightarrow \mathbb{R}$  be Palais–Smale. Let  $\alpha \in H^*(M)$ ,  $\alpha \neq 0$ , and  $\iota_\lambda : f^\lambda \hookrightarrow M$ . Then

$$\gamma(\alpha, f) := \inf \left\{ \lambda \mid \iota_\lambda^* \alpha \neq 0 \right\} \tag{16}$$

is a critical value for  $f$ .

*Proof.* If  $c$  is not a critical value, by Palais–Smale condition, there exists  $\varepsilon > 0$  such that  $f^{-1}([c - \varepsilon, c + \varepsilon])$  does not contain critical points. It is possible to deform  $f^{c+\varepsilon}$  in  $f^{c-\varepsilon}$ , by means of the flow of the vector field

$$X(x) := -\frac{\nabla f(x)}{|\nabla f(x)|^2}, \quad \text{for } x \in f^{c+\varepsilon} \setminus f^{c-\varepsilon},$$

which could be smoothly extended to all  $M$  to the zero vector field outside of a neighborhood of  $f^{c+\varepsilon} \setminus f^{c-\varepsilon}$ . One can easily check that if  $\varphi_t$  is the flow of  $X(x)$ , then

$$f(\varphi_t(x)) = f(x) - t.$$

By means of the diffeomorphism  $\varphi_{2\varepsilon} : f^{c+\varepsilon} \rightarrow f^{c-\varepsilon}$ ,  $H^*(f^{c+\varepsilon}) \cong H^*(f^{c-\varepsilon})$ , therefore if  $\alpha|_{f^{c+\varepsilon}} \neq 0$ , it cannot be  $\alpha|_{f^{c-\varepsilon}} = 0$ . □

Note that different cohomology classes could correspond to the same critical value.

**4.2. Quasi–quadratic functions.** The theory of the Generating Functions Quadratic at Infinity (GFQI) has been developed by M. Chaperon, J.C. Sikorav and C. Viterbo, in the context of Hamiltonian dynamics (see [7, 8, 15, 20, 21, 22, 24]). In the symplectic topology environment, D. Théret [22], and later C. Gol [15], revised carefully results claimed by C. Viterbo [25] adopting apparently more general definitions for GFQI. In the following we will stick to the Viterbo–Théret format.

**DEFINITION 6 (GFQI).** *We say that a function  $S : \mathbb{R}^n \rightarrow \mathbb{R}$  is quadratic at infinity (GFQI) if there exists a non-degenerate quadratic form  $\langle Qu, u \rangle$  and a constant  $K > 0$  such that*

$$\|S(u) - \langle Qu, u \rangle\|_{C^1} = \|S(u) - \langle Qu, u \rangle\| + \|S'(u) - 2Qu\| \leq K \tag{17}$$

For this class of functions the Palais-Smale property does hold, and for sufficiently large  $c > 0$ , the sub-level sets  $S^c := \{x : S(x) \leq c\}$  of a GFQI function  $S$  are diffeomorphic to the sub-level sets of the related quadratic form  $Q$ , leading as a consequence to the equivalence of the relative cohomology classes:

$$H^*(F^{-c}, F^c) \cong H^*(Q^{-c}, Q^c).$$

In this Section we will give an explicit proof of the equivalence  $S^{\pm c} \cong Q^{\pm c}$  along a rather different line of thought than [22]. We start by noticing that all the critical points of  $S$  are in a compact neighborhood of the origin. Indeed,

$$S'(\bar{x}) = 0, \quad \Rightarrow \quad |2Q\bar{x}| \leq K, \quad \Rightarrow \quad \bar{x} \in B\left(0, \frac{K}{2 \min|\text{Spec } Q|}\right).$$

Moreover, GFQI are Palais-Smale. Indeed, let us consider a P-S sequence  $\{x_i\}_{i \in \mathbb{N}}$ , i.e.  $S(x_i)$  bounded,  $S'(x_i) \rightarrow 0$ . For  $i$  sufficiently large we have  $|S'(x_i)| < K$ , and by quasi–quadraticity,  $|2Qx_i| < 2K$ , then  $x_i \in B\left(0, \frac{K}{\min|\text{Spec } Q|}\right)$ . Being the sequence definitively confined in a compact set, it is possible to extract a converging subsequence. We are ready now for proving the main result of this section.

**THEOREM 7.** For  $c > 0$  sufficiently large,

$$S^{\pm c} \stackrel{\text{diffeo}}{\cong} Q^{\pm c}.$$

*Proof.* Assume that  $c$  is large enough to surpass every critical value of  $S$ , and that at least one of the eigenvalues of  $Q$  is positive, otherwise we easily get  $S^c = Q^c = \mathbb{R}^n$ . Because of (17), we have

$$\dots \subseteq S^c \subseteq Q^{c+K} \subseteq S^{c+2K} \subseteq Q^{c+3K} \subseteq \dots$$

and also

$$S^c \cong S^{c+K} \cong \dots \cong S^{c+M} \quad \text{and} \quad Q^c \cong Q^{c+K} \cong \dots \cong Q^{c+M}.$$

Therefore we only need to prove

$$S^c \cong Q^{c+M}$$

for some  $c, M > 0$ . To this end we define a smooth function  $f$  such that  $f^c = S^c$  and  $f^{c+M} = Q^{c+M}$ , without critical values in  $[c, c + M]$ . This can be done by means of an Urysohn function  $\varphi$ ,

$$\varphi(x) = \begin{cases} 0 & \text{if } S(x) \leq c, \\ 1 & \text{if } c + M \leq Q(x), \end{cases} \tag{18}$$

and setting

$$f = (1 - \varphi)S + \varphi Q.$$

By doing so the derivative of  $f$ :

$$|f'| = |[ (1 - \varphi)S' + \varphi Q' ] - [\varphi'(Q - S)]| \geq \underbrace{|(1 - \varphi)S' + \varphi Q'|}_{\geq |Q'|-K} - \underbrace{|\varphi'| \cdot |Q - S|}_{< K}.$$

is non vanishing if  $|\varphi'|$  is bounded, i.e.,  $|\varphi'| < \frac{|Q'|-K}{K}$ . A continuous  $\varphi$  can be defined as follows:

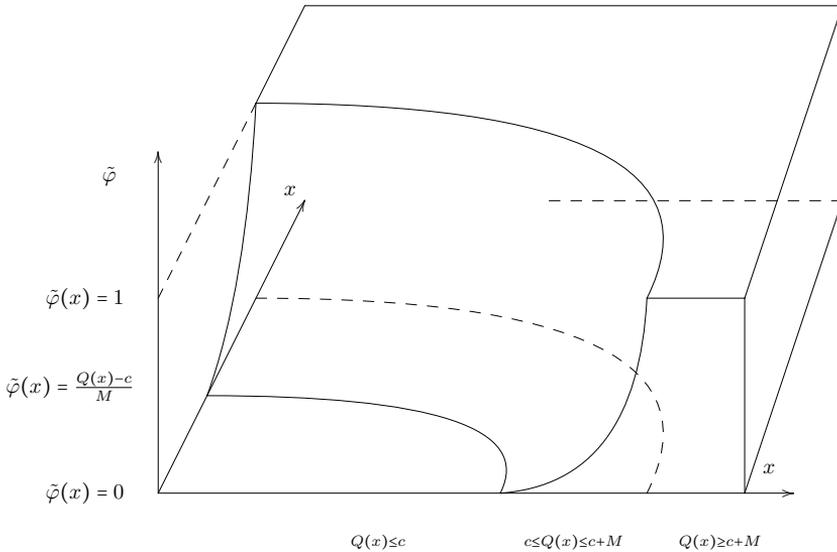
$$\varphi(x) = \begin{cases} 0 & \text{if } Q(x) \leq c, \\ \frac{Q(x)-c}{M} & \text{if } c \leq Q(x) \leq c + M, \\ 1 & \text{if } c + M \leq Q(x). \end{cases}$$

(see Figure 2), indeed  $|\varphi'(x)| = \frac{|Q'(x)|}{M} < \frac{|Q'(x)|-K}{K}$  wherever  $\varphi$  is differentiable. An everywhere smooth Urysohn function can be obtained from  $\varphi$  by mollification. In the Appendix we provide an explicit construction and verify that the constraint on the derivative still holds. It remains to prove that  $|Q'|$  restricted to  $\{Q = c\}$  is  $> K$ , provided that  $c$  is sufficiently large. We set  $v := Q'(x) = 2Qx$  and the problem is to minimize  $\frac{|v|^2}{2}$  for  $v^T \frac{Q^{-T}}{4} v = c$ . Lagrange multipliers format gives:

$$\mathcal{F}(v, \lambda) = \frac{|v|^2}{2} - \lambda \left( v^T \frac{Q^{-T}}{4} v - c \right),$$

i.e.,

$$0 = \frac{\partial \mathcal{F}}{\partial v} = v - \frac{\lambda}{2} Q^{-T} v.$$



**Figure 2.** A continuous Urysohn function  $\varphi$ , s.t.  $\varphi \equiv 0$  on  $S^c$  and  $\varphi \equiv 1$  on  $Q^{c+M}$ .

Therefore,  $v = 2Qx$  must be searched among the eigenvectors of  $Q^{-T}$ , or equivalently  $x$  among the eigenvectors  $u_i$  of  $Q$ . If  $x = \alpha u_i$  on the level  $c$ , i.e.,  $\alpha = \sqrt{\frac{c}{\lambda_i}}$ , then we have

$$Q'(x) = 2Qx = 2\sqrt{\frac{c}{\lambda_i}}Qu_i = 2\sqrt{c}\sqrt{\lambda_i}u_i \quad \Rightarrow \quad |Q'(x)| = 2\sqrt{c}\sqrt{\lambda_i},$$

and the minimum is realized for the smallest positive eigenvalue of  $Q$ , showing that  $|Q'(x)|$  grows at least as  $\sqrt{c}$  on  $Q = c$ . □

**4.3. Existence of a critical point for  $S$ .** Clearly,  $x = 0$  is the unique critical point of  $Q$  with Morse index equal to  $q = \text{ind}Q = \#\{\text{negative eigenvalues of } Q\}$ . Since

$$H^k(Q^c, Q^{-c}) = \begin{cases} \mathbb{R}, & \text{if } k = q, \\ 0, & \text{if } k \neq q, \end{cases} \tag{19}$$

(see, e.g., [14], page 188), we have, for  $c$  sufficiently large,

$$H^*(S^c, S^{-c}) \cong H^*(Q^c, Q^{-c}) \neq 0$$

where also we have used the fact that (relative) cohomology is invariant under diffeomorphisms. Thus, there exists a critical value for  $S$  between  $-c$  and  $c$ , corresponding –by Theorem 5– to the generator of  $H^*(Q^c, Q^{-c})$ .

**5. Quasi-quadratic reduced energy functional**

In this section we will investigate around suitable conditions on  $F$  in (1), in order that the associated reduced variational principle being a generating function quasi quadratic at infinity, thus admitting at least a critical point.

We write more precisely (13) pinpointing the quadratic and the nonquadratic part in  $u$ :

$$\begin{aligned}
 E(u) &= \int_0^1 \langle N(tu), u \rangle dt = \\
 &= - \left( \int_0^1 \langle Ltu, u \rangle dt + \lambda \int_0^1 \langle tu, u \rangle dt \right) - \int_0^1 \langle \varepsilon F(tu), u \rangle dt = \\
 &=: E^{L+\lambda}(u) + E^F(u).
 \end{aligned}$$

The reduced functional (15) splits consequently,

$$\tilde{E}(\mu) = E(u(m, \mu)) = E^{L+\lambda}(u(\mu)) + E^F(u(\mu)) = \tilde{E}^{L+\lambda}(\mu) + \tilde{E}^F(\mu), \quad (20)$$

and, furthermore, a quadratic form in the  $\mu_j$ 's can be isolated:

$$\begin{aligned}
 E^{L+\lambda}(u) &= E^{L+\lambda}(gv) = \sum_{j=1}^{\infty} \frac{1}{\lambda_j} v_j^2 - \lambda \left( \sum_{j=1}^{\infty} \frac{1}{\lambda_j^2} v_j^2 \right) = \\
 &= \sum_{j=1}^m \frac{\mu_j^2}{\lambda_j} + \sum_{j=m+1}^{\infty} \frac{\tilde{\eta}_j^2(\mu)}{\lambda_j} - \lambda \left( \sum_{j=1}^m \frac{\mu_j^2}{\lambda_j^2} + \sum_{j=m+1}^{\infty} \frac{\tilde{\eta}_j^2(\mu)}{\lambda_j^2} \right) = \\
 &= \sum_{j=1}^m \frac{\lambda_j - \lambda}{\lambda_j^2} \mu_j^2 + \sum_{j=m+1}^{\infty} \frac{\lambda_j - \lambda}{\lambda_j^2} \tilde{\eta}_j^2(\mu) = \\
 &= \tilde{E}_q^{L+\lambda}(\mu) + \tilde{E}_{nq}^{L+\lambda}(\mu). \quad (21)
 \end{aligned}$$

Thus we can set:

$$\begin{aligned}
 \tilde{E}_q^{L+\lambda}(\mu) &:= \sum_{j=1}^m \frac{\lambda_j - \lambda}{\lambda_j^2} \mu_j^2 \quad (=: \langle Q\mu, \mu \rangle), \\
 \tilde{E}_{nq}^{L+\lambda}(\mu) &:= \sum_{j=m+1}^{\infty} \frac{\lambda_j - \lambda}{\lambda_j^2} \tilde{\eta}_j^2(\mu).
 \end{aligned}$$

In the general case, there is no reason for  $\tilde{E}_{nq}^{L+\lambda}(\mu)$  and  $\tilde{E}^F(\mu)$  to be quadratic. With an additional hypothesis on  $F$ , it is possible to prove the following theorem.

**THEOREM 8.** *Let the nonlinear functional  $F : H \rightarrow H$  be a Nemitski operator,  $F(u) = f \circ u$ , associated to  $f : \mathbb{R} \rightarrow \mathbb{R}$ . If  $f$ , its derivative  $f'$  and one of its primitives  $\bar{f}$  are bounded, then, the reduced functional  $\tilde{E}(\mu)$  is a GFQI, i.e.*

$$\|\tilde{E}(\mu) - \langle Q\mu, \mu \rangle\|_{C^1} \leq const \quad (22)$$

where

$$Q := \begin{pmatrix} \frac{\lambda - \lambda_1}{\lambda_1^2} & & & & \textcircled{0} \\ & \frac{\lambda - \lambda_2}{\lambda_2^2} & & & \\ & & \textcircled{0} & & \\ & & & \ddots & \\ & & & & \frac{\lambda - \lambda_m}{\lambda_m^2} \end{pmatrix}.$$

As a result, problem (1) admits at least one solution.

*Note 9.* The request on  $f$  to be bounded along with its derivative and primitive could seem at first overly restrictive to include sufficiently general nonlinearities. However, it should be noticed that these restrictions apply to the strictly nonlinear part of the operator, and we

point out that our class of nonlinearities include  $C_0^\infty$  functions, Schwartz functions, as well zero-mean periodic functions, e.g.,  $\sin(x)$ .

*Proof.* Because of (20) and (21), we have that

$$\tilde{E}(\mu) - \langle Q\mu, \mu \rangle = \tilde{E}(\mu) - \tilde{E}_q^{L+\lambda}(\mu) = \tilde{E}_{nq}^{L+\lambda}(\mu) + \tilde{E}^F(\mu), \tag{23}$$

therefore we have to verify that  $\tilde{E}_{nq}^{L+\lambda}(\mu)$  and  $\tilde{E}^F(\mu)$  are bounded along with their derivatives. Let  $K > 0$  be the constant such that

$$|f|, |\bar{f}|, |f'| \leq K.$$

1.  $|\tilde{E}^F| \leq \text{cost.}$ : Whenever  $u(x) \neq 0$ ,

$$\int_0^1 f(tu(x)) dt = \frac{1}{u(x)} \int_0^{u(x)} f(\tau) d\tau,$$

thus, denoting by  $\Omega' = \{x \in \Omega \mid u(x) \neq 0\}$ ,

$$\begin{aligned} |\tilde{E}^F| &= \left| \left( \int_0^1 \varepsilon F(tu) dt, u \right) \right| = \left| \int_{\Omega'} \varepsilon \left( \frac{1}{u(x)} \int_0^{u(x)} f(\tau) d\tau \cdot u \right) dx \right| = \\ &= \varepsilon \left| \int_{\Omega'} \left( \int_0^{u(x)} f(\tau) d\tau \right) dx \right| = \varepsilon \left| \int_{\Omega'} \bar{f}(u(x)) dx \right| \leq \\ &\leq \varepsilon K \text{ meas}(\Omega). \end{aligned}$$

2.  $|\tilde{E}_{nq}^{L+\lambda}| \leq \text{cost.}$ : We have that

$$\tilde{E}_{nq}^{L+\lambda}(\mu) = \sum_{j=m+1}^\infty \frac{\lambda_j - \lambda}{\lambda_j^2} \tilde{\eta}_j(m, \mu)^2$$

Being  $\lambda$  fixed, while  $m$  can be chosen arbitrarily large, it is non restrictive to suppose  $\lambda \ll \lambda_j, \forall j > m$ .

$$|\tilde{E}_{nq}^{L+\lambda}(\mu)| \leq \frac{1}{\lambda_{m+1}} \sum_{j=m+1}^\infty \left| \frac{\lambda_j - \lambda}{\lambda_j} \right| |\tilde{\eta}_j(m, \mu)|^2 \leq \frac{1}{\lambda_{m+1}} \|\tilde{\eta}\|^2.$$

On the other hand, being  $\tilde{\eta}$  obtained by means of the fixed point technique, it satisfies

$$\tilde{\eta} = (\mathbb{I} - \lambda g)^{-1} \varepsilon \mathbb{Q}_m F g(\mu + \tilde{\eta}).$$

We prove that  $\|(\mathbb{I} - \lambda g)^{-1}\| < \frac{\lambda_{m+1}}{\lambda_{m+1} - \lambda}$ . Indeed,

$$\begin{aligned} (\mathbb{I} - \lambda g)x &= \left\{ \left( \mathbb{I} - \frac{\lambda}{\lambda_j} x_j \right) \right\}_{j=m+1}^\infty, \quad \forall x \in \mathbb{Q}_m H., \\ (\mathbb{I} - \lambda g)^{-1}x &= \left\{ \left( 1 - \frac{\lambda}{\lambda_j} \right)^{-1} x_j \right\}_{j=m+1}^\infty = \left\{ \frac{\lambda_j}{\lambda_j - \lambda} x_j \right\}_{j=m+1}^\infty, \end{aligned}$$

and if we suppose  $\lambda \ll \lambda_m \leq \lambda_j$ ,

$$\frac{\lambda_{m+1}}{\lambda_{m+1} - \lambda} \geq \frac{\lambda_j}{\lambda_j - \lambda} \xrightarrow{j \rightarrow \infty} 1.$$

Thus,

$$\|\tilde{\eta}\| \leq \frac{\lambda_{m+1}}{\lambda_{m+1} - \lambda} \varepsilon \|F(g(\mu + \tilde{\eta}))\| \leq \frac{\lambda_{m+1}}{\lambda_{m+1} - \lambda} \varepsilon K,$$

as claimed.

To give an estimate on the first derivatives of  $\tilde{E}^F$  and  $E_{ng}^{L+\lambda}$ , we start by considering  $\tilde{\eta}'(\mu)$  and  $u'(\mu)$ . Because of the fixed point equation defining  $\tilde{\eta}$ , we have that

$$\frac{\partial \tilde{\eta}}{\partial \mu_h} = (\mathbb{I} - \lambda g)^{-1} \varepsilon \mathbb{Q}_m F'(u(\mu)) \cdot \left( g \left( \frac{\partial \mu}{\partial \mu_h} + \frac{\partial \tilde{\eta}}{\partial \mu_h} \right) \right), \quad h = 1, \dots, m.$$

$$\begin{aligned} \left\| \frac{\partial \tilde{\eta}}{\partial \mu_h} \right\| &= \left\| (\mathbb{I} - \lambda g)^{-1} \varepsilon \mathbb{Q}_m F'(u(\mu)) \cdot \left( g \left( \frac{\partial \mu}{\partial \mu_h} + \frac{\partial \tilde{\eta}}{\partial \mu_h} \right) \right) \right\| \leq \\ &\leq \|(\mathbb{I} - \lambda g)^{-1}\| \varepsilon \|F'\| \cdot \left( \frac{1}{\lambda_h} + \frac{1}{\lambda_{m+1}} \left\| \frac{\partial \tilde{\eta}}{\partial \mu_h} \right\| \right) \leq \\ &\leq \varepsilon K \frac{\lambda_{m+1}}{\lambda_{m+1} - \lambda} \left( \frac{1}{\lambda_1} + \frac{1}{\lambda_{m+1}} \left\| \frac{\partial \tilde{\eta}}{\partial \mu_h} \right\| \right), \end{aligned}$$

thus,

$$\begin{aligned} \left\| \frac{\partial \tilde{\eta}}{\partial \mu_h} \right\| \left( 1 - \frac{\varepsilon K}{\lambda_{m+1} - \lambda} \right) &\leq \frac{\varepsilon K}{\lambda_1} \frac{\lambda_{m+1}}{\lambda_{m+1} - \lambda}, \\ \|\tilde{\eta}'\| &\leq \frac{\varepsilon K}{\lambda_1} \cdot \frac{\lambda_{m+1}}{\lambda_{m+1} - \lambda} \cdot \frac{1}{1 - \varepsilon K \frac{1}{\lambda_{m+1} - \lambda}} = \frac{\varepsilon K}{\lambda_1} \cdot \frac{\lambda_{m+1}}{\lambda_{m+1} - \lambda - \varepsilon K}, \end{aligned}$$

where the rightmost fraction is well defined and positive for  $m$  sufficiently large. We obtain also an estimate for  $u'$ ,

$$\begin{aligned} u(\mu) = g(\mu + \tilde{\eta}(\mu)) &\Rightarrow u' = g(\mu' + \tilde{\eta}'), \\ \|u'\| = \|g(\mu' + \tilde{\eta}')\| &\leq \|g\| (1 + \|\tilde{\eta}'\|), \|u'\| \leq \frac{1}{\lambda_1} (1 + \|\tilde{\eta}'\|). \end{aligned}$$

3.  $\left| \frac{\partial}{\partial \mu} \tilde{E}^F \right| \leq \text{cost.}$ : Recalling that

$$\left( \int_0^1 F(tu) dt, u \right) = \int_{\Omega} \bar{f}(u(x)) dx,$$

it is easy to check that

$$\begin{aligned} \left| \frac{\partial}{\partial \mu} \tilde{E}^F(\mu) \right| &= \left| \frac{\partial}{\partial \mu} \left( \int_0^1 F(tu(\mu)) dt, u \right) \right| = \\ &= \left| \frac{\partial}{\partial \mu} \int_{\Omega} \bar{f}(u(x)) dx \right| = \left| \int_{\Omega} f(u(x)) \frac{\partial u}{\partial \mu}(x) dx \right| \leq \\ &\leq \int_{\Omega} |f| \|u'\| dx \leq \|u'\| K \text{ meas}(\Omega). \end{aligned}$$

4.  $\left| \frac{\partial}{\partial \mu} \tilde{E}_{nq}^{L+\lambda} \right| \leq \mathbf{cost.}$ : By definition we have

$$\frac{\partial \tilde{E}_{nq}^{L+\lambda}}{\partial \mu} = \frac{\partial}{\partial \mu} \left( \sum_{j=m+1}^{\infty} \frac{\lambda_j - \lambda}{\lambda_j^2} \tilde{\eta}_j^2(\mu) \right) = 2 \sum_{j=m+1}^{\infty} \left( \frac{\lambda_j - \lambda}{\lambda_j} \frac{\partial \tilde{\eta}_j(\mu)}{\partial \mu} \right) \cdot \left( \frac{\tilde{\eta}_j(\mu)}{\lambda_j} \right),$$

then

$$\begin{aligned} \left\| \frac{\partial}{\partial \mu} \tilde{E}_{nq}^{L+\lambda}(\mu) \right\| &\leq 2 \left\| \left\{ \frac{\lambda_j - \lambda}{\lambda_j} \frac{\partial \tilde{\eta}_j}{\partial \mu} \right\}_{m+1}^{\infty} \right\| \cdot \frac{\|\tilde{\eta}\|}{\lambda_{m+1}} \leq \\ &\leq 2 \frac{1}{\lambda_{m+1}} \sup_{j>m} \left| \frac{\lambda_j - \lambda}{\lambda_j} \right| \|\tilde{\eta}'\| \cdot \|\tilde{\eta}\| \leq \frac{2}{\lambda_{m+1}} \|\tilde{\eta}'\| \cdot \|\tilde{\eta}\|. \end{aligned}$$

We can conclude that  $\|\tilde{E}(\mu) - \langle Q\mu, \mu \rangle\|_{C^1}$  is bounded by a function depending only on  $\varepsilon, m, K, \text{meas}(\Omega), \lambda_1$  and  $\lambda_{m+1}$ . □

### 6. Optimal mechanical interpretation of the reduction of a continuous system

**6.1. A model for the nonlinear elastic string.** In this section we consider a specific continuous problem, whose linear part could be obtained as the thermodynamic limit of a sequence of discrete mechanical systems. After having applied AZ reduction, by means of a suitable change of variables, we restore a discrete mechanical system

- (i) variationally equivalent to the continuum and
- (ii) physically comparable to an element of the sequence involved in the above thermodynamic limit.

Let us consider the one dimensional stationary problem of the *nonlinear elastic string*:

$$\begin{cases} \sigma u + \tau u'' = F(u), & u : [0, L] \longrightarrow \mathbb{R}, \\ u(0) = u(L) = 0. \end{cases} \tag{24}$$

where  $u$  is the vertical displacement of an infinitesimal element of the string,  $F$  is, as before, a globally Lipschitz nonlinear Nemitski operator,  $\sigma \geq 0$  or  $< 0$  represents a vertical attractive or repelling elastic force, while  $\tau > 0$  controls the elastic attraction among neighboring infinitesimal pieces of the string. The eigensystem of  $-\tau \frac{\partial^2}{\partial x^2}$  is given by

$$\begin{aligned} 0 < \lambda_1 < \lambda_2 < \dots & \quad \lambda_k = \tau \left( \frac{\pi k}{L} \right)^2, \quad 1 \leq k, \\ \hat{u}_k(x) &= \sqrt{\frac{L}{2\tau}} \frac{1}{\pi k} \sin \left( \frac{\pi k}{L} x \right). \end{aligned} \tag{25}$$

The above AZ construction leads to the finite variational principle  $dE(\mu) = 0$ , where

$$\begin{aligned} E(\mu_1, \dots, \mu_N) &= E^{lin}(\mu) + E^{nlin}(\mu), \\ E^{lin}(\mu) &= \sum_{j=0}^N \frac{\lambda_j + \sigma}{\lambda_j} \mu_j^2, \quad \text{with } \frac{\|F\|}{\lambda_{N+1}} < 1. \end{aligned} \tag{26}$$

This exact finite spectral formulation does correspond to the description of a large number of more or less realistic finite dynamical systems. The direct reconnaissance of (26) shows a system of  $N$  harmonic oscillators, whose positions are described by the spectral variables

$\mu_j$ , with elastic constants  $\frac{\lambda_j + \sigma}{\lambda_j}$ ; the term  $E^{nlin}(\mu)$  represents a further nonlinear coupling, incorporating all the nonlinearities of the system.

Our aim is to look for a mechanical discrete system, described by genuine physical variables, such that, at least when the linear part is concerned, its diagonalization –or spectral representation– is proposing precisely the quadratic part  $E^{lin}(\mu)$  of (26). Among the large number of possible systems satisfying this requirement, we will pick the fitting element from the thermodynamic sequence converging to the linear part of the continuous system (24).

According to this program, we consider the elastic chain of  $N + 2$  oscillating beads (see [13, Chapter 4] and [16]) with fixed extrema:

$$\begin{cases} \bar{\sigma}y_i + \tau(\Delta y)_i = (F(y))_i, & i = 1, \dots, N, \\ y_0 = y_{N+1} = 0, \end{cases} \quad (27)$$

where  $y_i$  represents the  $i$ -th vertical displacement, and the *finite differences Laplace operator*:

$$(\Delta y)_i := \frac{y_{i+1} - 2y_i + y_{i-1}}{a^2}, \quad a = \frac{L}{N + 1}, \quad i = 1, \dots, N \quad (28)$$

represents the bond among neighboring beads. Each oscillator is constrained to move only vertically, is bound to the neighboring beads with a linear spring with elastic constant  $\tau$  and also to its rest position at  $y_i = 0$  with another spring with constant  $\bar{\sigma}$ . In analogy with our continuous problem, we also introduce a nonlinear Lipschitz force  $F$  depending on the vertical displacements  $y$ .

The role of the variational principle is played by the potential energy, which can be written as

$$U(y) := \sum_{i=1}^N \frac{\bar{\sigma}}{2} ay_i^2 + \sum_{i=0}^N \frac{\tau}{2} a \frac{(y_i - y_{i+1})^2}{a^2} + U^F(y) = \frac{1}{2} y^\top Ay + U^F(y). \quad (29)$$

Next, we want to exhibit a suitable linear change of variables

$$y = y(\mu) = \Upsilon \mu, \quad (30)$$

such that the potential energy (29) assumes the form of (26). More precisely, we will choose  $\Upsilon$  in order to make coincide the quadratic parts of (29) and (26):

$$\frac{1}{2} y(\mu)^\top A y(\mu) = E^{lin}(\mu), \quad (31)$$

and the remainder representing the overall nonlinear coupling will be defined consequently:

$$U^F(y(\mu)) := E^{nlin}(\mu). \quad (32)$$

Doing so, we would have obtained a discrete system variationally equivalent to the nonlinear continuous model and with additionally a strong physical correspondence, in the sense that the discrete –described by (29), and in (39) below– simulates the microscopic granular structure of the continuum.

We carry over explicit computations in the following.

**6.2. Equivalence among a chain of springs and the elastic string.** We can compute the normal modes of the linear part of the system by providing the full solution set of the homogeneous version of equation (27), which is obtained via the eigensystem of the finite difference Laplace operator (28):

$$\begin{aligned} \bar{v}_i^k &= \sin\left(\frac{\pi}{N+1}ki\right), \\ \bar{\lambda}_k &= \tau \frac{2}{a^2} \left(1 - \cos \frac{\pi}{N+1}k\right) = \tau \frac{4(N+1)^2}{L^2} \sin^2\left(\frac{\pi}{2(N+1)}k\right). \end{aligned} \tag{33}$$

It is possible to write the potential energy (29) in spectral coordinates  $\zeta_k$ , obtaining an expression analogous to (26):

$$\begin{aligned} y(\zeta) := Y\zeta &\Rightarrow y_i(\zeta) = \sum Y_i^k \zeta_k = \sum_{k=1}^N \bar{v}_i^k \zeta_k, \\ \tilde{U}(\zeta) := U(y(\zeta)) &= \frac{a}{2} \sum_{k=1}^N (\bar{\sigma} + \bar{\lambda}_k) \zeta_k^2 + \tilde{U}^F(\zeta) \\ &= \frac{1}{2} \zeta^\top \tilde{A} \zeta + \tilde{U}^F(\zeta), \end{aligned} \tag{34}$$

where now  $\tilde{A}$  is diagonal. Approaching the thermodynamic limit (for  $N$  large), the ratio  $\frac{k}{2(N+1)} \simeq 0$  for any finite  $k$ . Therefore we can substitute the sinus with its argument in (33):

$$\bar{\lambda}_k = \tau \frac{4(N+1)^2}{L^2} \sin^2\left(\frac{\pi}{2(N+1)}k\right) \xrightarrow{N \rightarrow \infty} \tau \frac{\pi^2}{L^2} k^2 = \lambda_k. \tag{35}$$

Therefore, for large  $N$ , the first eigenvalues of the chain approximate the first eigenvalues of the continuous elastic string (25).

If we further apply the following diagonal change of coordinates:

$$\begin{aligned} \zeta(\mu) := Z\mu, &\Rightarrow \zeta_k = \sqrt{\frac{1}{\bar{\sigma} + \bar{\lambda}_k} \frac{\sigma + \lambda_k}{\lambda_k}} \mu_k, \\ \bar{U}(\mu) := \tilde{U}(\zeta(\mu)) &= \frac{a}{2} \sum_k \frac{\sigma + \lambda_k}{\lambda_k} \mu_k^2 + \bar{U}^F(\mu), \end{aligned} \tag{36}$$

we obtain a potential energy where the quadratic part now *coincides* with  $E^{lin}(\mu)$  in (26).

For the good definition of this transformation  $\bar{\sigma}$  has to be chosen such that the ratio  $\frac{\sigma + \lambda_k}{\bar{\sigma} + \lambda_k} > 0$  for every  $k$ . In other words, if  $\sigma$  is negative and  $\lambda_i < -\sigma < \lambda_{i+1}$ ,  $-\bar{\sigma}$  should be placed among the corresponding pair of eigenvalues for the chain:  $\bar{\lambda}_i < -\bar{\sigma} < \bar{\lambda}_{i+1}$ . If  $\sigma$  is positive it suffices that also  $\bar{\sigma}$  be positive.

Finally, composing the two linear coordinate changes (34) and (36):

$$y = Y\zeta = YZ\mu =: \Upsilon\mu, \quad \mu = \Upsilon^{-1}y = Z^{-1}Y^{-1}y. \tag{37}$$

and by substituting in (26), we obtain the finite dimensional non nonlinear energy function

$$U(y) = E(\Upsilon^{-1}y) \tag{38}$$

coinciding with the potential energy of a chain of  $N + 2$  nonlinearly coupled oscillators with fixed extrema as in (29):

$$U(y) = \sum_{i=1}^N \frac{\bar{\sigma}}{2} a y_i^2 + \sum_{i=0}^N \frac{\tau}{2} a \frac{(y_i - y_{i+1})^2}{a^2} + U^F(y), \tag{39}$$

$$U^F(y) := E^{nlin}(\mu(y)) = E^{nlin}(\Upsilon^{-1}y).$$

**Appendix: A smooth Urysohn function**

We provide an explicit construction of an everywhere differentiable version of the Urysohn function in (18).

LEMMA 10. *Let  $Q(x)$  a differentiable function without critical values between  $c$  and  $c+M$ . Then for every  $\varepsilon > 0$  there always exists a differentiable function  $\varphi$  such that*

$$\varphi(x) = \begin{cases} 0 & \text{if } Q(x) \leq c, \\ 1 & \text{if } c + M \leq Q(x), \end{cases}$$

and furthermore that  $|\varphi'(x)| \leq \frac{|Q'(x)|}{M-\varepsilon}$  for every  $x$ .

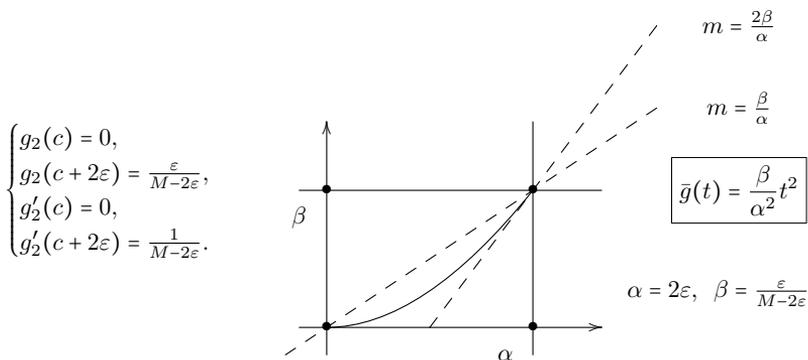
*Proof.* By composing  $Q(x)$  with a suitable differentiable function  $g : \mathbb{R} \rightarrow [0, 1]$ , strictly increasing in the interval  $[c, c + M]$ , we will maintain the same sublevel sets, and furthermore we are able to control the magnitude of the derivatives.

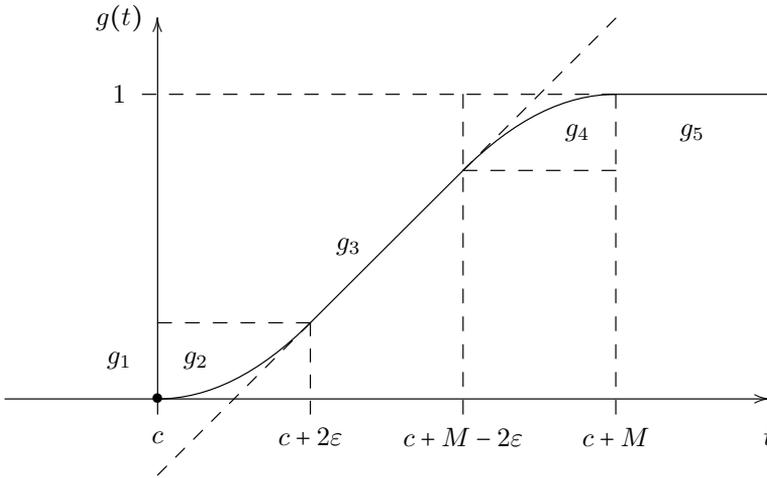
In order to define  $g$ , we divide the interval  $[c, c + M]$  in three subintervals, as in Figure 3:

$$g(t) = \begin{cases} g_1 \equiv 0 & \text{if } t \leq c, \\ g_3 \equiv \frac{t-c-\varepsilon}{M-2\varepsilon} & \text{if } c + 2\varepsilon < t < c + M - 2\varepsilon, \\ g_5 \equiv 1 & \text{if } c + M \leq t. \end{cases}$$

$g_2$  (and similarly  $g_4$ ) should fill the gap with continuous derivatives:

$$\begin{cases} g_2(c) = 0, \\ g_2(c + 2\varepsilon) = \frac{\varepsilon}{M-2\varepsilon}, \\ g_2'(c) = 0, \\ g_2'(c + 2\varepsilon) = \frac{1}{M-2\varepsilon}. \end{cases} \tag{40}$$





**Figure 3.** Detailed description of the function  $g$ . The maximal slope is realized in the central interval  $g_3$ .

These conditions are met by a unique second order polynomial:

$$g_2(t) := \frac{1}{4\varepsilon(M - 2\varepsilon)}(t - c)^2, \quad g_4 := c + M - \frac{1}{4\varepsilon(M - 2\varepsilon)}(t - c - M)^2.$$

We eventually check the bound on the derivatives:

$$\varphi'(x) = \begin{cases} g'_1 \cdot Q' \equiv 0 & \text{if } Q(x) \leq c, \\ g'_2 \cdot Q' = \frac{Q-c}{2\varepsilon(M-2\varepsilon)} \cdot Q' & \text{if } c < Q(x) < c + 2\varepsilon, \\ g'_3 \cdot Q' = \frac{Q'}{M-2\varepsilon} & \text{if } c + 2\varepsilon < Q(x) < c + M - 2\varepsilon, \\ g'_4 \cdot Q' = \frac{c+M-Q}{2\varepsilon(M-2\varepsilon)} \cdot Q' & \text{if } c + M - 2\varepsilon < Q(x) < c + M, \\ g'_5 \cdot Q' \equiv 0 & \text{if } c + M \leq Q(x). \end{cases}$$

Therefore we have

$$|\varphi'(x)| = |g'(Q(x))| \cdot |Q'(x)| \leq \frac{1}{(M - 2\varepsilon)} \cdot |Q'(x)|,$$

as wanted. □

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Article contributed to the Festschrift volume in honour of Giuseppe Grioli on the occasion of his 100th birthday.

Communicated 13 April 2012; published online 29 January 2013

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