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SUPERPOSITIONS IN PRIGOGINE'S APPROACH TO IRREVERSIBILITY FOR PHYSICAL AND FINANCIAL APPLICATIONS

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ABSTRACT. In this paper we apply the theory of superpositions for Radon measures on compact subsets of the real euclidean n-space $\mathbb{R}^n$ to Prigogine's approach in the study of irreversible processes, which emerge in Physics and in Economics; showing that the superposition is a natural rigorous tool feasible to face the problem.

1. Introduction

In section 2 we give a definition of discrete dynamical system that will be generalized in the paper. In section 3 we expose a particularly important kind of discrete dynamical systems: the systems generated by a function from a set to itself; we show their basic properties. In section 4 we define the probabilistic dynamical systems and explain the necessity of a generalization of the Prigogine's intuitive settings. In section 5, 6 and 8 we develop the foundation of superpositions theory on compact subsets and on locally compact subspaces of the real euclidean n-space $\mathbb{R}^n$. In section 7 we use superpositions to define the probabilistic system generated by a map. In section 9 we prove the main result of the paper: a generalization of one Prigogine's result. For what concerns the origin of the paper, it can be found in [1]-[6], for the theory of Radon measures see [7] and [8].

2. Discrete dynamical systems

Definition (of abstract discrete dynamical system). *A discrete (non-reversible) dynamical system on a non-empty set X is an action of the additive monoid of natural numbers on the set X; in other terms, it is a function $s : \mathbb{N}_0 \times X \to X$ such that $s(0, x_0) = x_0$, for every $x_0 \in X$ and such that $s(m+n, x_0) = s(m, s(n, x_0))$, for every pair of natural numbers (m, n).*

Terminology. Let s be a discrete dynamical system on X. The set X is called the space of states or the space of phases of the system s. Every element x of X is called a state of the system s. Every natural number n is called a time of the system s.

Remark. An action a of a monoid on a set X induces on the set a preorder in a natural way: a state x is said to precede a state y if there is an element m of the monoid that sends

x into y, i.e, such that $a(m,x) = y$. In general it is possible to find not-comparable states. If the monoid is a group then this preorder is an equivalence relation on the set.

Definition (of orbit). *For every time $n \in \mathbb{N}$ and for every state $x_0 \in X$, the state $s(n, x_0)$ is called the state of s at the time n determined by the initial condition (state) x_0. Fixed the state x_0, the sequence $(s(n, x_0))_{n\in\mathbb{N}}$ is called the orbit or trajectory of the system starting from the initial state x_0.*

Definition (of equilibria). *A state x_e is said an equilibrium-state for the system s if the orbit starting from x_e is constant of value x_e, that is $s(n, x_e) = x_e$, for every time n. If τ is a topology on the state space X, a state x_e is said a τ-asintotic equilibrium state for s under the initial condition x_0 if the orbit starting from x_0 τ-converges to x_e:*

$$^{\tau}\lim_{n\to+\infty} s(n, x_0) = x_e.$$

Remark. Clearly, an equilibrium state is a τ-asintotic equilibrium, for every topology τ on the set X.

Interpretation. A discrete dynamical system is the model for physical systems whose state depends on time, considered as a discrete quantity, and on its initial condition (state), that is the state at time 0.

Let us explain the definitions with an elementary example.

Example. Let $X = \mathbb{R}$ be the space of states of the dynamical system

$$s : \mathbb{N}_0 \times \mathbb{R} \to \mathbb{R} : s(n, x_0) = x_0^{n+1}.$$

Let us consider the orbits of the system starting from different initial conditions:

1) if $|x_0| < 1$, we have $\lim_{n\to+\infty} s(n, x_0) = 0$. The orbit starting from x_0 converges to the state 0, thus, with Prigogine, we say that the system s converges asintotically to the equilibrium-state $x_e = 0$ starting from state x_0;

2) if $x_0 > 1$, we have $\lim_{n\to+\infty} s(n, x_0) = +\infty$, in this case the orbit $s(\cdot, x_0)$ is divergent (in the sense of real sequences);

3) if $x_0 \leq -1$, the orbit $(s(n, x_0))_{n\in\mathbb{N}}$ is oscillating (in the sense of real sequences), and moreover, if $x_0 < -1$, it is absolutely divergent $\lim_{n\to+\infty} |s(n, x_0)| = +\infty$;

4) if $x_0 = 0$, we have $s(n, x_0) = 0$, for every $n \in \mathbb{N}_0$; thus the state $x_0 = 0$ is an equilibrium state of the system.

5) if $x_0 = 1$, we have $s(n, x_0) = 1$, for every $n \in \mathbb{N}_0$, thus also $x_0 = 1$ is an equilibrium state of the system s.

Remark. Let a be an action of a monoid on a set X. We recall that a point e of a preordered space is said strongly maximal if the relation "x is weakly greater than e" implies that x is equal to e. It is clear that a point e of the set X is an equilibrium of the dynamical system if and only if it is a strongly maximal point of X with respect to the preorder induced by the action on the set. An element x of the set X is said a torsion element with respect to the action if there is an element of the monoid that sends x into itself. The class of indifference of a torsion element coincides with its orbit and an element is a torsion element with respect to the action if and only if it is a maximal element with respect to the preorder induced by the action (for the proofs and a complete treatment see [9]).

3. Discrete dynamical systems generated by maps.

Definition (of discrete dynamical system generated by a map). *Let X be a non-empty set (that will be the space of states of our dynamical system) and let $f : X \to X$ a function. Then we consider the mapping*

$$s_f : \mathbb{N}_0 \times X \to X : (n, x_0) \mapsto f^n (x_0) ,$$

where by f^n we shall denote the composition of f with f itself n times. The mapping s_f is clearly a discrete dynamical system and it is called the system generated by the map f.

The equilibrium-states of a dynamical system generated by a map f are connected with the fixed point of f, as it is shown by the following theorems.

Theorem. *Let $f : X \to X$ be a map. Then,*
i) every fixed point x_0 of f is an equilibrium point associated with x_0;
ii) every fixed point of f in an orbit $s_f(\cdot, x)$ is an equilibrium associated with x;
iii) every equilibrium of s_f is a fixed point of f;
iv) if $s_f(n, x) = s_f(n+1, x)$ for some n and x, $s_f(n, x)$ is an equilibrium under x.

Proof. i) If $x_0 \in \mathrm{Fix}(f)$, we have $s_f(n, x_0) = f^n(x_0) = x_0$; hence x_0 is an equilibrium-state of the system s associated with the initial state x_0, and more the orbit starting from x_0 has a single state.

ii) If for some $m \in \mathbb{N}_0$, we have $s_f(m, x) \in \mathrm{Fix}(f)$, for every $n \geq m$, we have

$$s_f (n, x) = s_f (m, x) ,$$

and then $s_f (m, x)$ is an equilibrium of s_f associated with the initial state x.

iii) If $\overline{x} = s_f (m, x_0)$ is an equilibrium-state associated with the initial condition x_0, we have $s_f (m + 1, x_0) = s_f (m, x_0) = \overline{x}$, and taking into account that

$$s_f (m + 1, x_0) = f (s_f (m, x_0)) = f (\overline{x}) ,$$

we conclude $\overline{x} \in \mathrm{Fix}(f)$.

iv) We have

$$f (s_f (m, x_0)) = s_f (n + 1, x_0) = s_f (n, x_0) ,$$

so $s_f (m, x_0)$ is a fixed point of f belonging to the orbit $s_f (\cdot, x_0)$, and then by (ii) an equilibrium associated with x_0. ■

Theorem. *Let (X, τ) be a topological space and let $f : X \to X$ be a τ -continuous function. Let s_f be the discrete dynamical system generated by f, let x_0 be a state such that the orbit $s_f (\cdot, x_0)$ converges and let*

$$L := \lim_{n \to +\infty} s_f (n, x_0) .$$

Then, the asintotic equilibrium L of the system s_f is a fixed point of f.

Proof. We have

$$\begin{aligned} f(L) &= f\left(\lim_{n\to+\infty} s_f(n,x_0)\right) = \lim_{n\to+\infty} f(s_f(n,x_0)) = \\ &= \lim_{n\to+\infty} f(f^n(x_0)) = \lim_{n\to+\infty} f^{n+1}(x_0) = \\ &= \lim_{n\to+\infty} s_f(n+1,x_0) = L. \end{aligned}$$

As we desire. ■

Theorem. *Let (X,d) be a complete metric space and let $f : X \to X$ be a contraction on (X,d). Let s_f be the discrete dynamical system generated by f. Then, there exists only an equilibrium state L for s_f and, for every state x_0, the orbit $s_f(\cdot, x_0)$ converges to L, that is the only fixed point of f.*

Proof. It follows from the Banach fixed-point theorem. ■

We can introduce the main example of the paper.

Definition (of Bernoulli shift). *We define (deterministic) Bernoulli shift the system*

$$s_f : \mathbb{N}_0 \times \mathbb{R} \to \mathbb{R} : (n,x_0) \mapsto f^n(x_0),$$

generated by the map

$$f : [0,1] \to [0,1] : x \mapsto \begin{cases} 2x & \text{if} \quad x \in [0,1/2[\\ 2x-1 & \text{if} \quad x \in [1/2,1] \end{cases} .$$

4. Prigogine probabilistic systems and superpositions

Our aim, following Prigogine, is to associate with the Bernoully shift s_f, and more generally with a discrete dynamical system, a probabilistic dynamical system, i.e., a dynamical system whose space of states is a space of probability measures. To this end we must define rigorously the concept of probabilistic discrete dynamical system on a compact subset of $\mathbb{R}^n$. Our definition will be extremely natural. In the following we shall denote by $P(K)$ the set of probability measure on a compact subset K of $\mathbb{R}^n$, that is the subset of the dual of the Banach space $(C^0(K), \|\cdot\|_\infty)$ containing those non-negative functionals p whose value on the constant unitary real function 1_K is 1.

Definition (of probabilistic discrete dynamical system). *A probabilistic discrete dynamical system on a compact subset K of $\mathbb{R}^n$ is an action of the additive monoid of the natural numbers on the space of the Radon probability measures on the compact, in other terms, it is a discrete dynamical system on the space of the Radon probability measures on the compact; explicitly, it is a function $s : \mathbb{N}_0 \times P(K) \to P(K)$ such that $s(0,p_0) = p_0$, for every $p_0 \in P(K)$ and such that $s(m+n,p_0) = s(m, s(n,p_0))$, for every pair of natural numbers (m,n).*

We desire the classic dynamical systems on K correspond to particular probabilistic systems; to this aim note that, for every x_0 belonging to K, the Dirac measure δ_{x_0} means the certainty to find the system in the state x_0. For example, if s_f is the system generated by a function $f : [0,1] \to [0,1]$, the condition that the system s_f is in the state $f(x_0)$ at time 1 is $s_f(1, x_0) = f(x_0)$, this can be expressed, according to Prigogine, by the "integral" equality

$$\delta(x - f(x_0)) = \int_0^1 \delta(y - x_0)\, \delta(x - f(y))\, dy, \qquad (1)$$

unfortunately this expression does not have a precise mathematical sense, we shall give it a sense by the concept of superposition of a family of measures defined on a compact subset of $\mathbb{R}^n$. Moreover, the Prigogine's idea is the following: generalizing, if at a time $n \in \mathbb{N}_0$ we have a probability distribution of states p_n on $[0,1]$, at the time $n+1$ the system shall be in the probabilistic state p_{n+1} given by

$$U(p_n)(x) := p_{n+1}(x) = \int_0^1 p_n(y)\delta(x - f(y))\, dy,$$

regrettably, this expression holds, in the sense of distributions, when p_n and p_{n+1} are functions (and not too strange), but so we cannot use this expression for not-regular measures, and in particular we cannot generalize the equality (1). A correct and not-restrictive way to solve the problem can be found in the following section.

5. Superpositions of C^0-family in the space $C^{0\prime}(K)$

In the following, if H is a compact subset of $\mathbb{R}^n$, we denote by $C^{0\prime}(H)$ the dual of the Banach space $\left(C^0(H), \|\cdot\|_\infty\right)$.

Definition (families of class C^0). *Let $K \subseteq \mathbb{R}^n$ and $H \subseteq \mathbb{R}^m$ be compact subsets. Let $v = (v_i)_{i \in H}$ be a family in $C^{0\prime}(K)$ indexed by H. We define image of a continuous test function ϕ on K by the family v, the function*

$$v(\phi) : H \to \mathbb{R} : v(\phi)(i) = v_i(\phi).$$

If the image $v(\phi)$ lies in $C^0(H)$, for every $\phi \in C^0(K)$, we say that v is a family of class C^0. In these conditions we call the operator

$$\widehat{v} : C^0(K) \to C^0(H) : \phi \mapsto v(\phi),$$

operator generated by v.

Definition (superposition of measures). *Let $K \subseteq \mathbb{R}^n$ and $H \subseteq \mathbb{R}^m$ be compact subsets. Let $a \in C^{0\prime}(H)$, i.e., let a be a Radon-measure on H and let v be a family of class C^0 in $C^{0\prime}(K)$. The functional $a \circ \widehat{v}$ is denoted by*

$$\int_H av,$$

and is called superposition of v by the system of coefficients a. Moreover, if a is a probabilistic system of coefficients, i.e., if a is non-negative and with unitary integral, the superposition $\int_H av$ is called a convex superposition of v.

Theorem. *Let $K \subseteq \mathbb{R}^n$ and $H \subseteq \mathbb{R}^m$ be compact subsets. Let $a \in C^{0\prime}(H)$, and let v be a family of class C^0 in the space $C^{0\prime}(K)$ indexed by the compact H. Then the functional*

$$\int_H av : C^0(K) \to \mathbb{R},$$

is a Radon-measure, i.e., it is a continuous linear functional on $C^0(K)$. Moreover, if v is a family of probabilistic Radon-measures, every convex superposition of v is a probabilistic Radon-measure too.

Proof. Let ϕ be a continuous test function on K, we have, for some $j \in H$,

$$\begin{aligned}\left|\left(\int_H av\right)(\phi)\right| &= |a(v(\phi))| \leq \|a\|_{C^{0\prime}(H)} \|v(\phi)\|_{C^0(H)} = \|a\|_{C^{0\prime}(H)} \max_{i\in H} |v_i(\phi)| = \\ &= \|a\|_{C^{0\prime}(H)} |v_j(\phi)| \leq \|a\|_{C^{0\prime}(H)} \|v_j\|_{C^{0\prime}(K)} \|\phi\|_{C^0(K)},\end{aligned}$$

so $\int_H av$ is a bounded-linear functional and then it is continuous.

Let $v = (v_i)_{i\in H}$ be a C^0-family of probability measures on K, i.e., let v be a family of Radon-measures on K such that every v_i is normalized and non-negative, i.e., $v_i(1_K) = 1$ for every $i \in H$, and $v_i(\phi) \geq 0$, for every $\phi \in C^0(K)$ such that $\phi \geq 0$. Let $a \in C^{0\prime}(H)$ be (analogously) such that $a(1_H) = 1$ and $a \geq 0$.

Normalization. We have

$$\left(\int_H av\right)(1_K) = a(v(1_K));$$

now, for every i in H, we see $v(1_K)(i) = v_i(1_K) = 1$, that is equivalent to $v(1_K) = 1_H$, thus

$$\left(\int_H av\right)(1_K) = a(1_H) = 1.$$

Non-negativity. We have

$$\left(\int_H av\right)(\phi) = a(v(\phi));$$

now $v(\phi)(i) = v_i(\phi) \geq 0$, for every $i \in H$, by non-negativity of v_i, hence $v(\phi)$ is a non negative function, and then by non-negativity of a, we desume $a(v(\phi)) \geq 0$, hence every C^0 -convex superposition of v is a probability measure. ■

Theorem. *Let $f : H \to \mathbb{R}$ be a continuous function, let K be the range of f and let δ_f be the family $(\delta_{f(x)})_{x\in H}$ in the space $C^{0\prime}(K)$. Then, δ_f is a continuous family; moreover, for every measure $c \in C^{0\prime}(H)$, the superposition $\int_H c\delta_f$ is a measure in $C^{0\prime}(K)$ and we have*

$$\left(\int_H c\delta_f\right)(\phi) = \int_H \phi \circ f \, d\mu_c.$$

Proof. Let us compute the image $\delta_f(\phi)$ for every $\phi \in C^0(K)$; we have

$$\delta_f(\phi) : H \to \mathbb{R}, \quad \delta_f(\phi)(x) = \delta_{f(x)}(\phi) = \phi(f(x)) = (\phi \circ f)(x),$$

hence $\delta_f(\phi) = \phi \circ f$; since $\phi \circ f \in C^0(H)$, δ_f is a C^0-family. Let $c \in C^{0\prime}(H)$, we have so

$$\left(\int_H c\delta_f \right)(\phi) = c(\phi \circ f) . \blacksquare$$

6. Superpositions of measurable-families in the space $C^{0\prime}(K)$

We desire to extend the definition of superposition to families v indexed by compact intervalls of the real line and piecewise-continuous. A function $f : [a, b] \to \mathbb{R}$ is said piecewise-continuous if it is continuous but in a finite set of points in which the right and left limits there exist finite and in which f is right or left continuous. To this end we shall define a larger collection of families: the class of measurable families.

Definition (measurable families). *Let $K \subseteq \mathbb{R}^n$ and $H \subseteq \mathbb{R}^m$ be compact subsets and let $v = (v_i)_{i \in H}$ be a family in $C^{0\prime}(K)$ indexed by H. The image of a continuous test function ϕ on K under the family v, is again the function*

$$v(\phi) : H \to \mathbb{R} : v(\phi)(i) = v_i(\phi) .$$

If the image of $v(\phi)$ is measurable in the sense of Borel (retro-image of open sets are Borel-measurable sets), for every $\phi \in C^0(K)$, we say that v is a measurable family. In these conditions, if $\mathcal{B}(H, \mathbb{R})$ is the set of Borel-measurable real functions on H, we call the operator

$$\widehat{v} : C^0(K) \to \mathcal{B}(H, \mathbb{R}) : \phi \mapsto v(\phi) ,$$

operator generated by v.

If v is a measurable family and if $a \in C^{0\prime}(H)$, then the functional

$$\int_H av : C^0(K) \to \mathbb{R} : \left(\int_H av \right)(\phi) := \int_H v(\phi) \; d\mu_a,$$

where μ_a is the unique Borel measure associated with the functional a by the Riesz Representation Theorem, is called superposition of v under the system of coefficients a.

Theorem. *Let $a \in C^{0\prime}(H)$, and let v be a measurable family in the space $C^{0\prime}(K)$ indexed by the compact H. Then, if for every $\phi \in C^0(K)$ the function $|v(\phi)|$ attains his maximum, the functional*

$$\int_H av : C^0(K) \to \mathbb{R},$$

is a mensure-distribution, i.e., a continuous linear functional on $C^0(K)$. Moreover, if v is a family of probabilistic mensural distributions, every convex superposition of v is a probabilistic mensural distribution too.

Proof. Let ϕ be a continuous test function on K, we have, for some $j \in H$,

$$\begin{aligned} \left| \left(\int_H av \right)(\phi) \right| &= \left| \int_H v(\phi) \, d\mu_a \right| \le \int_H |v(\phi)| \, d|\mu_a| \le |\mu_a|(H) \sup |v(\phi)| = \\ &= \|a\|_{C^{0\prime}(H)} \max_{i \in H} |v_i(\phi)| = \|a\|_{C^{0\prime}(H)} |v_j(\phi)| \le \\ &\le \|a\|_{C^{0\prime}(H)} \|v_j\|_{C^{0\prime}(K)} \|\phi\|_{C^0(K)} , \end{aligned}$$

so $\int_H av$ is a bounded-linear functional and then it is continuous.

Let $v = (v_i)_{i\in H}$ be a C^0-family of probability distributions on K, i.e., be such that every v_i is normalized and non-negative, i.e., $v_i(1_K) = 1$, for every $i \in H$, and $v_i(\phi) \geq 0$, for every $\phi \in C^0(K)$ such that $\phi \geq 0$. Let $a \in C^{0\prime}(H)$ be (analogously) such that $a(1_H) = 1$ and $a \geq 0$.

Normalization. We have

$$\left(\int_H av\right)(1_K) = \int_H v(1_K)\, d\mu_a;$$

now, for every i in H, we see $v(1_K)(i) = v_i(1_K) = 1$, that is equivalent to $v(1_K) = 1_H$, thus

$$\left(\int_H av\right)(1_K) = \int_H 1_H\, d\mu_a = a(1_H) = 1.$$

Non-negativity. We have

$$\left(\int_H av\right)(\phi) = \int_H v(\phi)\, d\mu_a;$$

now $v(\phi)(i) = v_i(\phi) \geq 0$, for every $i \in H$, by non-negativity of v_i, hence $v(\phi)$ is a non negative function, and then by non-negativity of a,

$$\int_H v(\phi)\, d\mu_a \geq 0.$$

hence every C^0-convex superposition of v is a probability distribution. ■

Theorem. *Let $f : H \to \mathbb{R}$ be a Borel-measurable function with compact range K and let δ_f be the family $(\delta_{f(x)})_{x\in H}$ in the space $C^{0\prime}(K)$. Then, δ_f is a measurable family, moreover, for every $c \in C^{0\prime}(H)$ the superposition $\int_H c\delta_f$ is a measure and in particular we have*

$$\left(\int_H c\delta_f\right)(\phi) = \int_H \phi \circ f\, d\mu_c.$$

Proof. For every $\phi \in C^0(K)$, we have

$$\delta_f(\phi) : H \to \mathbb{R},\ \ \delta_f(\phi) = \phi \circ f \in \mathcal{B}(H, \mathbb{R}),$$

where $\mathcal{B}(H, \mathbb{R})$ is the set of Borel-measurable real functions on H. Thus δ_f is a measurable-family. Note that since $|\phi|$ is continuous it attains its maximum on the compact $f(H)$, and then $|\phi \circ f|$ attains its maximum on H. Concluding, by the preceding theorem, for every $c \in C^{0\prime}(H)$, the superposition $\int_H c\delta_f$ is a measure and we have

$$\left(\int_H c\delta_f\right)(\phi) = \int_H \phi \circ f\, d\mu_c. \blacksquare$$

7. Probabilistic systems via superpositions

Recall that our aim is to associate with the classic dynamical systems on a compact subset K probabilistic dynamical systems on the same subset. In particular we desire to associate with the dynamical system generated by a mapping $f : K \to K$ a probabilistic dynamical system on K. For instance, as we said, if s_f is the system generated by a function $f : [0,1] \to [0,1]$, the condition that the system s_f can be found in the state $f(x_0)$ at time 1 is $s_f\,(1, x_0) = f\,(x_0)$, this can be expressed, at last rigorously, in terms of superposition

$$\delta_{f(x_0)} = \int_{[0,1]} \delta_{x_0} \left(\delta_{f(y)}\right)_{y\in[0,1]},$$

we can read: the certainty to find the system s_f in the state $f\,(x_0)$ at time 1 is the superposition of the family $\left(\delta_{f(y)}\right)_{y\in[0,1]}$ with respect to the system of coefficients δ_{x_0}. Generalizing, if at the time $n \in \mathbb{N}_0$ we have a probability distribution of states p_n on $[0,1]$, at the time $n+1$ the system shall be in the probabilistic state p_{n+1} given by

$$U\,(p_n) := p_{n+1} = \int_{[0,1]} p_n \left(\delta_{f(y)}\right)_{y\in[0,1]}.$$

These considerations allow us to define a probabilistic system generated by a map.

Definition (of probabilistic discrete dynamical system generated by a map). *A probabilistic discrete dynamical system on a compact subset K of $\mathbb{R}^n$ generated by a Borel-measurable function $f : K \to K$ with compact range is defined to be the dynamical system*

$$s_f : \mathbb{N}_0 \times P(K) \to P(K)$$

defined recursively by

$$s_f(0,p) = p\ , \quad s_f(n+1,p) = \int_K s_f(n,p)(\delta_{f(y)})_{y\in K},$$

for every probabilistic state p and every time n. The evolution operator of s_f is the operator

$$U : P(K) \to P(K) : U(p) = \int_K p(\delta_{f(y)})_{y\in K}.$$

Remark. Note that the compactness of the range of f guarantees the measurability of δ_f.

Remark. In other terms, the probabilistic discrete dynamical system on a compact subset K of $\mathbb{R}^n$ generated by a Borel-measurable function $f : K \to K$ with compact range is defined to be the dynamical system generated by the function $F : P(K) \to P(K)$ defined by

$$F(p) = \int_K p(\delta_{f(y)})_{y\in K}.$$

8. Superpositions of C^0_c-families in the space $C^{0\prime}_c(T)$

In the following we denote by $C^0_c(T)$ the space of continuous functions with compact support on a locally compact space T, the dual of $C^0_c(T)$ with respect to its standard locally convex topology is the space of measures on T and it's simply denoted by $C^{0\prime}_c(T)$.

Definition (families of class C_c^0). *Let $T \subseteq \mathbb{R}^n$ and $H \subseteq \mathbb{R}^m$ be locally compact subsets. Let $v = (v_i)_{i \in H}$ be a family in the space of measures $C_c^{0\prime}(T)$ indexed by H. We define image of a continuous test function ϕ with compact support on T by the family v, the function*

$$v(\phi) : H \to \mathbb{R} : v(\phi)(i) = v_i(\phi).$$

If the image of $v(\phi)$ lies in $C_c^0(H)$, for every $\phi \in C_c^0(T)$, we say that v is a family of class C_c^0. In this condition we define the operator

$$\widehat{v} : C_c^0(T) \to C_c^0(H) : \phi \mapsto v(\phi),$$

and we call it the operator generated by v.

Definition (superposition of measures). *In the above conditions, let $a \in C_c^{0\prime}(H)$, i.e, let a be a measure on H and let v be a family of class C_c^0 in the space $C_c^{0\prime}(K)$. The functional $a \circ \widehat{v}$ is denoted by*

$$\int_H av,$$

and is called superposition of v by the system of coefficient a. Moreover, if a is a probabilistic system of coefficients, i.e., if a is non-negative and with unitary integral, the superposition $\int_H av$ is called a convex superposition of v.

Theorem. *Let $a \in C_c^{0\prime}(H)$, and let v be a family of class C_c^0 in the space $C_c^{0\prime}(T)$ indexed by the locally compact H. Then the functional*

$$\int_H av : C_c^0(T) \to \mathbb{R},$$

is a meansure, i.e., a linear continuous functional on $C_c^0(K)$.

Proof. Let K be a compact subset of T, and let ϕ be a real continuous function on T, with compact support contained in K, moreover let H' the support (compact) of $v(\phi)$, we have, for some $j \in H$,

$$\begin{aligned} \left| \left(\int_H av \right)(\phi) \right| &= |a(v(\phi))| \leq M_{H'}^a \|v(\phi)\|_{C^0(H)} = M_{H'}^a \max_{i \in H} |v_i(\phi)| = \\ &= M_{H'}^a \cdot |v_j(\phi)| \leq M_{H'}^a \cdot M_K^{v_j} \cdot \|\phi\|_{C^0(K)}, \end{aligned}$$

so $\int_H av$ is a bounded-linear functional and then it is continuous. ■

Theorem. *Let I, J be intervalls of the real line, $f : I \to J$ be a continuous bijective function and let δ_f be the family $(\delta_{f(x)})_{x \in I}$ in the space $C_c^{0\prime}(J)$. Then, δ_f is a family of class C_c^0; moreover, for every measure $c \in C_c^{0\prime}(I)$, the superposition $\int_I c\delta_f$ is a measure and we have*

$$\left(\int_I c\delta_f \right)(\phi) = \int_I \phi \circ f \, d\mu_c.$$

Proof. Note that f is a homeomorphism of I onto J (by a classic result of Analysis). Let us compute $\delta_f(\phi)$ for every $\phi \in C_c^0(J)$; we have

$$\delta_f(\phi) : I \to \mathbb{R}, \quad \delta_f(\phi)(x) = \delta_{f(x)}(\phi) = \phi(f(x)) = (\phi \circ f)(x),$$

hence $\delta_f(\phi) = \phi \circ f$; note that $\phi \circ f \in C_c^0(I)$, infact,

$$\operatorname{supp}(\phi \circ f) = f^-(\operatorname{supp}\phi).$$

Thus δ_f is a C_c^0-family in $C_c^{0\prime}(J)$. Let $c \in C^{0\prime}(I)$, we have so

$$\left(\int_I c\delta_f\right)(\phi) = c(\phi \circ f). \blacksquare$$

9. The generalized Perron-Frobenius operator

In this section we shall prove a generalization of the following result of Prigogine:

Theorem (Prigogine). *Let Bernoully shift at time 0 be in a probabilistic-state p_0, and let p_0 be a regular measure generated by a certain real continuous function g_0 defined on $[0,1]$. Then, the orbit $(g_n)_{n\in\mathbb{N}_0}$ starting from g_0 verifies*

$$g_{n+1}(x) = \frac{1}{2}\left(g_n\left(\frac{x}{2}\right) + g_n\left(\frac{x+1}{2}\right)\right),$$

for every time $n \in \mathbb{N}_0$.

First, we have to consider a generalization of composition of a measure with a function. Recall that

Definition (of composition in $C_c^{0\prime}(\Omega)$ with a diffeomorphism). *Let Ω,Ω' be two open sets of $\mathbb{R}^n$ and let $F \in \operatorname{Diff}^1(\Omega',\Omega)$, that is F is bijective, $F \in C^1(\Omega',\Omega)$, $F^- \in C^1(\Omega,\Omega')$. Then, for every $\psi \in C_c^0(\Omega')$, the function $(\psi \circ F^-)|\det J_{F^-}|$ belongs to the space $C_c^0(\Omega)$, moreover, for every $u \in C_c^{0\prime}(\Omega)$, the functional*

$$u \circ F : C_c^0(\Omega') \to \mathbb{K} : \psi \mapsto u\left((\psi \circ F^-)|\det J_{F^-}|\right)$$

is a measure on Ω' called the composition of u with F.

In our context we have to define composition in the space $C_c^{0\prime}(T)$, with T locally compact subspace of $\mathbb{R}^n$. We recall that T is a locally compact subspace of $\mathbb{R}^n$ if and only if it is the intersection of an open subset with a closed subset of $\mathbb{R}^n$. The extension is immediate:

Definition (of composition in $C_c^{0\prime}(T)$ with a diffeomorphism). *Let T, T' be two locally compact subsets of $\mathbb{R}^n$, let $F \in \operatorname{Diff}^1(T',T)$, that is there exists a function $F_0 \in \operatorname{Diff}^1(\Omega',\Omega)$, with Ω, Ω' open sets of $\mathbb{R}^n$ containing T and T' respectively, and such that*

$$(F_0)_{|(T',T)} = F.$$

Then, for every $\psi \in C_c^0(T')$, the function $(\psi \circ F^-)|\det J_{F^-}|$ belongs to the space $C_c^0(T)$ and moreover, for every $u \in C_c^{0\prime}(T)$, the functional

$$u \circ F : C_c^0(T') \to \mathbb{K} : \psi \mapsto u\left((\psi \circ F^-)|\det J_{F^-}|\right)$$

is a measure on T' called the composition of u with F.

Recall that a real function f, defined on an intervall of the real line, is said piecewise-affine if there is a (ordered) partition $I = (I_j)_{j=1}^k$ of its domain such that each set of I is an intervall and f is affine on every set of I. If f_j is the restriction of f to I_j and L_j the slope of f_j, the systems $(f_j)_{j=1}^k$ and $(L_j)_{j=1}^k$ are defined the system of restrictions and the system of slopes of f with respect to I.

Theorem. *Let φ_0 be a measure on $[0,1]$, and let φ be the orbit starting from φ_0 of the probabilistic dynamical system generated by a piecewise-affine mapping $f : [0,1] \to [0,1]$ of partition $(I_j)_{j=1}^k$ and associated system of restrictions $(f_j)_{j=1}^k$. Assume f with compact image (widely) contained in $[0,1]$ and assume the slopes different from 0. Then, for every time n, we have*

$$\varphi_{n+1}(\phi) = \sum_{j=1}^{k} \frac{1}{|L_j|} \left(\varphi_n \circ f_j^{-}\right) \left(\phi_{|f_j(I_j)}\right),$$

for every $\phi \in C^0([0,1])$, where L_j is the slope of f_j.

Proof. Let $\phi \in C^0([0,1])$, note that the superposition $\int_{[0,1]} \varphi_n \delta_f$ is a measure by a previous result, since f has compact image and it is measurable. Moreover, for each n,

$$\begin{aligned} \varphi_{n+1}(\phi) &= \left(\int_{[0,1]} \varphi_n \delta_f\right)(\phi) = \int_{[0,1]} \phi \circ f \, d\varphi_n = \sum_{j=1}^{k} \int_{I_j} \phi \circ f \, d\varphi_n = \\ &= \sum_{j=1}^{k} \int_{I_j} \phi_{|f_j(I_j)} \circ f_i \, d(\varphi_n)_{|I_j} = \sum_{j=1}^{k} \left(\int_{I_j} (\varphi_n)_{|I_j} \delta_{f_j}\right) (\phi_{|f_j(I_j)}). \end{aligned}$$

To justify the last equality note that, for every $\psi \in C_c^0(f_j(I_j))$,

$$\left(\int_{I_j} (\varphi_n)_{|I_j} \delta_{f_j}\right)(\psi) = (\varphi_n)_{|I_j} \left(\widehat{\delta}_{f_j}(\psi)\right),$$

in fact, $\delta_{f_j}(\psi)$ is a continuous function with compact support, since, for every y in I_j,

$$\delta_{f_j}(\psi)(y) = \delta_{f_j(y)}(\psi) = \psi(f_j(y)),$$

hence $\delta_{f_j}(\psi) = \psi \circ f_j$, moreover f_j is a homeomorphism, so $f_j^{-}(\mathrm{supp}\psi)$ is compact and it is the support of $\psi \circ f_j$. So the family δ_{f_j} is a family of class C_c^0. Concluding, the superposition

$$\mu := \int_{I_j} (\varphi_n)_{|I_j} \delta_{f_j},$$

is a measure on I_j. Moreover, since for every $\psi \in C_c^0(f_j(I_j))$ it is

$$\left|(\varphi_n)_{|I_j} (\widehat{\delta}_{f_j}(\psi))\right| \leq |\varphi_n|([0,1]) \, \|\psi\|_{C_b^0(I_j)},$$

μ is a continuous linear functional on $C_c^0(f(I_j))$. Note that, although the function $\phi_{|f_j(I_j)}$ can be not with compact support, it is certainly in $\mathcal{L}_\mu^1(f_j(I_j))$ (it is continuous and bounded on $f_j(I_j)$, i.e., of class C_b^0), and then the expression $\mu(\phi_{|f_j(I_j)})$ is well defined.

Note being the function $f_j : I_j \to f_j(I_j) : x \mapsto L_j x$ an invertible function, by definition of composition of a distribution with a diffeomorphism, we have

$$\left(\int_{I_j} \varphi_n \delta_{f_j}\right)(\psi) = \varphi_n(\psi \circ f_j) = \frac{1}{|L_j|}\left(\varphi_n \circ f_j^{-1}\right)(\psi),$$

in other words,

$$\int_{I_j} \varphi_n \delta_{f_j} = \frac{1}{|L_j|}(\varphi_n)_{|I_j} \circ f_j^{-1} = \frac{1}{|L_j|}\varphi_n \circ f_j^{-1},$$

and the theorem is completely proved. ■

We conclude the argumentation generalizing the Perron-Frobenius operator.

Definition (the generalized Perron-Frobenius operator). *Let f be a piecewise-affine function on an intervall $[a,b]$ with partition $(I_j)_{j=1}^k$, system of restrictions $(f_j)_{j=1}^k$ and system of slopes $(L_j)_{j=1}^k$. Assume f with compact image (widely) contained in $[a,b]$ and assume the slopes different from 0. We call the operator*

$$P : C^{0\prime}([a,b]) \to C^{0\prime}([a,b])$$

defined by

$$P(\mu)(\phi) = \sum_{j=1}^{k} \frac{1}{|L_j|}\left(\mu \circ f_j^{-}\right)\left(\phi_{|f_j(I_j)}\right),$$

for every measure μ on $[a,b]$, the generalized Perron-Frobenius operator associated with the piecewise-affine function f.

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